Localisation and Curve Veering: A Different Perspective on Modal Interactions

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ABSTRACT

The interaction of vibration modes subject to parametric variation has long been observed to cause curious effects, notably those of localisation and frequency curve veering. These phenomena have great significance in practical applications: localisation is responsible for unexpectedly high levels of response in periodic structures, for example in turbine rotor assemblies, and frequency curve veering has shown correlation with a variety of critical behaviours, for example in the stability analysis of helicopter vibrations. This paper presents an in-depth analysis of the behaviour and derives new system properties with respect to parameter perturbations. The properties offer a different perspective on the interaction of vibration modes and the fresh insight is demonstrated with reference to some example problems. Two normalised criteria are suggested to standardise the approach to such problems. Combining these criteria produces a veering index which is proposed as a definitive measure of the presence and intensity of curve veering.

Nomenclature

α	angle between arbitrary reference eigenvectors and transformed eigenvectors in the normal coordinate system
β	angle between arbitrary eigenvectors and reference datum eigenvectors in the normal coordinate system
δ_j	arbitrary parameter
δ_K	parameter representing linear stiffness matrix variation
κ_{ijk}	coupling between i^{th} and k^{th} modes with respect to variation of parameter δ_j, alternatively "cross-sensitivity"
λ_i	i^{th} eigenvalue
ϕ_i	i^{th} mass-normalised eigenvector
σ_{iji}	sensitivity of i^{th} eigenvalue to parameter δ_j
$\Delta\lambda_{ik}$	separation of i^{th} and k^{th} eigenvalues, $\lambda_i - \lambda_k$
$\Delta\sigma_{kji}$	difference in eigenvalue sensitivies with respect to δ_j for k^{th} and i^{th} modes, $\sigma_{kjk} - \sigma_{iji}$
$\mathbf{\Lambda}$	diagonal matrix of eigenvalues
$\mathbf{\Lambda}_{ik}$	diagonal matrix of i^{th} and k^{th} eigenvalues
$\mathbf{\Phi}$	full eigenvector matrix
$\mathbf{\Phi}_{ik}$	matrix of i^{th} and k^{th} eigenvectors
$\mathbf{\Psi}_{ik}$	matrix of i^{th} and k^{th} stiffness-normalised eigenvectors (in explicit contexts normalised using $\mathbf{A}$ instead of $\mathbf{K}$)
$\mathbf{\Sigma}_j$	sensitivity matrix, where diagonal terms are eigenvalue sensitivities and off-diagonals are modal coupling with respect to parameter δ_j
$\mathbf{\Sigma}_{ijk}$	submatrix of full sensitivity matrix, containing rows and columns corresponding to i^{th} and k^{th} modes
k	lumped stiffness
m	lumped mass

s	small coupling spring stiffness
$\mathbf{A}$	arbitrary constant matrix
CSQ_{ijk}	cross-sensitivity quotient for modes i and k with respect to variation of parameter δ_j
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
δ_M	parameter representing linear mass matrix variation
$\mathbf{M}$	mass matrix
MDF_{ijk}	modal dependence factor for modes i and k with respect to variation of parameter j
Q_{ijk}	quantity describing the validity of the assumption of a fixed subspace as modes i and j vary under variation of parameter δ_j
$\mathbf{T}$	rotational transform matrix
VI_{ijk}	veering index for modes i and k with respect to parameter variation j
$'$	denotes properties of modes after parameter variation
$\bar{\bullet}$	bar denotes reference datum quantities
$*$	denotes properties adapted to stiffness-normalised analysis

1 INTRODUCTION

Localisation is a phenomenon that has oft been observed in periodic structures, where small disturbances in the periodicity may lead to the confinement of one or more vibration modes to small regions. The first rigorous experimental demonstration was that of Hodges and Woodhouse [1]. Significantly, the behaviour can be induced through almost imperceptible perturbations to the symmetry yet leads to far greater vibration loads than those predicted analytically. Such problems are encountered commonly, for example, in bladed disc assemblies in turbomachinery [2].

Eigenvalue curve veering has an interesting history of discovery, with the rigour of early calculations (for example, [3]) called into question [4] before exact solutions were found exhibiting the behaviour [5]. Its most straightforward manifestation is observed when two eigenvalue loci, traced by a system under parametric variation, converge upon one another only to veer abruptly away again. The loci swap trajectories and the modes swap properties. The proliferation of detailed parametric studies facilitated by the advent of modern computing power has produced a slew of examples in the literature (for example the surface veering of Plaut et al. [6]) and an experimental correlation is presented by du Bois et al. [7]. It has been shown in recent studies to be useful in predicting stability in rotorcraft and other rotating blade assemblies [8].

Veering and localisation are intrinsically linked; while it is possible to observe veering without localisation, the reverse is rarely true. In fact, it was the gross distortions of otherwise predictable mode shapes in transition regions that apparently attracted much of the criticism directed towards the early theoretical predictions of veering and it is the self same distortions that lead to localisation, or more accurately the lack thereof. Thus, the authors believe that a holistic approach as adopted in works such as those of Pierre [9] and Triantafyllou and Triantafyllou [10] will aid in the understanding of both phenomena.

