Topics in Applied Dynamical Systems

Background material

J.H.P. Dawes

Throughout this summary, definitions are indicated by underlined words. References (to books on the lists of suggested reading) are given in *italics* at the beginning of sections.

Vectors are not underlined - the context should make it clear whether a variable is one-dimensional or multidimensional. If in doubt, assume multidimensional - most of the time the notation is the same.

1 Local Bifurcation Theory

1.1 Definitions

References: Kuznetsov, chapters 1–5; Glendinning, chapters 1–4, 8; Guckenheimer & Holmes, chapters 1–3.

A dynamical system is an object whose state at a future time depends deterministically on

- its present state, and
- a law that governs its evolution through time.

Phase space (also called state space) is the set X of all possible states x of the system. There is an evolution operator $\phi(x,t)$ which acts on X:

$$\phi : G \times X \to X$$

where G is a group that parametrises 'time'. We will consider only the cases $G = \mathbb{R}$ (continuous time) and $G = \mathbb{Z}$ (discrete time). Often we'll take $X = \mathbb{R}^n$ but it could well be a more general space. The evolution operator $\phi_t \equiv \phi(t, \cdot)$ obeys the composition rule

$$\phi_{t_1+t_2}(x) \quad = \quad \phi_{t_1}(\phi_{t_2}(x)).$$

Note that ϕ_t is not necessarily invertible: the evolution operators ϕ_t have a semigroup structure, and they form a group only if inverses exist.

In continuous time, the solution curve $x(t) = \phi_t(x_0)$ is defined indirectly (by integration) from a vector field $\dot{x} = f(x)$ (a set of first-order ODEs) and is called a flow. We assume f does not depend explicitly on time t (i.e. the dynamics are autonomous).

In discrete time we specify the dynamics by a map $x_{n+1} = F(x_n)$, hence $\phi_1(x) = F(x)$, $\phi_2(x) = F(F(x)) = F^2(x)$ (note notation denoting the repeated composition of F) and so on.

Examples

- Simple pendulum $\ddot{\theta} = -\sin\theta$, or equivalently

$$\begin{aligned}\dot{\theta} &= p \\ \dot{p} &= -\sin\theta\end{aligned}$$

with $X = [0, 2\pi) \times \mathbb{R}$. Continuous time.

- Predator-prey (Lotka–Volterra) dynamics, $X = \mathbb{R}^2_+$

$$\begin{aligned}\dot{N}_1 &= aN_1 - bN_1N_2 \qquad \text{(prey)} \\ \dot{N}_2 &= -cN_2 + dN_1N_2 \qquad \text{(predators)}\end{aligned}$$

where $a, b, c, d > 0$. Each term has a biological interpretation.

- Logistic map $x_{n+1} = F(x_n) \equiv \lambda x_n(1 - x_n)$. $X = [0, 1]$

- Reaction-diffusion PDEs, e.g.

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \nabla \cdot (D\nabla \mathbf{u})$$

where $\mathbf{u} = (u_1(x, t), u_2(x, t), \ldots, u_n(x, t))$ gives the concentrations of n species in the spatial domain $\Omega = [0, 1]^3$. Here X is a function space, maybe $L^2(\Omega, \mathbb{R}^n)$ with norm $|\mathbf{u}| = (\int_\Omega |\mathbf{u}|^2 \mathrm{d}x)^{1/2}$. Solutions in this particular function space are called 'weak' solutions. We'd actually like something more, for example $\mathbf{u} \in C^2(\Omega, \mathbb{R}^n)$ and continuous in time. But this raises the question, if $\mathbf{u}(x, 0) \in C^2(\Omega, \mathbb{R}^n)$, does it stay C^2 as time evolves, i.e. does $\mathbf{u}(x, t)$ even remain in this phase space? We won't pursue these infinite-dimensional examples further, except to mention

- Navier-Stokes equations in 3D.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathcal{F}$$

where $\mathcal{F}$ is some external forcing. Again our phase space X will be infinite-dimensional, e.g. $X = \{\mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0\}$. General questions about the existence and uniqueness of solutions, and the possibility of finite-time blowup, remain open.

For ODEs in $\mathbb{R}^n$ there are well-known theorems giving necessary and sufficient conditions for the existence and uniqueness of solutions, and continuity of the solution with respect to initial conditions.

The trajectory (or orbit) of a point $x_0 \in X$ is

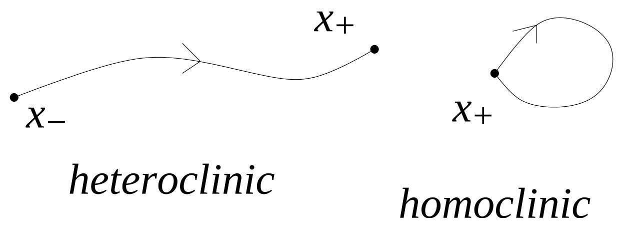
$$O(x_0) = \{\phi_t(x_0) \text{ for all } t \geq 0\}$$

if ϕ_t is invertible then the trajectory includes $\phi_t(x_0)$ for all negative t as well.

