J. Appl. Prob. **28**, 717–726 (1991)
Printed in Israel

WINDING ANGLE AND MAXIMUM WINDING ANGLE OF THE TWO-DIMENSIONAL RANDOM WALK

CLAUDE BÉLISLE AND
JULIAN FARAWAY,* *University of Michigan*

Abstract

Recent results on the winding angle of the ordinary two-dimensional random walk on the integer lattice are reviewed. The difference between the Brownian motion winding angle and the random walk winding angle is discussed. Other functionals of the random walk, such as the maximum winding angle, are also considered and new results on their asymptotic behavior, as the number of steps increases, are presented. Results of computer simulations are presented, indicating how well the asymptotic distributions fit the exact distributions for random walks with 10^m steps, for $m = 2, 3, 4, 5, 6, 7$.

BROWNIAN MOTION; RANDOM WALK ON INTEGER LATTICE

1. Introduction

In recent years there has been an increasing interest in the study of the winding angle of two-dimensional random walks (Bélisle (1986), (1989), (1990), Berger (1987), Berger and Roberts (1988), Brereton and Butler (1987), Duplantier and Saleur (1988), Fisher et al. (1984), Rudnick and Hu (1987), (1988)). Via rigorous mathematical analysis, via heuristic arguments, and via computer simulations, researchers have investigated the probability distribution of the total angle wound around the origin by the two-dimensional random walk. The ordinary (free) random walk and the self-avoiding random walk have both been studied. The distribution of the winding angle after a finite number of steps, and the behavior of that distribution as the number of steps increases, have both been investigated.

The present paper is concerned with the ordinary two-dimensional random walk on the integer lattice, although the results stated hold for a large collection of two-dimensional random walks (Bélisle (1986), (1989), (1990)). In Section 2, we recall the theorem of Spitzer (1958) on the winding angle of the two-dimensional Brownian motion. We also discuss a theorem, due to Messulam and Yor (1982), on the so-called big winding angle of the two-dimensional Brownian motion, i.e. the total winding occurring while the Brownian motion is outside the unit disk centered at the origin. The two results are drastically different. In Spitzer's theorem, the limit distribution of the

Received 14 May 1990; revision received 3 October 1990.
* Postal address: Department of Statistics, University of Michigan, Ann Arbor, MI 48109, USA.

(normalized) winding angle is a Cauchy distribution. In the Messulam and Yor result, the limit distribution of the (normalized) big winding angle is a hyperbolic secant distribution. In Section 3 we review the recent results of Bélisle (1986), (1989), (1990) on the winding angle of two-dimensional random walks. It turns out that the random walk winding angle behaves asymptotically like the Brownian motion big winding angle. Thus the limit distribution of the (normalized) random walk winding angle is a hyperbolic secant distribution. We also present new results involving the maximum winding angle, the maximum absolute winding angle, and the range of the winding angle. The limiting distributions turn out to be relatives of the hyperbolic secant distribution. In Section 4, we discuss some properties of the limiting distributions obtained in Section 3. In particular, their moments are computed and expressed in terms of Euler's numbers and alternating series of reciprocals of powers of integers. Finally, in Section 5 we present the results of computer simulations indicating how well the asymptotic distributions fit the exact distributions for random walks with 10^m steps, for $m = 2, 3, 4, 5, 6, 7$.

2. Windings of Brownian motion

Consider a standard two-dimensional Brownian motion starting at a point z_0 other than the origin and let $\Theta(t)$ denote the total continuous angle wound around the origin up to time t. Spitzer (1958) has shown that the distribution of $2\Theta(t)/\log t$ is asymptotically standard Cauchy as $t \to \infty$, i.e., for every θ,

$$\lim_{t\to\infty} \boldsymbol{P}\left[\frac{2\Theta(t)}{\log t} \leqq \theta\right] = \int_{-\infty}^{\theta} \frac{1}{\pi(1+u^2)}\,du. \tag{1}$$

In local form, (1) would be written as

$$\boldsymbol{P}[\theta \leqq \Theta(t) \leqq \theta + \Delta\theta] \approx \frac{\frac{1}{2}\log t}{\pi((\frac{1}{2}\log t)^2 + \theta^2)}\,\Delta\theta.$$

Spitzer's proof involves a computation of the Fourier transform of the cumulative distribution function of $\Theta(t)$. A variation of his proof can be found in Itô and McKean (1965).