Critically, the authors believe that quantitative measures are required if the behaviour is to be studied, manipulated or exploited successfully. Several authors have made efforts to quantify veering. An adept methodology is set out by Perkins and Mote [5], who derive "modal coupling factors" to determine the behaviour of converging modes. While the term "modal coupling" is sometimes associated with the coupling induced through non-proportional damping, the systems discussed here are undamped. Paradoxically, by definition, modes are uncoupled within an undamped system. The parametric variations extend the scope of the system, however, and within this metasystem the modal properties are coupled. The coupling factors provide a good qualitative description of the metasystem and as will be seen they form important quantities in the analysis, but on their own the values do not provide significant insight. Recognising a need to identify veering regions, Liu [11] proposed using the eigenvector derivatives or the second derivatives of the eigenvalues to quantify the behaviour. Once again, however, the significance of the values would need to be interpreted on a case by case basis.

The approach taken in this paper is to use quantities derived from the modal coupling in conjuction with an analysis of the eigenvector transformations to produce a universal description of veering. Two non-dimensional quantities are derived to describe the behaviour in terms of tangible effects and when combined these provide a single, unambiguous measure of the presence and severity of veering. It is thought that this will facilitate the advancement of both veering and localisation studies.

The present analysis is limited to self-adjoint, undamped systems with linear matrix variations, with the intention of generalising the method in a later paper. Each concept is demonstrated first for variations of the stiffness matrix, where the motives behind the approach are clearest, before introducing the analogous methods for mass matrix variation.

2 MODAL COUPLING

Consider a self-adjoint, discrete, undamped structural dynamic eigenproblem. Fox and Kapoor [12] derive the eigenvalue sensitivity to a parameter δ_j as

$$\frac{d\lambda_i}{d\delta_j} = \phi_i^T \left(\frac{d\mathbf{K}}{d\delta_j} - \lambda_i \frac{d\mathbf{M}}{d\delta_j} \right) \phi_i \tag{1}$$

where λ_i and ϕ_i are the eigenvalue and mass normalised eigenvector of the i^{th} mode and $\mathbf{M}$ and $\mathbf{K}$ are the system mass and stiffness matrices. The corresponding eigenvector sensitivity is given as

$$\frac{d\phi_i}{d\delta_j} = -\frac{\phi_i^T \frac{d\mathbf{M}}{d\delta_j} \phi_i}{2} \phi_i + \sum_{r \neq i} \frac{\phi_r^T \left(\frac{d\mathbf{K}}{d\delta_j} - \lambda_i \frac{d\mathbf{M}}{d\delta_j} \right) \phi_i}{\Delta\lambda_{ir}} \phi_r \tag{2}$$

where $\Delta\lambda_{ir} = \lambda_i - \lambda_r$. Differentiating eqn. (1) with respect to δ_j and using eqn. (2) yields

$$\frac{d^2\lambda_i}{d\delta_j^2} = \phi_i^T \left(\frac{d^2\mathbf{K}}{d\delta_j^2} - \lambda_i \frac{d^2\mathbf{M}}{d\delta_j^2} - 2\frac{d\lambda_i}{d\delta_j}\frac{d\mathbf{M}}{d\delta_j} \right) \phi_i + 2\sum_{r \neq i} \frac{\left[\phi_r^T \left(\frac{d\mathbf{K}}{d\delta_j} - \lambda_i \frac{d\mathbf{M}}{d\delta_j} \right) \phi_i \right]^2}{\Delta\lambda_{ir}} \tag{3}$$

where $\frac{d^2\lambda_i}{d\delta_j^2}$ is the second derivative, or curvature, of the eigenvalue. If the i^{th} and k^{th} eigenvalues become close such that $\Delta\lambda_{ik}$ is very small then the expression for curvature is dominated by the corresponding term in the summation where $r = k$, and it is this term that is responsible for the veering of the eigenvalue loci. The numerator of that term is $2\left[\phi_k^T \left(\frac{d\mathbf{K}}{d\delta_j} - \lambda_i \frac{d\mathbf{M}}{d\delta_j} \right) \phi_i \right]^2$, which is analogous to Perkins and Mote's "coupling factor" [5]. For the purposes of this paper the "modal coupling" shall be defined slightly differently as

$$\kappa_{ijk} = \phi_k^T \left(\frac{d\mathbf{K}}{d\delta_j} - \lambda_i \frac{d\mathbf{M}}{d\delta_j} \right) \phi_i. \tag{4}$$

Expanding this to the full set of modes, a sensitivity matrix can be defined:

$$\mathbf{\Sigma}_j = \mathbf{\Phi}^T \frac{d\mathbf{K}}{d\delta_j} \mathbf{\Phi} - \mathbf{\Phi}^T \frac{d\mathbf{M}}{d\delta_j} \mathbf{\Phi} \mathbf{\Lambda} \tag{5}$$

where $\mathbf{\Phi}$ is the complete matrix of eigenvectors, $[\phi_1, \phi_2 ... \phi_N]$, and $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues. The diagonal terms in $\mathbf{\Sigma}_j$ are the eigenvalue sensitivities and the off-diagonal terms are the modal coupling, which can be interpreted as cross-sensitivities representing the influence of each mode on the derivatives of the other modes' properties.