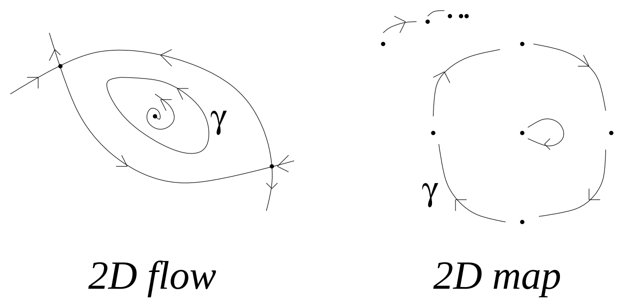
Special orbits include equilibrium points (in continuous time), fixed points (discrete time) and periodic orbits. We say $x^* \in X$ is an equilibrium point for the flow $\dot{x} = f(x)$ if $f(x^*) = 0$. $x^* \in X$ is a fixed point for the map $x_{n+1} = F(x_n)$ if $F(x^*) = x^*$. A periodic orbit (of least period T) is a non-equilibrium orbit γ such that each point $x \in \gamma$

satisfies $\phi_T(x) = x$ and $\phi_t(x) \neq x$ for any $0 < t < T$. In discrete time this implies $F^T(x) = x \; \forall \; x \in \gamma$, i.e. x is a fixed point of the map F^T.

A trajectory which tends towards equilibria $x_\pm$ as $t \to \pm\infty$ respectively is a heteroclinic or connecting orbit. If $x_+ = x_-$ it is a homoclinic orbit.



Examples in 2D:



An invariant set $I \subset X$ is a collection of points with the property that $\phi_t(x) \in I$ whenever $x \in I$ and ϕ_t is defined. Whole trajectories are invariant sets, and especially those corresponding to equilibria and periodic orbits. The 'dynamical systems viewpoint' is to ignore transient behaviour, asking the question 'what happens as $t \to \infty$?' For a single point this large-time behaviour is given by the ω-limit set:

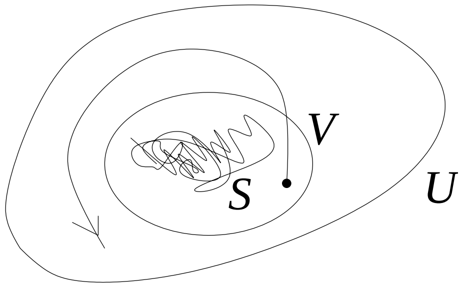
$$\omega(x) \quad = \quad \{y : \; \exists t_n \to \infty \quad \text{such that } \phi_{t_n}(x) \to y\}$$

Exercise 1 *Prove that for a continuous flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$, if $\phi_t(x) \subset K$ a compact set, for all $t \geq 0$, then $\omega(x)$ is non-empty, closed, invariant and connected.*

1.1.1 Stability

Stability turns out to be a slightly tricky issue - in all the cases we will consider it will be clear cut, but a few subtleties are hidden in the definitions we make below.

An invariant set S is Lyapunov stable ('start near, stay near') if, for any sufficiently small neighbourhood $U \supset S$, $\exists$ a neighbourhood $V \supset S$ such that $x \in V \Rightarrow \phi_t(x) \in U$ $\forall \; t > 0$.



Lyapunov stability

S is <u>quasi-asymptotically stable</u> ('tend towards') if $\exists$ a neighbourhood $U \supset S$ such that $\phi_t(x) \to S$ for all $x \in U$ as $t \to \infty$.

Note: neither of these definitions implies the other! (A linear centre in 2D is Lyapunov stable but not quasi-asymptotically stable, a saddle-node equilibrium with a homoclinic connection is quasi-asymptotically stable but not Lyapunov stable). S is <u>asymptotically stable</u> if it is both Lyapunov and quasi-asymptotically stable. In such a case S is an <u>attracting set</u>. But in order to make a good definition of an 'attractor' we need to know slightly more; consider the 2D flow given by

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y. \end{aligned}$$

Although circles centred on the origin with radius $r > 1$ are attracting sets, they contain the origin (which is clearly unstable since points of the form $(x, 0)$ move away for small $|x|$). This problem is solved by introducing a condition of 'indecomposability' - topological transitivity. A closed invariant set S is <u>topologically transitive</u> if, for any two open sets $U, V \subset S$: $\exists\, t \in \mathbb{R}$ such that $\phi_t(U) \cap V \neq \emptyset$. Then we define an <u>attractor</u> to be a topologically transitive attracting set.

The <u>basin of attraction</u> of an invariant set S is $\mathcal{B}(S) = \bigcup_{t \leq 0} \phi_t(S)$. Asymptotically stable sets have 'large' basins of attraction containing whole neighbourhoods. But much weaker definitions of attractor exist: possibly the weakest is the idea of a Milnor attractor. An invariant set M is a <u>Milnor attractor</u> if $\mathcal{B}(M)$ has positive (Lebesgue) measure.

The stability of equilibria and fixed points is usually determined by the eigenvalues of the <u>Jacobian matrix</u>

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \vdots \\ \vdots & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \equiv Df$$

evaluated at the equilibrium or fixed point. In more detail:

(a) Continuous time: $\dot{x} = f(x)$. Near an equilibrium $0 = f(x^*)$ we write $x(t) = x^* + \delta(t)$ and use Taylor series:

$$\dot{x} = \dot{\delta} = f(x^* + \delta) = f(x^*) + Df|_{x^*}\delta + O(|\delta|^2)$$

if $|\delta| \ll 1$ then we have the linearised equation $\dot{\delta} = J\delta$ describing the evolution of initial conditions close to $x = x^*$. The solution is $\delta(t) = \mathrm{e}^{Jt}\delta_0$ where $\mathrm{e}^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$ formally defines the exponential of the matrix A.

We can guarantee that the point x^* is asymptotically stable when all the eigenvalues λ of J satisfy $\text{Re}(\lambda) < 0$.