A more powerful approach to the study of the winding angle of Brownian motion was introduced by Williams (1974) and Durrett (1982). It involves a conformal invariance argument. Here is a brief outline. Start the Brownian motion $(Z_t; t \geqq 0)$ at the point $(1, 0)$. The time at which the Brownian motion hits the circle of radius $\sqrt{t}$ centered at the origin, say $T(\sqrt{t})$, is of the order of t. Thus

$$\frac{2\Theta(t)}{\log t} \approx \frac{2\Theta(T(\sqrt{t}))}{\log t}.$$

(More precisely, one shows that the difference between the above two quantities goes to 0 in probability as $t \to \infty$.) Now conformal invariance (see e.g. Durrett (1984)) says that the process $\log Z_t = \log \| Z_t \| + i\Theta(t)$ is a time-changed Brownian motion. More precisely, the process $\log Z_{t(u)}$ is a standard two-dimensional Brownian motion starting at the origin, where $t(u) = \inf\{t \geqq 0 : \int_0^t \| Z_s \|^{-2} ds = u\}$. It follows that

$$\frac{2\Theta(T(\sqrt{t}))}{\log t} = \frac{\Theta(T(\sqrt{t}))}{\log \sqrt{t}} = \frac{Y(\sigma_{\log \sqrt{t}})}{\log \sqrt{t}}$$

where $((X(u), Y(u)); u \geqq 0) = ((\log \| Z_{t(u)} \|, \Theta(t(u)); u \geqq 0)$ and where $\sigma_v = \inf\{u \geqq 0 : X(u) = v\}$. (Below, the process $((X(u), Y(u)); u \geqq 0)$ is referred to as the log-scaling limit process.) By scaling, $Y(\sigma_{\log \sqrt{t}})/\log \sqrt{t}$ is equal in distribution to $Y(\sigma_1)$ and a standard computation shows that the distribution of $Y(\sigma_1)$ is standard Cauchy. Thus $2\Theta(t)/\log t$ is asymptotically standard Cauchy.

This approach led to further investigations. In particular Messulam and Yor (1982) have shown that if $\Theta_+(t)$ denotes the total winding occurring while the Brownian motion path is outside the unit disk centered at the origin, that is

$$\Theta_+(t) = \int_0^t 1_{\{\| Z_s \| > 1\}} \, d\Theta(s),$$

then, as $t \to \infty$, the distribution of $2\Theta_+(t)/\log t$ is asymptotically hyperbolic secant; i.e. for every θ

$$\lim_{t \to \infty} P\left[\frac{2\Theta_+(t)}{\log t} \leqq \theta\right] = \int_{-\infty}^{\theta} \frac{1}{2} \operatorname{sech}\left(\frac{\pi u}{2}\right) du. \tag{2}$$

The process $(\Theta_+(t); t \geqq 0)$ is usually referred to as the big winding process. The probability distribution with density $(1/2)\operatorname{sech}(\pi u/2)$ has mean 0 and variance 1; we refer to it as the standard hyperbolic secant distribution. Note that unlike the Cauchy distribution, the hyperbolic secant distribution has finite moments of all orders. Thus a comparison of (1) and (2) shows that it is the excursions of the Brownian motion path near the origin that give the distribution of $2\Theta(t)/\log t$ its heavy tails.