3 EIGENVECTOR ROTATION

For proximate modes i and k, if $\Delta\lambda_{ik} << \Delta\lambda_{ir}$ for all $r \neq i, k$ then eqn. (2) can be approximated by

$$\frac{d\phi_i}{d\delta_j} \approx -\left(\frac{1}{2} \phi_i^T \frac{d\mathbf{M}}{d\delta_j} \phi_i \right) \phi_i + \left(\frac{\kappa_{ijk}}{\Delta\lambda_{ik}} \right) \phi_k. \tag{6}$$

From this equation (and the equivalent expression for $\frac{d\phi_k}{d\delta_j}$) it is seen that the two vectors throughout veering can always be represented by a linear combination of a single pair of vectors; as they transform, they always remain in the same plane or subspace. Furthermore, the validity of this assumption can be quantified for each mode by comparing the ℓ^2-norms of eqns. (6) and (2) within the normal basis:

$$Q_{ijk} = \sqrt{\frac{-\left(\frac{1}{2}\phi_i^T \frac{d\mathbf{M}}{d\delta_j}\phi_i\right)^2 + \left(\frac{\kappa_{ijk}}{\Delta\lambda_{ik}}\right)^2}{-\left(\frac{1}{2}\phi_i^T \frac{d\mathbf{M}}{d\delta_j}\phi_i\right)^2 + \sum_{r\neq i}\left(\frac{\kappa_{ijr}}{\Delta\lambda_{ir}}\right)^2}} \tag{7}$$

and noting that the summed term in the denominator is easily computed using a single column of the sensitivity matrix in eqn. (5).

Suppose that a constant matrix, $\mathbf{A}$, can be found such that $\mathbf{\Psi}_{ik}^T\mathbf{A}\mathbf{\Psi}_{ik} = \mathbf{I}$ for all values of δ_j, where $\mathbf{I}$ is an identity matrix and $\mathbf{\Psi}_{ik}$ is the $N\times 2$ matrix of $\mathbf{A}$-normalised eigenvectors, $[\psi_i, \psi_k]$. In this case, the two eigenvectors will always form an orthonormal basis with respect to $\mathbf{A}$, and their magnitude and orientation within the subspace can be defined relative to a set of reference vectors by a single angle. This is illustrated in fig. 1, and can be expressed

$$\mathbf{\Psi}'_{ik}(\mathbf{\Psi}_{ik},\alpha) = \mathbf{\Psi}_{ik}\mathbf{T}, \qquad \mathbf{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{8}$$

This is a generalisation of the system described by Balmés [13] and demonstrates that his observations may be *directly* extrapolated to any veering system, contingent on the existence of an appropriate orthonormalising matrix and satisfactory agreement with eqn. (6). The latter is generally true for proximate modes. The former is achieved most readily by keeping either the mass or stiffness matrix constant and these two scenarios will be considered in the sections that follow.

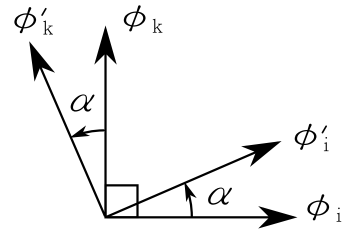


Figure 1: Orthogonal mode shape vectors, for jth and kth modes, transforming into new modes within their subspace.

4 CROSS-SENSITIVITY QUOTIENT

In this section the variation of the modal coupling throughout veering is investigated. A reduced sensitivity matrix for modes i and k shall be defined as

$$\mathbf{\Sigma}_{ijk} = \mathbf{\Phi}_{ik}^T\frac{d\mathbf{K}}{d\delta_j}\mathbf{\Phi}_{ik} - \mathbf{\Phi}_{ik}^T\frac{d\mathbf{M}}{d\delta_j}\mathbf{\Phi}_{ik}\mathbf{\Lambda}_{ik} = \begin{bmatrix} \sigma_{iji} & \kappa_{ijk} \\ \kappa_{kji} & \sigma_{kjk} \end{bmatrix} \tag{9}$$

where σ_{iji} is equivalent to the eigenvalue sensitivity, $\lambda_{i,j}$. Considering a linear variation in the stiffness matrix, represented by δ_K, the mass matrix remains constant and serves as an orthonormalising matrix, allowing the substitution of eqn. (8) in eqn. (9) using $\mathbf{\Psi}_{ik} = \mathbf{\Phi}_{ik}$. Noting that $\frac{d\mathbf{M}}{d\delta_K} \equiv 0$ and $\frac{d^2\mathbf{K}}{d\delta_K^2} \equiv 0$ so $\frac{d\mathbf{K}}{d\delta_K} = \frac{d\mathbf{K}}{d\delta_K}'$, this substitution produces

$$\mathbf{\Sigma}'_{iKk} = \mathbf{\Phi}'^T_{ik}\frac{d\mathbf{K}}{d\delta_K}'\mathbf{\Phi}'_{ik} = \mathbf{T}^T\mathbf{\Phi}_{ik}^T\frac{d\mathbf{K}}{d\delta_K}\mathbf{\Phi}_{ik}\mathbf{T} = \mathbf{T}^T\mathbf{\Sigma}_{iKk}\mathbf{T} \tag{10}$$

where $\mathbf{\Sigma}'_{iKk}$ and $\frac{d\mathbf{K}}{d\delta_K}'$ are the sensitivity and stiffness matrices corresponding to the eigenvectors $\mathbf{\Phi}'_{ik}$ produced by a change in δ_K. $\mathbf{\Sigma}'_{iKk}$ is a symmetric 2×2 matrix in which the off-diagonal elements are equal:

$$\begin{aligned} \kappa'_{iKk} = \kappa'_{kKi} &= \kappa_{iKk}(\cos^2\alpha - \sin^2\alpha) + (\sigma_{kKk} - \sigma_{iKi})\cos\alpha\,\sin\alpha \\ &= \overline{\kappa}_{iKk}\cos\,(2\alpha - 2\beta) \end{aligned} \tag{11}$$

