Topics in Applied Dynamical Systems **J.H.P. Dawes**
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Problem Sheet 1

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1. Identify the bifurcations in the following systems and sketch the bifurcation diagrams. Co-ordinate changes to move bifurcation points to the origin may be useful in order to explicitly demonstrate the normal forms for the bifurcations.

   (a) $\dot{x} = \mu - 2x - 2x^2$

   (b) $\dot{x} = 2\mu - (2 + \mu)x + x^2$

   (c) $\dot{x} = (\mu - 2) + \mu x + 3x^2 + x^3$

   (d) $$\begin{aligned} \dot{x} &= (\mu + 2)x + 2y - (2x^2 + 2xy + y^2)x \\ \dot{y} &= -4x + (\mu - 2)y - (2x^2 + 2xy + y^2)y \end{aligned}$$

   In (d), first make the linear change of co-ordinates that brings the equation into normal form, then sketch phase portraits for μ above and below the bifurcation point.

2. Identify the bifurcation in the following system and sketch the bifurcation diagram and phase portraits:

$$\begin{aligned} \dot{x} &= \mu y - x - 2x^3, \\ \dot{y} &= x - y - y^3. \end{aligned}$$

   Compute the *extended* centre manifold near the bifurcation point to determine the nature of the bifurcation. Remember to shift the bifurcation parameter so that the bifurcation occurs when it is zero. Hint: if you wish you can make a linear change of co-ordinates to diagonalise the linear part of the problem first (so that $\mathbb{E}^c$ is an axis). Otherwise, just start by constructing W^c_{loc} to be tangent to $\mathbb{E}^c$ in the original co-ordinates.

3. (a) The Lorenz equations

$$\begin{aligned} \dot{a} &= \sigma(-a + rb), \\ \dot{b} &= a - b - ac, \\ \dot{c} &= \varpi(-c + ab), \end{aligned}$$

   clearly have an equilibrium point at the origin (the 'trivial' equilibrium). For what range of values of the parameters r, σ and ϖ (all positive) do other (non-trivial) equilibria exist? At what parameter values are there local bifurcations? Compute the centre manifold at $r = 1$ and hence determine whether the bifurcation is subcritical or supercritical.

(b) Now analyse the bifurcation at $r = 1$ using adiabatic elimination, as follows. Write $r = 1 + \mu$. Since c decays fast at $r = 1$ (so $\dot{c} \approx 0$), scale $a = \varepsilon a'$, $b = \varepsilon b'$ but $c = \varepsilon^2 c'$ to balance $c' \sim a'b'$ in the third equation. Then also substitute

$$\frac{d}{dt} = \varepsilon^{\alpha} \frac{d}{dt'}, \qquad \mu = \varepsilon^{\beta} \mu',$$

for some (as yet undetermined) positive constants α, β. Hence, re-arranging the c and b equations and substituting them into themselves we get, dropping the primes:

$$\begin{aligned} c &= ab - \varepsilon^{\alpha} \dot{c}/\varpi, \\ b &= a - \varepsilon^{\alpha} \dot{a} - \varepsilon^2 a^3 + O(\varepsilon^{\alpha+2}, \varepsilon^4). \end{aligned}$$

Substitute these into the scaled $\dot{a}$ equation and choose appropriate values for α and β to balance terms and obtain

$$\dot{a} = \frac{\sigma}{1+\sigma}(\mu a - a^3) + O(\varepsilon^2)$$

which should agree with your answer to part (a).

4. Find choices of the coefficients α_1 etc for the near–identity transformation

$$\begin{aligned} x &= \xi + \alpha_1 \xi^2 + \beta_1 \xi \eta + \gamma_1 \eta^2, \\ y &= \eta + \alpha_2 \xi^2 + \beta_2 \xi \eta + \gamma_2 \eta^2, \end{aligned}$$

which reduce the equations

$$\begin{aligned} \dot{x} &= y + a_1 x^2 + b_1 xy + c_1 y^2, \\ \dot{y} &= a_2 x^2 + b_2 xy + c_2 y^2, \end{aligned}$$

(where the coefficients $a_1, \dots, c_2$ are given constants) to each of the simpler forms that follow:

(a) The version used by Takens:

$$\begin{aligned} \dot{\xi} &= \eta + A\xi^2 + O(3) \\ \dot{\eta} &= B\xi^2 + O(3) \end{aligned}$$

where A, B are constants, in fact linear combinations of the coefficients $a_1, \dots, c_2$ and $O(3)$ denotes third and higher-order terms.

(b) The version used by Bogdanov:

$$\begin{aligned} \dot{\xi} &= \eta + O(3) \\ \dot{\eta} &= C\xi^2 + D\xi\eta + O(3) \end{aligned}$$

where C, D are again linear combinations of $a_1, \dots, c_2$.

Check that (a) is transformed into (b) with coefficients $C = D = 1$ by the (nonlinear) co-ordinate transformation

$$\begin{aligned} \xi' &= \frac{4A^2}{B}\xi \\ \eta' &= \frac{8A^3}{B^2}(\eta + A\xi^2) \\ t' &= \frac{B}{2A}t, \quad \text{hence} \quad \frac{d}{dt} = \frac{B}{2A}\frac{d}{dt'} \end{aligned}$$

which involves reversing the direction of time if $AB < 0$. This demonstrates the topological equivalence of (a) and (b).

(c) The equivariant normal form. From the theorem at the end of section 1.5 in the notes there must exist a version of the normal form which commutes with $\exp(sL^T)$ where $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the linearisation of the original ODEs for (x, y), hence $\exp(sL^T) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ (check). This normal form is

$$\begin{aligned} \dot{\xi} &= \eta + A\xi^2 + O(3) \\ \dot{\eta} &= A\xi\eta + B\xi^2 + O(3) \end{aligned}$$

where A, B are linear combinations of $a_1, \ldots, c_2$. Find choices of the coefficients $\alpha_1, \ldots, \gamma_2$ which produce this normal form and verify the equivariance condition

$$\mathbf{f}\left(\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\mathbf{f}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$\mathbf{f}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = A\begin{pmatrix} \xi^2 \\ \xi\eta \end{pmatrix} + B\begin{pmatrix} 0 \\ \xi^2 \end{pmatrix}.$$

Note that in this example, unfortunately, the linear term is not equivariant and so the entire normal form is not $\exp(sL^T)$-equivariant.

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