Pitman and Yor ((1986), section 8, pp. 761–769; (1989), section 2, pp. 970–975) took a giant step further. They obtained a large class of limit theorems for various functionals of the two-dimensional Brownian motion. These so-called log-scaling laws include results (1) and (2) above. They include the famous Kallianpur–Robbins limit theorem for occupation times. They also include limit theorems for quantities such as the maximum winding $\max_{0 \leqq s \leqq t} \Theta(s)$, the maximum absolute winding $\max_{0 \leqq s \leqq t} |\Theta(s)|$, the winding range $\max_{0 \leqq s \leqq t} \Theta(s) - \min_{0 \leqq s \leqq t} \Theta(s)$, the maximum big winding $\max_{0 \leqq s \leqq t} \Theta_+(s)$, the maximum absolute big winding $\max_{0 \leqq s \leqq t} |\Theta_+(s)|$, and the big winding range $\max_{0 \leqq s \leqq t} \Theta_+(s) - \min_{0 \leqq s \leqq t} \Theta_+(s)$. In each case the appropriate normalizing constant is $2/\log t$, the approach is a refinement of the Williams–Durrett approach outlined above, and the limiting distribution is expressed in terms of the log-scaling limit process. Standard computations, involving refinements of the classical reflection principle for Brownian motion, allow us to compute the probability density function of the limiting distributions. In particular, for the big winding angle we can write the results in the following form. For every $\theta > 0$:

$$\lim_{t\to\infty} P\left[\frac{2}{\log t}\max_{0\le s\le t}\Theta_+(s)\le\theta\right]=\int_0^\theta \operatorname{sech}\left(\frac{\pi u}{2}\right)du;$$

$$\lim_{t\to\infty} P\left[\frac{2}{\log t}\max_{0\le s\le t}|\Theta_+(s)|\le\theta\right]$$

$$=\int_0^\theta 2\sum_{k=0}^{\infty}(-1)^k(2k+1)\operatorname{sech}\left(\frac{(2k+1)\pi u}{2}\right)du;$$

$$\lim_{t\to\infty} P\left[\frac{2}{\log t}\left(\max_{0\le s\le t}\Theta_+(s)-\min_{0\le s\le t}\Theta_+(s)\right)\le\theta\right]$$

$$=\int_0^\theta 4\sum_{k=0}^{\infty}(-1)^{k+1}k^2\operatorname{sech}\left(\frac{k\pi u}{2}\right)du.$$

3. Windings of random walks

Now let $S=(S_n;\, n\ge 0)$ denote the ordinary random walk on the two-dimensional integer lattice. Thus S_0 is the origin and $S_n=\Sigma_{j=1}^n X_j$, where $X_1, X_2, X_3,\cdots$ are independent random vectors with common distribution given by

$$P[X_j=(1,0)]=P[X_j=(0,1)]=P[X_j=(-1,0)]=P[X_j=(0,-1)]=1/4.$$

Let $\phi(n)$ denote the winding angle at time n. More precisely, $\phi(n)=\Sigma_{j=1}^n\lambda(j)$ where the winding increments (i.e. the $\lambda(j)$'s) are defined as follows. If S_{j-1}, S_j and the origin are collinear, then $\lambda(j)=0$. If S_{j-1}, S_j and the origin are not collinear, then $\lambda(j)$ is the unique number between $-\pi/4$ and $\pi/4$ such that

$$\frac{S_j}{\| S_j\|}=e^{i\lambda(j)}\frac{S_{j-1}}{\| S_{j-1}\|},$$

where $i=\sqrt{-1}$ and where $\|(u,v)\|=\sqrt{u^2+v^2}$. Bélisle (1986), (1989), (1990) has shown that as $n\to\infty$ the distribution of $2\phi(n)/\log n$ is asymptotically standard hyperbolic secant, i.e. we have the following result.

Theorem 1. *For every* θ,

$$\lim_{n\to\infty} P\left[\frac{2\phi(n)}{\log n}\le\theta\right]=\int_{-\infty}^\theta \frac{1}{2}\operatorname{sech}\left(\frac{\pi u}{2}\right)du. \tag{3}$$

In local form, (3) would be written as

$$P[\theta\le\phi(n)\le\theta+\Delta\theta]\approx\frac{\operatorname{sech}(\pi\theta/\log n)}{2\log n}\Delta\theta.$$

Theorem 1 holds for a large class of random walks (including the ordinary two-dimensional random walk on the integer lattice). The proof is presented in Bélisle (1989). It involves a decomposition of the random walk path into a series of short excursions near the origin and for which the winding angle contributions are of a smaller order than $\log n$ and a series of long excursions far from the origin and for which the winding angle contributions can be approximated by the big winding angle of a Brownian motion. The result (3) is then obtained from the result (2). For spherically symmetric random walks, such as the random walk studied by Berger and Roberts (1988), a simpler proof using Brownian embedding is presented in Bélisle (1990).