where

$$\tan\,(2\beta) = \Delta\sigma_{kKi}/2\kappa_{iKk} \tag{12}$$

$$\overline{\kappa}^2_{iKk} = {\kappa_{iKk}}^2 + (\Delta\sigma_{kKi}/2)^2 \tag{13}$$

and $\Delta\sigma_{kKi} = \sigma_{kKk} - \sigma_{iKi}$. The modal coupling is seen to vary harmonically with the orientation of the vectors. The maximum coupling is given by eqn. (13) and this is used to define a corresponding set of reference vectors, $\overline{\mathbf{\Phi}}_{ik}$. Setting $\mathbf{\Phi}'_{ik} = \overline{\mathbf{\Phi}}_{ik}$ gives $\kappa'_{iKk} = \overline{\kappa}_{iKk}$ and hence from eqn. (11) $\alpha = \beta$, so that eqn. (12) describes the angle between $\mathbf{\Phi}_{ik}$ and $\overline{\mathbf{\Phi}}_{ik}$. Setting $\mathbf{\Phi}'_{ik} = \mathbf{\Phi}_{ik}$ gives $\kappa'_{iKk} = \kappa_{iKk}$ and $\alpha = 0$, so that eqn. (11) produces

$$\kappa_{iKk} = \overline{\kappa}_{iKk}\cos\ (2\beta) \tag{14}$$

From eqn. (12) the angle β is zero when $\Delta\sigma_{kKi} = 0$ and the sensitivities of the two modes are equal: effectively the point where the eigenvalue loci swap trajectories. This corresponds to the point where the eigenvalues are closest, and since eqns. (3) and (6) can be written $\frac{d^2\lambda_i}{d\delta_K^2} \approx 2(\kappa_{iKk}^2/\Delta\lambda_{ik})$ and $\frac{d\phi_i}{d\delta_K} \approx (\kappa_{iKk}/\Delta\lambda_{ik})\phi_k$ for this case (since $\frac{d\mathbf{M}}{d\delta_K} = \frac{d^2\mathbf{M}}{d\delta_K^2} = \frac{d^2\mathbf{K}}{d\delta_K^2} = \mathbf{0}$) it is also the point where the eigenvalue curvature and eigenvector sensitivity are greatest. These reference vectors form a veering datum set where the modal coupling, or cross-sensitivity, is greatest. The cross-sensitivity thus provides a useful measure of the intensity of veering, its square being proportional to the eigenvalue curvature. The maximum cross-sensitivity over a range of δ_K is easily computed from the modal properties for any single value of δ_K, and it is convenient to define a cross-sensitivity quotient as $\mathrm{CSQ}_{iKk} = \left(\kappa_{iKk}/\overline{\kappa}_{iKk}\right)^2$. Using eqns. (13-14),

$$\mathrm{CSQ}_{iKk} = \cos^2(2\beta) = \frac{\kappa_{iKk}^2}{\kappa_{iKk}^2 + \left(\Delta\sigma_{kKi}/2\right)^2}. \tag{15}$$

A more general definition is afforded by examining the eigenvector rotations. As $\beta \to \pm\frac{\pi}{4}$, the modal coupling goes to zero and from eqn. (6) the vector rotation also halts. Thus for an idealised veering case (without interaction from other modes), the datum vectors are oriented exactly half way between their asymptotic limits. This definition is used to derive a CSQ for the case of mass matrix variation as follows.

Consider a linear variation in the mass matrix, represented by δ_M, with constant stiffness matrix. The stiffness matrix may be used as the orthonormalising matrix such that

$$\mathbf{\Psi}_{ik}^T\mathbf{K}\mathbf{\Psi}_{ik} = \mathbf{I}, \qquad \mathbf{\Psi}_{ik} = \mathbf{\Phi}_{ik}\mathbf{\Lambda}_{ik}^{-\frac{1}{2}} \tag{16}$$

where $\mathbf{\Lambda}$ is a diagonal matrix so the inverse square root needs no further clarification. Combining eqns. (9) and (16) while noting $\frac{d\mathbf{K}}{d\delta_M} = 0$ yields

$$\mathbf{\Sigma}_{iMk} = -\mathbf{\Lambda}_{ik}^{\frac{1}{2}}\mathbf{\Psi}_{ik}^T\frac{d\mathbf{M}}{d\delta_M}\mathbf{\Psi}_{ik}\mathbf{\Lambda}_{ik}^{\frac{3}{2}} \tag{17}$$

This matrix is not symmetric, and maximum values for κ_{iMk} and κ_{kMi} will not necessarily coincide. In order to define a cross-sensitivity quotient for the two modes in the same manner as before, a symmetric matrix is defined in the form of an adapted sensitivity matrix:

$$\mathbf{\Sigma}_{iMk}^* = \mathbf{\Lambda}_{ik}^{-\frac{1}{2}}\mathbf{\Sigma}_{iMk}\mathbf{\Lambda}_{ik}^{-\frac{3}{2}} = -\mathbf{\Psi}_{ik}^T\frac{d\mathbf{M}}{d\delta_M}\mathbf{\Psi}_{ik}. \tag{18}$$