(b) Discrete time: $x_{n+1} = F(x_n)$. At a fixed point $x^* = F(x^*)$. Looking locally, we write $x_n = x^* + \delta_n$ and Taylor expand as before:

$$\begin{aligned} x_{n+1} &= x^* + \delta_{n+1} = F(x^* + \delta_n) = F(x^*) + DF|_{x^*}\delta_n + O(|\delta_n|^2) \\ \Rightarrow \delta_{n+1} &= F(x^*) - x^* + J\delta_n + O(|\delta_n|^2) \\ \Rightarrow \delta_{n+1} &= J\delta_n \end{aligned}$$

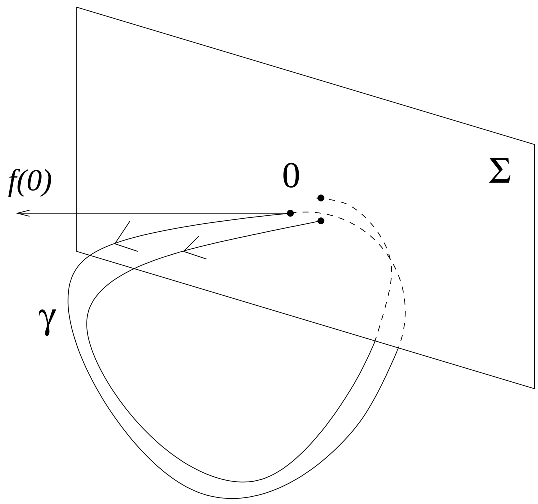
which is the linearised equation governing the behaviour of small perturbations δ_n under iteration.

Similar to the continuous time case, asymptotic stability of $x = x^*$ can be guaranteed if all eigenvalues λ of J satisfy $|\lambda| < 1$. In this case the eigenvalues are often called Floquet multipliers.

Notes:

- We may not always move closer on every iterate.
- These conditions fail in infinite dimensions.

Poincaré sections are an obvious connection between maps and flows; we can use maps of one dimension lower to understand the original flow. Consider trajectories near a periodic orbit γ with period t_0, for the flow $\dot{x} = f(x)$ in $\mathbb{R}^n$.



Moving coordinates so that γ intersects Σ at the origin 0, we may choose a plane Σ defined by $x \cdot f(0) = 0$ (or more generally any $(n-1)$-dimensional surface that intersects γ transversally at $x = 0$). Then points near $x = 0$ travel along trajectories close to γ and hit Σ after times close to t_0. This intuitively defines a map $F : \Sigma \to \Sigma$ which has a fixed point at $x = 0$. This is a Poincaré (return) map. To prove the existence of the map F we use the implicit function theorem (IFT), see the (non-examinable) handout.

1.2 Topological Equivalence and Structural Stability

In this section we discuss the relationship between the linearised flow near an equilibrium, and the original nonlinear flow. This leads us to consider the 'robustness' of vector fields to perturbations and the important ideas of hyperbolicity, structural stability and bifurcations.

Two flows ϕ_1, ϕ_2 in $\mathbb{R}^n$ are topologically equivalent if $\exists$ a homeomorphism (a continuous map with a continuous inverse) $h : \mathbb{R}^n \to \mathbb{R}^n$ and a continuous time reparametrisation function $\tau(t, x)$, $\tau : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is monotonically increasing in t for each fixed x, such that

$$h \circ \phi_1^{\tau(t,x)}(x) \quad = \quad \phi_2^t(h(x)).$$

Topological equivalence preserves information about the number, stability and topology of invariant sets, but may lose information about transients. Time reparametrisation deals with continuous changes in the period of a periodic orbit, for example. Topological equivalence is an equivalence relation on the space of vector fields.

The motivation for this discussion is that we would like to find simpler flows that are (maybe only locally) topologically equivalent to a given flow. The simplest possible flow is a linear one $\dot{x} = Df|_{x^*} x \equiv Jx$ with solution $x(t) = \mathrm{e}^{Jt} x_0$.

Subspaces for the linear flow

The linear unstable subspace $\mathbb{E}^u(x^*)$ for an equilibrium x^* is defined to be the invariant subspace of $\mathbb{R}^n$ spanned by the (generalised) eigenvectors of $Df|_{x^*}$ corresponding to eigenvalues λ with $\mathrm{Re}(\lambda) > 0$.

Similarly we define the linear stable subspace $\mathbb{E}^s(x^*)$ to be the invariant subspace spanned by the generalised eigenvectors of $Df|_{x^*}$ corresponding to eigenvalues λ with $\mathrm{Re}(\lambda) < 0$. The linear center subspace $\mathbb{E}^c(x^*)$ is the subspace corresponding to eigenvalues with $\mathrm{Re}(\lambda) = 0$. If there are no zero or purely imaginary eigenvalues then x^* is said to be a hyperbolic equilibrium point.

Manifolds for the nonlinear flow

Let U be a neighbourhood of x^*, then we define the local unstable manifold $W^u_{loc}(x^*)$ and the local stable manifold $W^s_{loc}(x^*)$:

$$\begin{aligned} W^u_{loc}(x^*) &= \{y \in U : \phi_t(y) \to x^* \text{ as } t \to -\infty \text{ and } \phi_t(y) \in U \ \forall\, t \leq 0\}, \\ W^s_{loc}(x^*) &= \{y \in U : \phi_t(y) \to x^* \text{ as } t \to \infty \text{ and } \phi_t(y) \in U \ \forall\, t \geq 0\}, \end{aligned}$$

i.e. $W^s_{loc}(x^*)$ is the set of points that remain within the neighbourhood U for all positive time, and tend towards x^* as $t \to \infty$. Removing the requirement that the points remain within the neighbourhood U leads to the global unstable and stable manifolds:

$$\begin{aligned} W^u(x^*) &= \{y : \phi_t(y) \to x^* \text{ as } t \to -\infty\}, \\ W^s(x^*) &= \{y : \phi_t(y) \to x^* \text{ as } t \to \infty\}. \end{aligned}$$