This approach can also be used to obtain the asymptotic distribution of the maximum winding angle $\max_{0\leq j\leq n}\phi(j)$, the maximum absolute winding angle $\max_{0\leq j\leq n}|\phi(j)|$ and the winding angle range $\max_{0\leq j\leq n}\phi(j)-\min_{0\leq j\leq n}\phi(j)$ from the analogous Brownian motion big winding results presented above. The results are summarized in Theorem 2 below. The proof is almost identical to the proof of Theorem 1.

Theorem 2. *For every positive* θ,

$$\lim_{n\to\infty} \boldsymbol{P}\left[\frac{2}{\log n}\max_{0\leq j\leq n}\phi(j)\leqq\theta\right]=\int_0^\theta \operatorname{sech}\left(\frac{\pi u}{2}\right)du; \tag{4}$$

$$\lim_{n\to\infty} \boldsymbol{P}\left[\frac{2}{\log n}\max_{0\leq j\leq n}|\phi(j)|\leqq\theta\right] =\int_0^\theta 2\sum_{k=0}^{\infty}(-1)^k(2k+1)\operatorname{sech}\left(\frac{(2k+1)\pi u}{2}\right)du; \tag{5}$$

$$\lim_{n\to\infty} \boldsymbol{P}\left[\frac{2}{\log n}\left(\max_{0\leq j\leq n}\phi(j)-\min_{0\leq j\leq n}\phi(j)\right)\leqq\theta\right] =\int_0^\theta 4\sum_{k=1}^{\infty}(-1)^{k+1}k^2\operatorname{sech}\left(\frac{k\pi u}{2}\right)du. \tag{6}$$

Finally, let us mention that the same approach can also be used to study the winding angle of a random walk reflected outside a disk, or a square, centered at the origin. For instance if $\phi_r(n)$ denotes the winding angle at time n for a random walk reflected outside a disk of radius r centered at the origin, then for every $r>0$ the results (3), (4), (5) and (6) are still valid with ϕ replaced by ϕ_r. The asymptotic behavior of the winding angle of the random walk kept away from the origin was also studied by Rudnick and Hu (1987), (1988). They considered the case where the random walk is kept away from the origin with a reflecting boundary and the case where it is kept away with an absorbing boundary. In the former case, their result is consistent with ours: see Rudnick and Hu (1988), p. 712, Equation 3. In the latter case they obtained a different asymptotic behavior: see Rudnick and Hu (1988), p. 712, Equation 2.

4. Some remarks on the hyperbolic secant distribution

Let us denote the limit probability density functions arising in (3), (4), (5) and (6) by $f_{[1]}(\theta)$, $f_{[2]}(\theta)$, $f_{[3]}(\theta)$ and $f_{[4]}(\theta)$ respectively, and the corresponding cumulative distribution functions by $F_{[1]}(\theta)$, $F_{[2]}(\theta)$, $F_{[3]}(\theta)$ and $F_{[4]}(\theta)$ respectively. Thus

$$f_{[1]}(\theta) = (1/2)\operatorname{sech}(\pi\theta/2), \qquad -\infty < \theta < \infty;$$

$$f_{[2]}(\theta) = \begin{cases} \operatorname{sech}(\pi\theta/2), & \theta \geqq 0 \\ 0, & \theta < 0; \end{cases}$$

$$f_{[3]}(\theta) = \begin{cases} 2\sum_{k=0}^{\infty}(-1)^k(2k+1)\operatorname{sech}((2k+1)\pi\theta/2), & \theta \geqq 0 \\ 0, & \theta < 0; \end{cases}$$

$$f_{[4]}(\theta) = \begin{cases} 4\sum_{k=1}^{\infty}(-1)^{k+1}k^2\operatorname{sech}(k\pi\theta/2), & \theta \geqq 0 \\ 0, & \theta < 0; \end{cases}$$

and, for $j = 1, 2, 3, 4$, $F_{[j]}(\theta) = \int_{-\infty}^{\theta} f_{[j]}(u)du$. In particular, the standard hyperbolic secant cumulative distribution function is given by

$$F_{[1]}(\theta) = \int_{-\infty}^{\theta} (1/2)\operatorname{sech}(\pi u/2)du = \frac{1}{2} + \frac{1}{\pi}\arctan(\sinh(\pi\theta/2)).$$