Substituting eqn. (8) and remembering $\frac{d\mathbf{M}}{d\delta_M} = \frac{d\mathbf{M}}{d\delta_M}'$,

$$\mathbf{\Sigma}_{iMk}^{*}{}' = -\mathbf{\Psi}_{ik}'^T\frac{d\mathbf{M}}{d\delta_M}'\mathbf{\Psi}'_{ik} = -\mathbf{T}^T\mathbf{\Psi}_{ik}^T\frac{d\mathbf{M}}{d\delta_M}\mathbf{\Psi}_{ik}\mathbf{T} = \mathbf{T}^T\mathbf{\Sigma}_{iMk}^*\mathbf{T}. \tag{19}$$

This is equivalent to eqn. (10) and, by analogy,

$$\mathrm{CSQ}_{iMk}^* = \frac{{\kappa_{iMk}^*}^2}{{\kappa_{iMk}^*}^2 + \left(\Delta\sigma_{kKi}^*/2\right)^2} = \frac{\frac{\kappa_{iMk}^2}{\lambda_i\lambda_k^3}}{\left(\frac{\kappa_{iMk}^2}{\lambda_i\lambda_k^3} + \frac{1}{4}\left(\frac{\sigma_k}{\lambda_k^2} - \frac{\sigma_i}{\lambda_i^2}\right)^2\right)}. \tag{20}$$

Note that the eigenvalues are generally close at veering, and if $\lambda_i \approx \lambda_k$ then $\kappa^* \approx \kappa$ and $\mathrm{CSQ}^* \approx \mathrm{CSQ}$. Eqns. (15) and (20) are valid for any symmetric, undamped structural eigenproblem with linear variation of the mass or stiffness matrices.

5 MODAL DEPENDENCE FACTOR

Veering is distiguished from other forms of parametric variation by the swapping of modal properties from one mode to another. This is effected by a transformation of the eigenvectors within a fixed subspace. If the vectors stray significantly outside their subspace, it is an indication that they are interacting with other modes. On this premise, a modal dependence factor (MDF) is derived below to quantify the contribution of the interaction between two modes to their total variation.

Eqn. (7) gives an exact measure of the conformity of the mass-normalised eigenvectors to their subspace. As before, considering a change in parameter δ_K causing a variation of the stiffness matrix such that $\frac{d\mathbf{M}}{d\delta_K} = 0$, eqn. (7) can be written

$$\mathrm{MDF}_{iKk} = \mathrm{Q}_{iKk}^2 = \frac{\left(\kappa_{iKk}/\Delta\lambda_{ik}\right)^2}{\sum_{r\neq i}\left(\kappa_{iKr}/\Delta\lambda_{ir}\right)^2} \tag{21}$$

This equation requires knowledge of the modal parameters for all the modes, but it is desirable that the modal dependence factor, as with the cross-sensitivity quotient, may be computed using only modal parameters for the two modes concerned. The eigenvector derivative, $\frac{d\phi_i}{d\delta_K}$ can be obtained in a computationally efficient manner using only modal properties for the i^{th} mode with Nelson's method [14]. Transposing eqn. (2), post-multiplying by $\mathbf{M}\phi_k$ and noting the orthogonality properties gives

$$\frac{d\phi_i}{d\delta_K}^T \mathbf{M}\phi_k = \kappa_{iKk}/\Delta\lambda_{ik}. \tag{22}$$

Post-multiplying eqn. (2) again, this time by $\mathbf{M}\frac{d\phi_i}{d\delta_K}$, and remembering $\dfrac{d\mathbf{M}}{d\delta_K} = 0$ gives

$$\frac{d\phi_i}{d\delta_K}^T \mathbf{M}\frac{d\phi_i}{d\delta_K} = \sum_{r\neq i}\left(\frac{\kappa_{iKr}}{\Delta\lambda_{ir}}\right)^2. \tag{23}$$

Combining eqns. (21-23) yields

$$\mathrm{MDF}_{iKk} = \frac{\left(\frac{d\phi_i}{d\delta_K}^T \mathbf{M}\phi_k\right)^2}{\frac{d\phi_i}{d\delta_K}^T \mathbf{M}\frac{d\phi_i}{d\delta_K}}, \tag{24}$$

giving the contribution of the k^{th} mode to the derivative of the i^{th} eigenvector. From vector algebra and inner products, this is seen to be equivalent to the cosine of the angle between the eigenvector derivative and the plane $\mathbf{\Phi}_{ik}$ in the normal coordinate system. The same approach may be taken for mass matrix variation with $\frac{d\mathbf{K}}{d\delta_M} = 0$, to produce

$$\mathrm{MDF}^*_{iMk} = \frac{\left(\frac{d\psi_i}{d\delta_M}^T \mathbf{K}\psi_k\right)^2}{\frac{d\psi_i}{d\delta_M}^T \mathbf{K}\frac{d\psi_i}{d\delta_M}}. \tag{25}$$

where ψ_i is once more the stiffness-normalised i^{th} eigenvector, and careful attention must be given to the correct normalisation of $\frac{d\psi_i}{d\delta_j}$ when using Nelson's scheme. In the case of several modes veering simultaneously, the MDFs may also be summed to quantify the confinement of the vector within the larger subspace.

6 VEERING INDEX

Veering was shown in section 2 to depend on two factors: the modal coupling and the eigenvalue separation. Veering occurs in the presence of strong modal coupling and proximate modes. Contrarily, the behaviour is most often observed in systems with weak modal coupling outside of veering regions. In these circumstances the eigenvalues must approach closely in order to induce veering, producing more rapid and hence more discernible instances of the effect. In contrast, the gradual veering resulting from higher modal coupling generally involves greater eigenvalue separation and wider parameter ranges such that veering is less remarkable and disrupted by other modal interactions.