Theorem 1 (Stable Manifold Theorem for flows) *Let x^* be a hyperbolic equilibrium for $\dot{x} = f(x)$ with linear stable and unstable subspaces $\mathbb{E}^s$ and $\mathbb{E}^u$. Then there exist local stable and unstable manifolds W^s_{loc} and W^u_{loc} of the same dimensions as $\mathbb{E}^s$ and $\mathbb{E}^u$ respectively. These manifolds are tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$ (respectively) at x^*, are as smooth as f and are flow-invariant.*

Sketch of proof: [by the graph transform method due to Hadamard.] Take co-ordinates $(x, y) \in \mathbb{E}^u \oplus \mathbb{E}^s$. The local unstable manifold is described as a graph over $\mathbb{E}^u$, of the form $y = h(x)$. Now consider the evolution of the initial condition (x_0, y_0) under the time–$t > 0$ map ϕ_t:

$$(x_0, h(x_0)) \quad \to \quad (x(t; x_0, h(x_0)), y(t; x_0, h(x_0))).$$

If $y = h(x)$ were the invariant unstable manifold then we would have

$$y(t; x_0, h(x_0)) \quad = \quad h(x(t; x_0, h(x_0)),$$

for any fixed t, i.e. the image point lies on the unstable manifold. But in general it will not, so we define the *graph transform* $\mathcal{G}$ which acts on a suitable space S of (Lipschitz) functions $h(x)$. $\mathcal{G}$ is defined as

$$\mathcal{G}(h)(x(t; x_0, h(x_0))) \quad \stackrel{\text{def}}{=} \quad y(t; x_0, h(x_0))$$

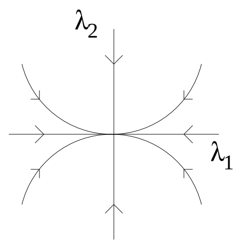
The idea now is to use the exponential contraction in the y-direction to show that $\mathcal{G}$ is a contraction mapping on the (complete metric) space S. Then, from the contraction mapping theorem, it follows that $\mathcal{G}$ has a unique fixed point and this fixed point is the required function $h(x)$.

Exactly the same argument applied to the map ϕ_{-t} proves the existence of the local stable manifold. More details can be found in the book by Wiggins, pp45–47.

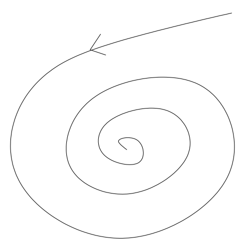
Theorem 2 (Hartman-Grobman) *Let x^* be a hyperbolic equilibrium for $\dot{x} = f(x)$, then there exists a neighbourhood U of x^* on which the flow is topologically equivalent to the linearised flow $\dot{x} = Df|_{x^*}x$.*

Morally, near a hyperbolic equilibrium the nonlinear flow 'looks like' the linearised flow. Moreover, $W^{s,u}_{loc}$ are tangent to $\mathbb{E}^{s,u}$ at x^*. This information is very useful when sketching phase portraits: we look at the linear behaviour local to each equilbrium and then complete the sketch in a sensible way. A further trick is to use $\mathrm{d}y/\mathrm{d}x = \dot{y}/\dot{x}$ to solve for trajectories explicitly if necessary.

In 2D there are three topologically distinct cases of hyperbolic equibrium points with distinct eigenvalues:

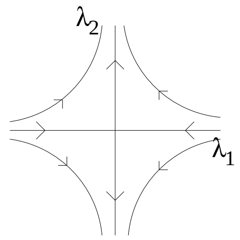


- $\lambda_2 < \lambda_1 < 0$: $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, <u>stable node</u>

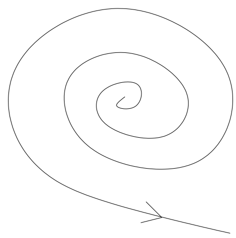


$\lambda \pm \mathrm{i}\omega$, with $\lambda < 0$: $J = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$, <u>stable focus</u>

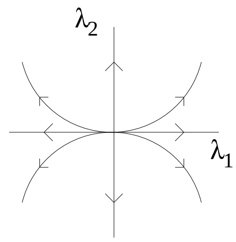
These two cases are topologically equivalent.



- $\lambda_1 < 0 < \lambda_2$: $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, <u>saddle point</u>



- $\lambda \pm \mathrm{i}\omega$, with $\lambda > 0$: $J = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$, <u>unstable focus</u>



$\lambda_2 > \lambda_1 > 0$: $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, <u>unstable node</u>

These two cases are also topologically equivalent.

Exercise 2 *Sketch trajectories of the linearised systems*

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

[Hint: draw the <u>nullclines</u> $\dot{x} = 0$ and $\dot{y} = 0$.]