A comparison with the standard Cauchy cumulative distribution function

$$F(\theta) = \int_{-\infty}^{\theta} \frac{1}{\pi(1+u^2)}du = \frac{1}{2} + \frac{1}{\pi}\arctan(\theta)$$

shows that if a random variable W is standard hyperbolic secant, then the random variable $\sinh(\pi W/2)$ is standard Cauchy and, equivalently, if a random variable V is standard Cauchy, then the random variable $(2/\pi)\operatorname{arcsinh}(V)$ is standard hyperbolic secant. Other interesting properties of the hyperbolic secant distribution are discussed in Manoukian and Nadeau (1988).

We now turn our attention to the moments of the above distributions. With the help of Gradshteyn and Ryznik (1980) we can, after tedious computations, express the moments as follows:

$$\int_{-\infty}^{\infty} \theta^k f_{[1]}(\theta)d\theta = \begin{cases} |E_k|, & k = 0, 2, 4, \cdots \\ 0, & k = 1, 3, 5, \cdots; \end{cases}$$

$$\int_{-\infty}^{\infty} \theta^k f_{[2]}(\theta)d\theta = \begin{cases} |E_k|, & k = 0, 2, 4, \cdots \\ 2(2/\pi)^{k+1}k!\, a_{k+1}, & k = 1, 3, 5, \cdots; \end{cases}$$

$$\int_{-\infty}^{\infty} \theta^k f_{[3]}(\theta)d\theta = \begin{cases} 2|E_k|a_k, & k = 0, 2, 4, \cdots \\ (4k/\pi)|E_{k-1}|a_{k+1}, & k = 1, 3, 5, \cdots; \end{cases}$$

$$\int_{-\infty}^{\infty} \theta^k f_{[4]}(\theta)d\theta = \begin{cases} 4|E_k|b_{k-1}, & k=0,2,4,\cdots \\ 8(2/\pi)^{k+1}k!\, b_{k-1}a_{k+1}, & k=1,3,5,\cdots. \end{cases}$$

Here E_k denotes the kth Euler number, $a_0 = 1/2$ and, for $k \geqq 1$, $a_k = \Sigma_{j=0}^{\infty}(-1)^j/(2j+1)^k$, $b_{-1} = 1/4$, $b_0 = 1/2$ and, for $k \geqq 1$, $b_k = \Sigma_{j=1}^{\infty}(-1)^{j+1}/j^k$. The first four moments are given in the last row of Tables 1, 2, 3 and 4 below, under the heading '∞'.

5. Results of computer simulations

Let $F_n(\theta)$ denote the cumulative distribution function of $2\phi(n)/\log n$, i.e. $F_n(\theta) = \boldsymbol{P}[2\phi(n)/\log n \leqq \theta]$. Then Theorem 1 says that for every $\theta \in \mathbb{R}$, $\lim_{n\to\infty} F_n(\theta) = F_{[1]}(\theta)$ or, equivalently (see e.g. Breiman (1968), p. 160),

$$\lim_{n\to\infty} \sup_{\theta\in\mathbf{R}} |F_n(\theta) - F_{[1]}(\theta)| = 0. \tag{7}$$

The theorem does not give any information as to how fast convergence takes place in (7). The purpose of this section is to get some empirical insight into this matter.

Consider k independent replications of an n-step simple symmetric random walk on the integer lattice $\mathbb{Z}^2$ and let $F_{n,k}$ denote the empirical cumulative distribution function of the (normalized) winding angle:

$$F_{n,k}(\theta) = \frac{N_{n,k}(\theta)}{k}$$

where $N_{n,k}(\theta)$ is the number of replications for which the observed $2\phi(n)/\log n$ is less than or equal to θ. Then, for every n, the strong law of large numbers gives

$$\boldsymbol{P}\left[\lim_{k\to\infty} |F_{n,k}(\theta) - F_n(\theta)| = 0\right] = 1$$

and since for each fixed n the distributions $F_n(\theta)$ and $F_{n,k}(\theta)$, $k = 1, 2, 3, \cdots$, are supported by the same finite set, an elementary discrete version of the Glivenko–Cantelli theorem (see e.g. Billingsley (1986), p. 275) yields

$$\boldsymbol{P}\left[\lim_{k\to\infty} \sup_{\theta\in\mathbf{R}} |F_{n,k}(\theta) - F_n(\theta)| = 0\right] = 1.$$