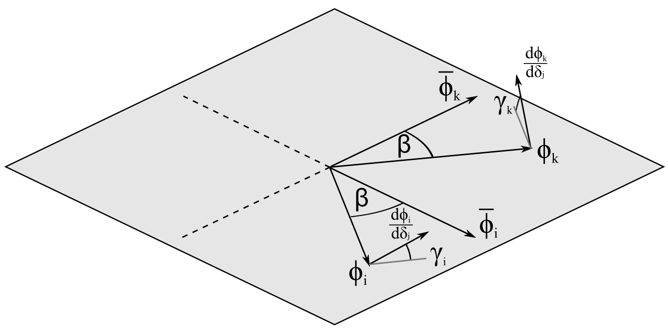


Figure 2: A geometric interpretation of the cross-sensitivity quotient and modal dependence factors described by $\mathrm{CSQ}_{ijk} = \cos^2(2\beta)$, $\mathrm{MDF}_{ijk} = \cos(\gamma_i)$ and $\mathrm{MDF}_{kji} = \cos(\gamma_k)$. Depicted is a plane or subspace in the normal coordinate system containing two eigenvectors ϕ_i and ϕ_k. These vectors are separated from the veering datum vectors for that subspace, $\overline{\phi}_i$ and $\overline{\phi}_k$, by angle β. The corresponding eigenvector derivatives are pictured forming angles γ_i and γ_k with the subspace.

The difficulty in quantifying the behaviour then lies in determining what values are considered to be strong modal coupling and small eigenvalue separation. The MDFs and CSQ provide two alternative descriptive quantities. The MDFs give a measure of the proximity of the modes in terms of their influence on each other relative to the influence of the other system modes, encompassing both the eigenvalue separation and the modal coupling. The CSQ indicates the proximity of two veering modes relative to their closest approach. These normalised quantities put the measurements into context and independently allow meaningful conclusions on the state of veering to be drawn.

A geometric interpretation is given in fig. 2. From this the MDFs are seen to describe the extent to which the eigenvector derivatives deviate from their subspace, while the CSQ describes the eigenvector orientation relative to the veering datum within that subspace. Thus the MDFs determine whether the modes *can* veer and the CSQ determines whether they *are* veering. It is necessary and sufficient that they are both close to unity to produce veering.

To determine the presence of veering, a veering index is proposed as the product of the CSQ and the two MDFs:

$$\mathrm{VI}_{iKk} = \mathrm{MDF}_{iKk} \times \mathrm{CSQ}_{iKk} \times \mathrm{MDF}_{kKi} \tag{26}$$

$$\mathrm{VI}^*_{iKk} = \mathrm{MDF}^*_{iKk} \times \mathrm{CSQ}^*_{iKk} \times \mathrm{MDF}^*_{kKi} \tag{27}$$

In essence, the veering indices computed with eqns. (9), (15), (24), (26), (20), (25) and (27) are a measure of the extent to which two modes are swapping properties with each other. In addition to the insight gained directly from the veering indices, they are expected to prove a useful aid in the interpretation of more observable results, an example of which is given in a separate paper found in these proceedings.

7 EXAMPLES

Two examples are presented here: the first is a simple 2 degree of freedom (DOF) system which will demonstrate the principles of the veering quotient. The second example has been chosen to demonstrate some of the more surprising results obtained with the veering index.

Fig. 3 shows the 2 DOF system, consisting of two grounded spring-mass arrangements and a light coupling spring between them. In this example $k_1 = k_2 >> s$. Away from veering, each mass dominates the motion for its respective vibration mode. As m_2 varies, the natural frequencies of the two modes converge and veer, forming two symmetrical mode shapes where $m_2 = m_1$. The eigenvalue loci are plotted in Fig. 4(a). Because there are only two modes in this system, the modal dependency factors MDF_{1m2} and MDF_{2m1} will always be unity. In this case, the cross-sensitivity quotient and the veering index are identical and are plotted using eqn. (20) in Fig. 4(b). They provide a clear indication of the intensity of veering. The "half-SCQ parameter bandwidth" has also been marked, denoting the region within which the SCQ exceeds 0.5. The effect of veering on the eigenvalue loci is most pronounced in this range.

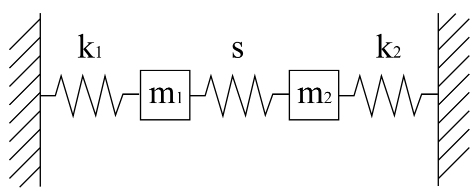
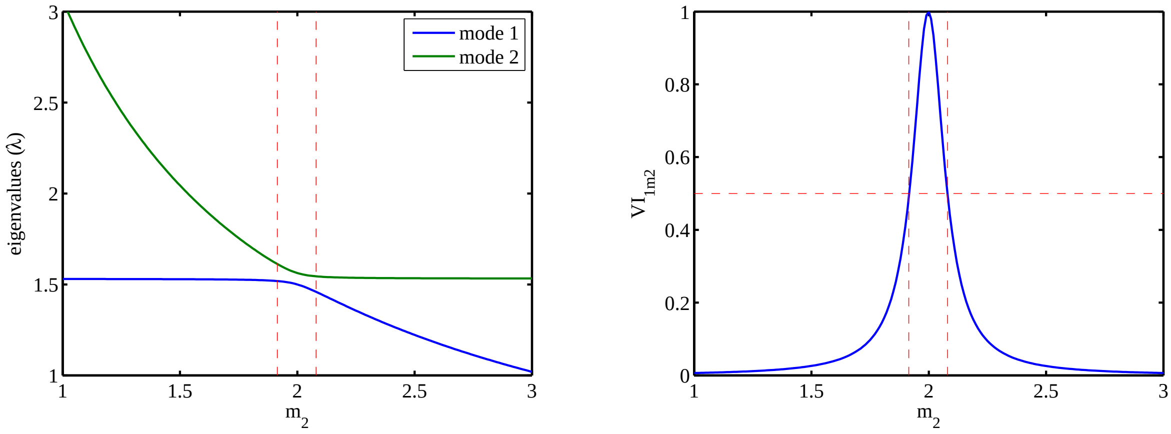