Worked example

Consider the 2D system

$$\begin{aligned} \dot{x} &= x(2 - y - x) \\ \dot{y} &= y(4x - x^2 - 3) \end{aligned}$$

in $x > 0$, $y > 0$. It is easy to compute that there are equilibria only at $(0,0)$, $(2,0)$ and $(1,1)$. To compute stability we find the Jacobian matrix:

$$J(x,y) = \begin{pmatrix} 2 - y - 2x & -x \\ y(4 - 2x) & 4x - x^2 - 3 \end{pmatrix}$$

hence

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix},$$

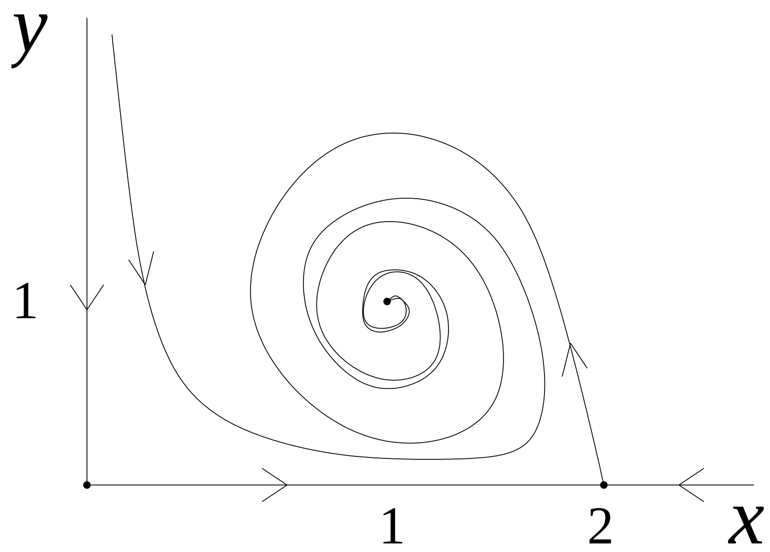
and $(0,0)$ is a saddle point. Similarly

$$J(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

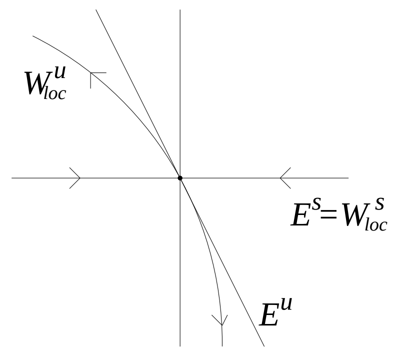
and $(2,0)$ is also a saddle point. Finally,

$$J(1,1) = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$$

and so $(1,1)$ is a stable focus. The complete phase portrait is



To interpret this phase portrait, we could imagine that x and y represent populations of prey and predators, respectively. Note that the behaviour of trajectories near $(2,0)$ can be worked out by computing the eigenvector $\binom{2}{-3}$ corresponding to the eigenvalue 1 there:



Structural stability

Structural stability concerns the robustness of vector fields to small perturbations: do the qualitative properties of the flow change under perturbation or not? In full generality this question turns out to be very delicate. Here we will not present a complete discussion, but one that is sufficient for non-chaotic flows. We first make more precise what we mean by 'perturbation'.

Let $f(x) \in C^\infty(\mathbb{R}^n)$ be a smooth vector field on $\mathbb{R}^n$. Then the vector field $v(x)$ is a C^1_ε perturbation of $f(x)$ if there exists a closed bounded set $K \subset \mathbb{R}^n$ such that

- $f(x) = v(x)$ on $\mathbb{R}^n \setminus K$, and
- $|f - v| + |\frac{\partial f}{\partial x_1} - \frac{\partial v}{\partial x_1}| + \cdots + |\frac{\partial f}{\partial x_n} - \frac{\partial v}{\partial x_n}| < \varepsilon$ inside K.

That is, $f(x)$ and $v(x)$ agree outside K, and their first derivatives and function values are close inside K. Then we can say that a smooth vector field $f(x)$ is structurally stable if there exists an $\varepsilon > 0$ such that all C^1_ε perturbations of $f(x)$ are topologically equivalent to f.

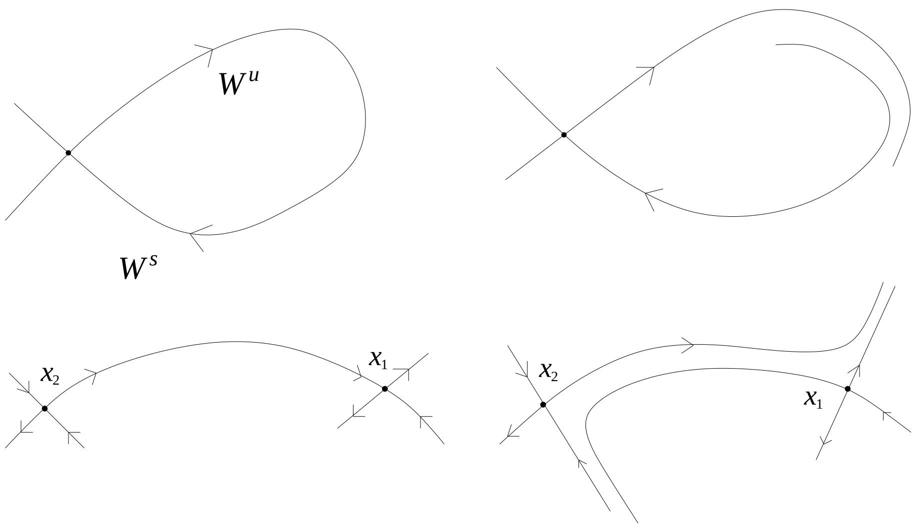
For ODEs $\dot{x} = f(x, \mu)$ depending on a parameter we can extend the definition above to require topological equivalence for any sufficiently close value of μ.

To show why the definition takes this slightly cumbersome form, it is instructive to consider $\dot{x} = f(x) = -x$ in 1D, which seems to be a natural candidate for a structurally stable flow, sensibly defined. It is clear that perturbations of the form $\varepsilon|x|^{1/2}$ or $\varepsilon|x|^{3/2}$ give flows that are, however, not topologically equivalent to $f(x)$ and so our definition must exclude them.