For large n the jumps of the cumulative distribution function $F_n(\theta)$ are small and the Kolmogorov–Smirnov theorem (see Breiman (1968), Section 13.6) tells us that the distance $\sup_{\theta\in\mathbf{R}} |F_{n,k}(\theta) - F_n(\theta)|$ is of the order of $k^{-1/2}$. Thus the distance $\sup_{\theta\in\mathbf{R}} |F_n(\theta) - F_{[1]}(\theta)|$ can be estimated via the inequality

$$\sup_{\theta\in\mathbf{R}} |F_n(\theta) - F_{[1]}(\theta)| \leqq \sup_{\theta\in\mathbf{R}} |F_{n,k}(\theta) - F_n(\theta)| + \sup_{\theta\in\mathbf{R}} |F_{n,k}(\theta) - F_{[1]}(\theta)|$$

by choosing k large enough so that the first term on the right-hand side of the above inequality is comfortably small and by empirically computing the second term.

We performed $k = 20000$ independent replications of the n-step simple symmetric random walk, for $n = 10^m$ with $m = 2, 3, 4, 5, 6, 7$. The distances $\Delta_n = \sup_{\theta \in \mathbf{R}} |F_{n,k}(\theta) - F_{[1]}(\theta)|$ are reported in Table 1 under the heading 'Sup'. We also

TABLE 1
Winding

n	Moments				
	1	2	3	4	Sup
10^2	−0.101	1.336	−0.467	6.699	0.103
10^3	−0.076	1.149	−0.352	5.431	0.064
10^4	−0.057	1.101	−0.191	5.264	0.047
10^5	−0.038	1.072	−0.083	5.214	0.035
10^6	−0.038	1.065	−0.122	5.159	0.033
10^7	−0.023	1.004	−0.067	4.390	0.028
SE	0.006	0.022	0.039	0.278	0.001
∞	0	1	0	5	0

computed the first four empirical moments of $2\phi(n)/\log n$ based on these 20000 replications. These empirical moments are reported in Table 1 where they can be compared with the corresponding moments of the limiting distribution computed in Section 4. Finally, we computed the estimated standard error of the first four moments and of the distance Δ_n for the case $n = 10^7$. These estimated standard errors are reported in the penultimate row of Table 1 under the heading 'SE'.

TABLE 2
Maximum winding

n	Moments				
	1	2	3	4	Sup
10^2	0.854	1.256	2.415	5.747	0.213
10^3	0.847	1.211	2.328	5.616	0.171
10^4	0.838	1.187	2.298	5.665	0.159
10^5	0.833	1.188	2.357	5.997	0.139
10^6	0.824	1.159	2.269	5.675	0.123
10^7	0.813	1.126	2.178	5.409	0.116
SE	0.006	0.019	0.068	0.279	0.003
∞	0.742	1	1.949	5	0

Tables 2, 3, and 4 present the results of similar simulations and computations for the maximum winding, the maximum absolute winding, and the winding range respectively. These simulations were performed on a DEC 3100 at the Statistics Department of the University of Michigan.

Looking at these tables, we observe that the sample moments appear to be converging to the moments of the corresponding limit distributions. However, this convergence is

TABLE 3
Maximum absolute winding

n	Moments				
	1	2	3	4	Sup
10^2	1.495	2.691	5.775	14.625	0.358
10^3	1.407	2.435	5.166	13.250	0.281
10^4	1.359	2.304	4.847	12.448	0.219
10^5	1.330	2.237	4.719	12.218	0.186
10^6	1.314	2.197	4.623	11.986	0.168
10^7	1.273	2.071	4.248	10.676	0.144
SE	0.007	0.032	0.129	0.530	0.002
∞	1.166	1.832	3.777	9.889	0