Figure 3: Two degree of freedom spring mass system.



(a) eigenvalues (b) veering index

Figure 4: 2 DOF system plotted for $k_1 = k_2 = 3$, $m_1 = 2$, $s = 0.0625$ and $m_2 = 1...3$. Dotted lines indicate the half-SCQ parameter bandwidth.

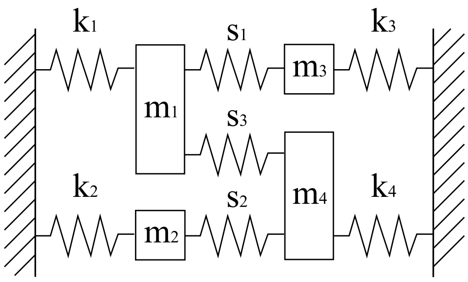


Figure 5: Four degree of freedom spring mass system with light spring couplings s_{1-3} between the masses.

The second example is illustrated in fig. 5. It consists of two pairs of lightly coupled spring-mass arrangements as used in the first example, with an even lighter spring coupling the two systems together. The masses are all equal in this example and the parameter change δ_j corresponds to an equal linear increase in the stiffnesses of k_1 and k_2. The initial spring stiffnesses, k_{1-4}, are chosen such that prior to veering modes 1-4 are dominated by the motion of DOFs 1-4 respectively. The coupling springs, s_{1-3}, introduce light modal coupling and $s_1 = s_2 >> s_3$. The eigenvalues are plotted in fig. 6(a), where modes 2 and 3 appear to veer away from each other. Before veering these modes correspond to the motion of DOFs 2 and 3, which have extremely low coupling in comparison to that between DOFs 1 and 3 and DOFs 2 and 4. Consequently, the veering observed in fig. 6(a) is in fact caused by the modes pairs 1-3 and 2-4. This is clearly indicated by the veering indices in fig. 6(b) where the only curves to rise substantially above zero are those corresponding to VI_{1j3} and VI_{2j4}. Examining the cross-sensitivity quotients in fig. 6(c) shows that as the two mode pairs veer the vectors swing close to the veering datums for other mode pair combinations; the sharp peaks at $\delta_j \approx 77$ correspond to pairs 2-3 and 1-4. Consultation of the modal dependency factors in fig. 6(d), however, confirms that while the MDFs for the veering mode pairs stay close to unity, those for the spurious mode combinations remain small. The high CSVs do not impact significantly on the modal variation and this is reflected in the veering indices.

Increasing the coupling between the two spring-mass systems so that s_3=s_2=s_1 produces similar eigenvalue loci, presented in fig 6(a). Referring to the veering indices in fig. 6(b), the observed curvature is now seen to be attributable to interaction between several modes, in three distinct phases. First modes 1 and 3 begin to veer. As mode 3 takes on the properties of mode 1 its coupling to mode 2 increases. At the same time the 2nd and 3rd eigenvalues get closer and the combination of these effects causes those two modes to veer, taking the dominant role in the variation. As these modes diverge again the 2nd mode starts to veer with the 4th and the corresponding veering index peaks. At no stage are any two modes interacting solely with one another and this is witnessed by the veering indices which are always significantly below unity.