Moreover, and importantly, a vector field in $\mathbb{R}^n$ cannot be structurally stable if either of the two following situations occurs:

1. There exist equilibria or periodic orbits that are not hyperbolic (e.g. consider $\dot{x} = x^3 - \varepsilon x$).
2. There exist saddle points x_1, x_2 (possibly equal) such that $W^s(x_1) \cap W^u(x_2) \neq \emptyset$ and $\dim(W^s(x_1)) + \dim(W^u(x_2)) \leq n$.

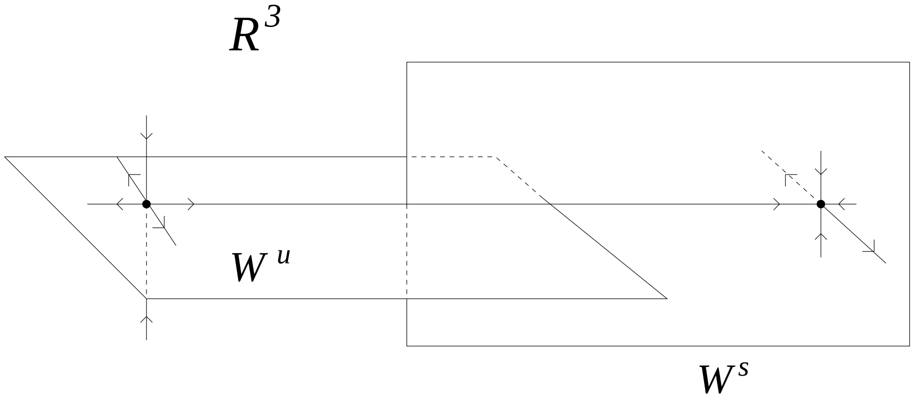
Case 2 means there are homoclinic or heteroclinic orbits that can be 'broken apart' by small perturbations, for example in 2D:



On closed bounded subsets of $\mathbb{R}^2$ these are also sufficient conditions for structural stability; this is Peixoto's theorem (see Glendinning, page 92). In higher dimensions life is more complicated, but the occurrence of either of 1 and 2 above is always enough to produce a flow that is not structurally stable.

Moreover, conditions 1 and 2 enable us to define bifurcations: in a parametrised family of vector fields, these are parameter values $\mu = \mu_0$ at which $f(x, \mu)$ is not structurally stable. We have a local bifurcation in case 1 (when an equilibrium or periodic orbit is non-hyperbolic), and a global bifurcation in case 2.

Note that some heteroclinic orbits are structurally stable, when $\dim(W^u(x_1)) + \dim(W^s(x_2)) > n$:



Here, the left-hand equilibrium has a 2D unstable manifold, and the right-hand equilibrium has a 2D stable manifold. So there is generically an intersection in a 1D curve, i.e. a heteroclinic orbit that persists under perturbations.

We will now investigate local bifurcations of equilibria; there are the distinct possibilities of zero eigenvalues (which give rise to steady-state bifurcations) and pairs of purely imaginary eigenvalues $\pm i\omega$ (which give rise to oscillatory bifurcations). We'll deal with the case of a single zero eigenvalue to start with.

1.3 Local bifurcations in 1D

For a 1D real ODE there is only the possibility of a steady-state bifurcation. The simplest possible flow is $\dot{x} = f(x, \mu)$ with $x \in \mathbb{R}$, $\mu \in \mathbb{R}$. Assume $f(0,0) = 0$ so that $x = 0$ is an equilibrium when $\mu = 0$, and assume also that this equilibrium is non-hyperbolic, i.e. $J = f_x(0,0) = 0$ where the subscript denotes partial differentiation. Assuming that f is smooth we can expand using Taylor series, looking locally:

$$\dot{x} \quad = \quad f_\mu \mu + \frac{1}{2} f_{\mu\mu}\mu^2 + f_{x\mu} x\mu + \frac{1}{2} f_{xx} x^2 + O(3),$$

where all partial derivatives are taken to be evaluated at $(0,0)$, and $O(3)$ indicates terms that are third order jointly in x and μ. Finding equilibria of this expression leads to

$$x \quad = \quad \frac{-\mu f_{x\mu} \pm \left[(f_{x\mu}\mu)^2 - 2 f_{xx}(f_{\mu\mu}\mu^2/2 + f_\mu \mu)\right]^{1/2}}{f_{xx}}$$

Taking $|\mu| \ll 1$ we find

$$x \approx \pm\sqrt{-\frac{2 f_\mu}{f_{xx}}\mu},$$

as long as $f_\mu \neq 0$, $f_{xx} \neq 0$. These conditions turn out to be very important. We'll call this

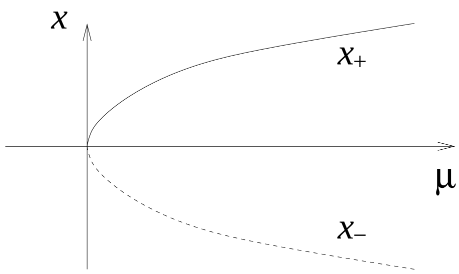
Case 1. Two equilibria exist on one side of the bifurcation point $\mu = 0$, and none on the other side, with the sides depending on the sign of $f_\mu f_{xx}$. This is a saddle-node bifurcation.

Example: $\dot{x} = \mu - x^2 = f(x, \mu)$

Equilibria are $x_\pm = \pm\sqrt{\mu}$ (exist in $\mu \geq 0$). Stability given by $J = \partial f/\partial x = -2x$ evaluated at $x_\pm$:

$$J(x_\pm) = \mp 2\sqrt{\mu}$$

so x_+ is stable and x_- is unstable. The bifurcation diagram is a plot of the location of solutions x as a function of μ:



In this diagram and in all that follow, solid lines denote stable solutions and dashed ones denote unstable ones.