TABLE 4
Winding range

n	Moments				
	1	2	3	4	Sup
10^2	1.891	4.083	9.982	27.553	0.344
10^3	1.823	3.853	9.433	26.605	0.285
10^4	1.770	3.672	8.900	25.037	0.227
10^5	1.735	3.563	8.633	24.470	0.182
10^6	1.714	3.497	8.475	24.172	0.168
10^7	1.665	3.319	7.875	21.979	0.147
SE	0.008	0.039	0.163	0.693	0.002
∞	1.485	2.773	6.413	18.034	0

slow. This is not surprising in view of the $\log n$ rate. The distances Δ_n also decrease with n although the actual values are quite large, even for large n. Again this is not surprising in view of the slow $\log n$ rate of growth of these various functionals.

Acknowledgements

The authors wish to thank the referee for helpful suggestions.

References

BÉLISLE, C. (1986) Limit Distribution of Windings of Planar Random Walks. Ph.D. thesis, Dept. of Statistics, Univ. of California, Berkeley.

BÉLISLE, C. (1989) Windings of random walks. *Ann. Prob.* **17**, 1377–1402.

BÉLISLE, C. (1990) Windings of spherically symmetric random walks via Brownian embedding. *Statist. Prob. Lett.* To appear.

BERGER, M. A. (1987) The random walk winding number problem: convergence to a diffusion process with excluded area. *J. Phys. A: Math. Gen.* **20**, 5949–5960.

BERGER, M. A. AND ROBERTS, P. H. (1988) On the winding number problem with finite steps. *Adv. Appl. Prob.* **20**, 261–274.

BILLINGSLEY, P. (1986) *Probability and Measure*, 2nd edn. Wiley, New York.

BREIMAN, L. (1968) *Probability*. Addison-Wesley, Reading, Mass.

BRERETON, M. G. AND BUTLER, C. (1987) A topological problem in polymer physics: configurational and mechanical properties of a random walk enclosing a constant area. *J. Phys. A: Math. Gen.* **20**, 3955–3968.

DUPLANTIER, B. AND SALEUR, H. (1988) Winding-angle distributions of two-dimensional self-avoiding walks from conformal invariance. *Phys. Rev. Lett.* **60**, 2343–2346.

DURRETT, R. (1982) A new proof of Spitzer's result on the winding of two dimensional Brownian motion. *Ann. Prob.* **10**, 244–246.

DURRETT, R. (1984) *Brownian Motion and Martingales in Analysis*. Wadsworth, Belmont, CA.

FISHER, M. E., PRIVMAN, V. AND REDNER, S. (1984) The winding angle of planar self-avoiding random walks. *J. Phys. A: Math. Gen.* **17**, L569–L578.

GRADSHTEYN, I. S. AND RYZHIK, I. M. (1980) *Table of Integrals, Series, and Products*. Academic Press, New York.

ITÔ, K. AND MCKEAN, H. P. (1965) *Diffusion Processes and their Sample Paths*. Springer-Verlag, Berlin.

MANOUKIAN, E. B. AND NADEAU, P. (1988) A note on the hyperbolic secant distribution. *Amer. Statistician* **42**, 77–79.

MESSULAM, P. AND YOR, M. (1982) On D. Williams' 'pinching method' and some applications. *J. London. Math. Soc.* (2) **26**, 348–364.

PITMAN, J. W. AND YOR, M. (1986) Asymptotic laws of planar Brownian motion. *Ann. Prob.* **14**, 733–779.

PITMAN, J. W. AND YOR, M. (1989) Further asymptotic laws of planar Brownian motion. *Ann. Prob.* **17**, 965–1011.

RUDNICK, J. AND HU, Y. (1987) The winding angle distribution for an ordinary random walk. *J. Phys. A: Math. Gen.* **20**, 4421–4438.

RUDNICK, J. AND HU, Y. (1988) Winding angle of a self-avoiding walk. *Phys. Rev. Lett.* **60**, 712–715.

SPITZER, F. (1958) Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, 187–197.

WILLIAMS, D. (1974) A simple geometric proof of Spitzer's winding number formula for 2-dimensional Brownian motion. Unpublished manuscript, University College, Swansea.