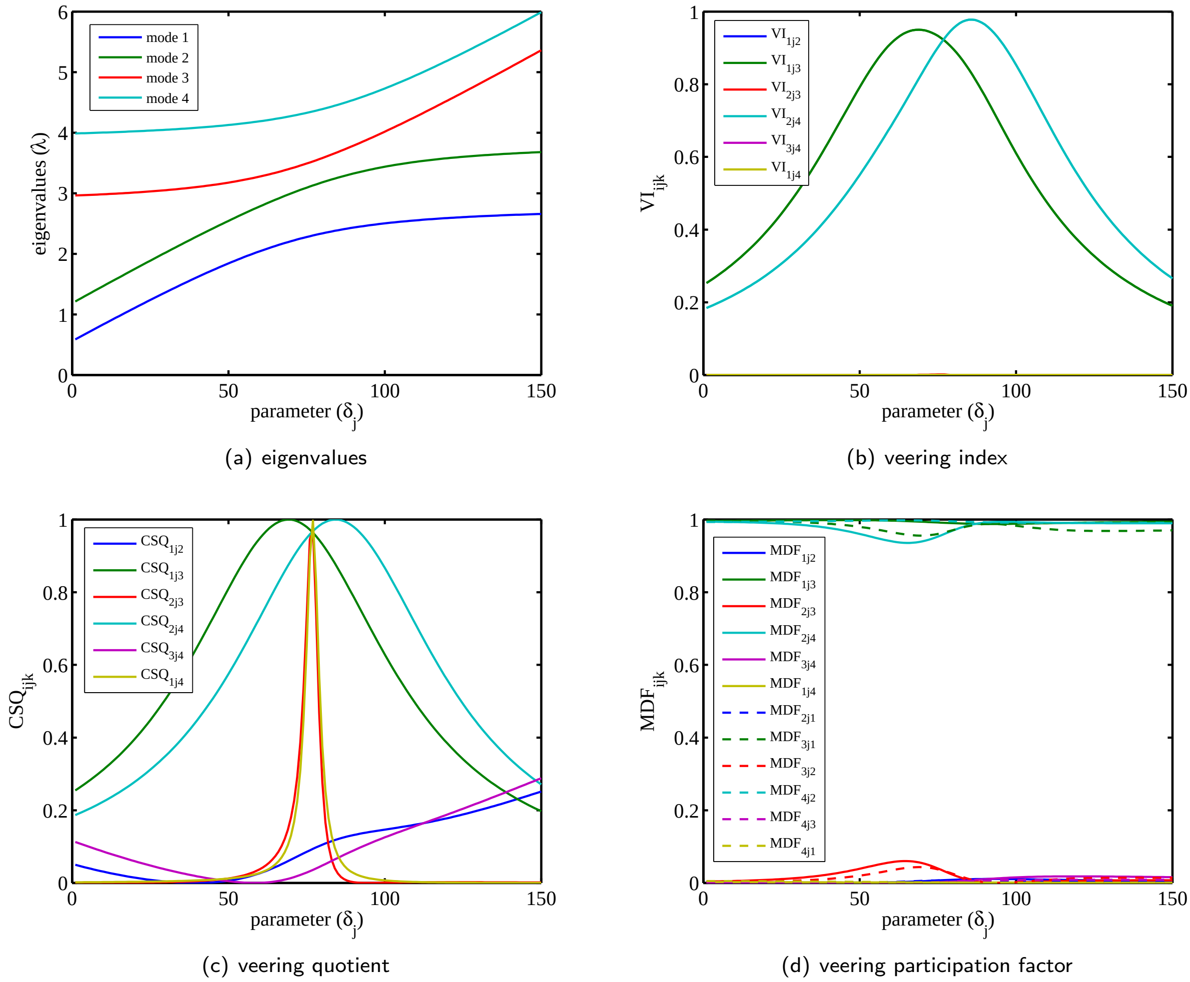
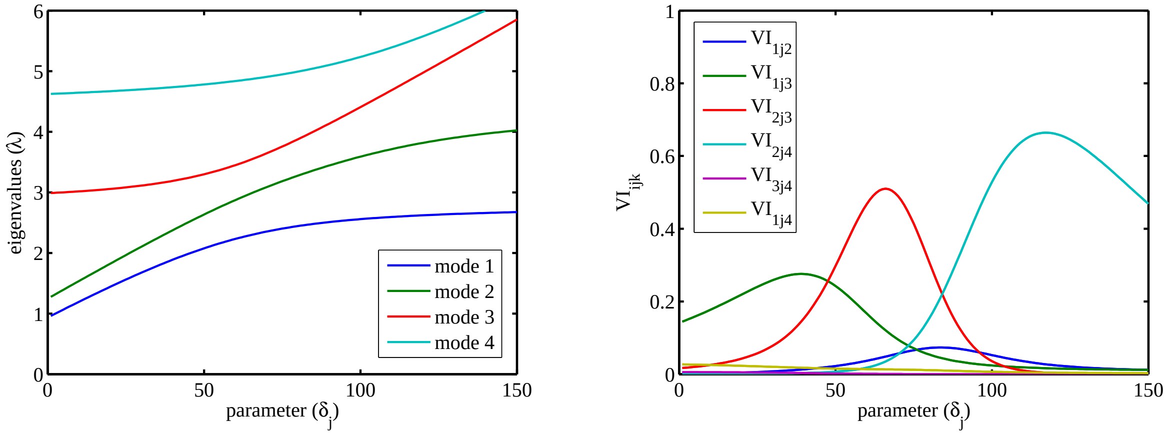


Figure 6: 4 DOF system plotted for $m_1 = m_2 = m_3 = m_4 = 1$**,** $s_1 = s_2 = 0.6$**,** $s_3 = 0.05$**,** $k_1 = 0.1 + 0.03\delta_j$**,** $k_2 = 0.75 + 0.03\delta_j$**,** $k_3 = 2.2$**,** $k_4 = 3.2$ **and** $\delta_j = 1...150$**.**

8 CONCLUSIONS

The distinguishing characteristic of eigenvalue curve veering lies in the swapping of properties from one mode to another. This is effected through the transformation of the eigenvectors within their subspace. The methods presented here allow analysis of the mechanisms through which veering is manifested, resulting in three normalised criteria. The *cross-sensitivity quotient* describes the state of veering of two modes within their subspace, the *modal dependence factor* identifies the conformity of the modes to that subspace, and the *veering index* combines the two to give a definitive quantification of mode veering. An important feature of the technique is that it requires only knowledge of the modal properties for the two modes concerned at a single parameter value. This method produces insightful results when used in isolation but its principal application is expected to be in the interpretation and extrapolation of less esoteric quantities, for example in localisation and stability studies. A further example is demonstrated by the authors with regard to model updating in a second paper in these proceedings.



(a) eigenvalues

(b) veering index

Figure 7: The 4 DOF system plotted for $s_1 = s_2 = s_3 = 0.6$.

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