Case 2. What happens if $f_\mu = 0$ and $f_{xx} \neq 0$?

Taking the previous expression, equilibria are expected to be found near

$$x = \frac{-f_{x\mu}\mu \pm \mu[(f_{x\mu})^2 - f_{xx}f_{\mu\mu}]^{1/2}}{f_{xx}}$$

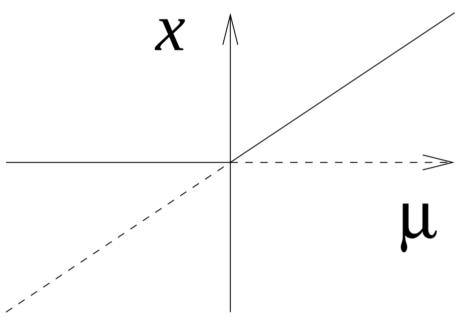
i.e.

$$x \approx -\mu\left(\frac{f_{x\mu} \pm \Delta}{f_{xx}}\right)$$

where $\Delta^2 = (f_{x\mu})^2 - f_{xx}f_{\mu\mu} > 0$. In this case we expect to find two branches of equilibria existing in both $\mu > 0$ and $\mu < 0$. This is a transcritical bifurcation.

Example: $\dot{x} = \mu x - x^2$

Equilibria are at $x = 0$ and $x = \mu$. $J = \mu - 2x$ so $J(x = 0) = \mu$ and $J(x = \mu) = -\mu$. The bifurcation diagram is



Exercise 3 *Sketch the bifurcation diagram for $\dot{x} = \mu x + x^2$.*

What happens if $f_\mu \neq 0$ and $f_{xx} = 0$? In this case the Taylor series expansion for $|x| \ll 1$ and $|\mu| \ll 1$ implies the existence of an equilibrium at $x \approx -f_\mu/f_{x\mu}$ that does not come close to the supposed non-hyperbolic equilibrium point at $x = \mu = 0$ as μ is varied. So there is no branch of equilibria extending from $x = 0$ in either direction, for small μ, unless $f_\mu = 0$ as well.

Case 3. What happens if $f_\mu = f_{xx} = 0$?

Then we take higher-order terms into account in the Taylor series:

$$\dot{x} \;=\; f_{x\mu}x\mu + \frac{1}{2}f_{\mu\mu}\mu^2 + \frac{1}{6}f_{xxx}x^3 + \frac{1}{2}f_{xx\mu}x^2\mu + \frac{1}{2}f_{x\mu\mu}x\mu^2 + \frac{1}{6}f_{\mu\mu\mu}\mu^3 + O(4)$$

This gives a cubic equation to solve for x in terms of μ when we look for equilibria. The important leading-order balances are:

- between the first two terms, i.e. $x \sim \mu$ as in case 2. This leads to

$$x \approx -\frac{f_{\mu\mu}}{2f_{x\mu}}\mu$$

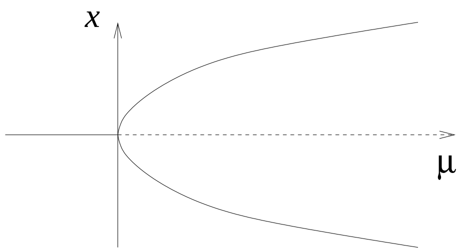
- between the first and third terms, i.e. $x \sim \sqrt{\mu}$ as in case 1 (note that in this case the second term is strictly smaller than the first and third terms). This leads to

$$x \sim \pm\sqrt{\frac{-6f_{x\mu}}{f_{xxx}}\mu}.$$

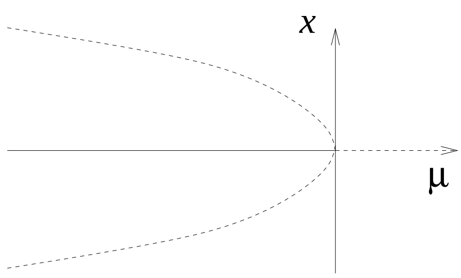
These equilibria exist as long as $f_{x\mu} \neq 0$ and $f_{xxx} \neq 0$, and this case is a pitchfork bifurcation.

Example: $\dot{x} = \mu x - x^3$

Equilibria exist at $x = 0$ and $x = \pm\sqrt{\mu}$. $J = \mu - 3x^2$ so the non-trivial branches are stable, and the bifurcation diagram looks like



Because of the stability of the non-trivial branches, this type of pitchfork bifurcation is called the supercritical case. The other type is a subcritical pitchfork bifurcation, illustrated by the ODE $\dot{x} = \mu x + x^3$:



These two cases are topologically different, unlike the cases examined above for the transcritical bifurcation $\dot{x} = \mu x \pm x^2$ which are essentially identical.

These three bifurcations are not all of the same standing - in order to derive transcritical and pitchfork bifurcations we had to assume extra conditions on some of the partial derivatives of f, and this leads us to suspect that they are not as 'likely' to occur in 'general' sets of ODEs as saddle-node bifurcations are. This intuition is correct, and leads to a discussion of what precisely 'generic' means. We will not dwell on this here for long, although we will return to these ideas when we consider local bifurcations with symmetry later on. However, a few more definitions will come in useful.

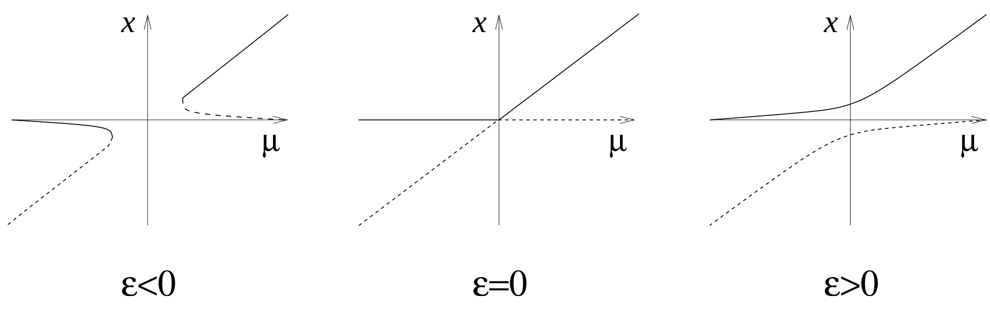
The codimension of a bifurcation is (loosely) the number of independent conditions that must be fixed for the bifurcation to occur. A (partial) unfolding of a bifurcation is a family $\tilde{f}(x, \mu, \varepsilon)$ which contains the original bifurcation structure, i.e. $\tilde{f}(x, \mu, 0) \equiv f(x, \mu)$. A universal unfolding contains all possible partial unfoldings while containing the smallest possible number of parameters.

Name	Codimension	Conditions
Saddle-node	1	$f_x = 0$
Transcritical	2	$f_x = 0 = f_\mu$
Pitchfork	3	$f_x = 0 = f_\mu = f_{xx}$

Equivalently, the codimension of a bifurcation is the smallest number of parameters that must be introduced to reveal all possible types of behaviour near the bifurcation:

Name	Unfolding	Parameters
Saddle-node	$\dot{x} = \mu - x^2$	μ
Transcritical	$\dot{x} = \varepsilon + \mu x - x^2$	μ, ε
Pitchfork	$\dot{x} = \varepsilon_1 + \mu x + \varepsilon_2 x^2 - x^3$	$\mu, \varepsilon_1, \varepsilon_2$

Example: a perturbed transcritical bifurcation splits into 2 or 0 saddle-node bifurcations: $\dot{x} = \varepsilon + \mu x - x^2$.

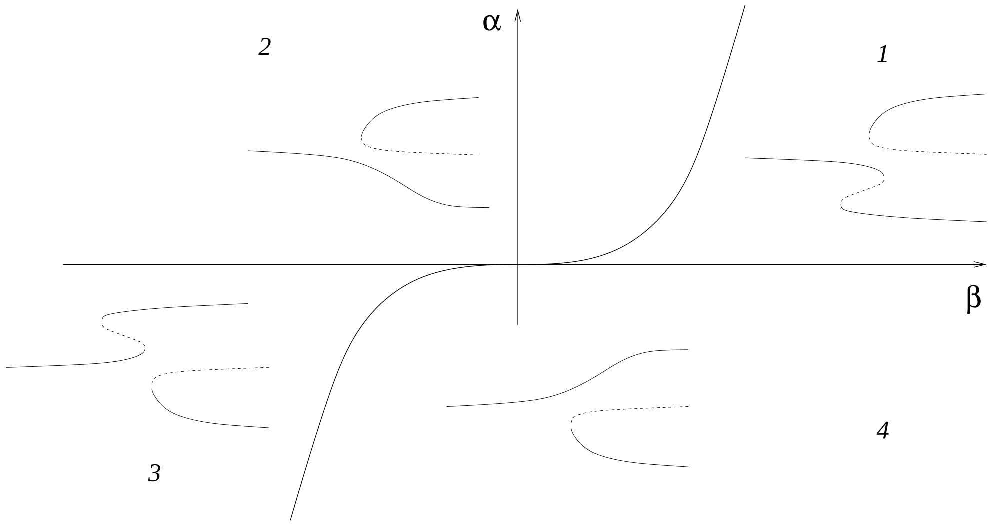


Exercise 4 *Consider the partial unfolding of the pitchfork bifurcation* $\dot{x} = \mu x - \beta x^2 - x^3$ *similarly. Examine how this partial unfolding fits into the universal unfolding given below.*

Exercise 5 *Show that the complete unfolding of the pitchfork bifurcation*

$$\dot{x} \quad = \quad f(x, \mu) = \mu x - \alpha - \beta x^2 - x^3$$

is summarised by the diagram below (small inset figures are diagrams in the (μ, x) *plane). Find the equation of the cubic curve that forms the boundary between regions 1 and 2 and between regions 3 and 4. [Answer:* $\alpha = \beta^3/27$*.]*



Remarks:

- Transcritical and pitchfork bifurcations become of lower codimension if we have extra information, e.g. we know that $x = 0$ is an equilibrium for all parameter values μ, or we assume there are symmetries in the system.
- Bifurcations of higher codimension occur through extra degeneracies in either the linear or nonlinear terms, see sections 3.2 and 3.1 respectively for examples (without involving the complications of symmetry).
- For simple bifurcations we can usually tell which of saddle-node, transcritical or pitchfork occurs by counting equilibria.

What about a bifurcation with a pair of purely imaginary eigenvalues?

Oscillatory (Hopf) bifurcation

Consider the system of ODEs

$$\begin{aligned}\dot{x} &= \mu x - \omega y - Ax(x^2+y^2) - By(x^2+y^2)\\ \dot{y} &= \omega x + \mu y - Ay(x^2+y^2) + Bx(x^2+y^2)\end{aligned}$$

which has an equilibrium at $x = y = 0$. Take $\omega > 0$. Look at stability:

$$J = \begin{pmatrix} \mu - 3Ax^2 - Ay^2 - 2Bxy & -\omega - 2Axy - Bx^2 - 3By^2 \\ \omega - 2Axy + 3Bx^2 + By^2 & \mu - 3Ay^2 - Ax^2 + 2Bxy \end{pmatrix},$$

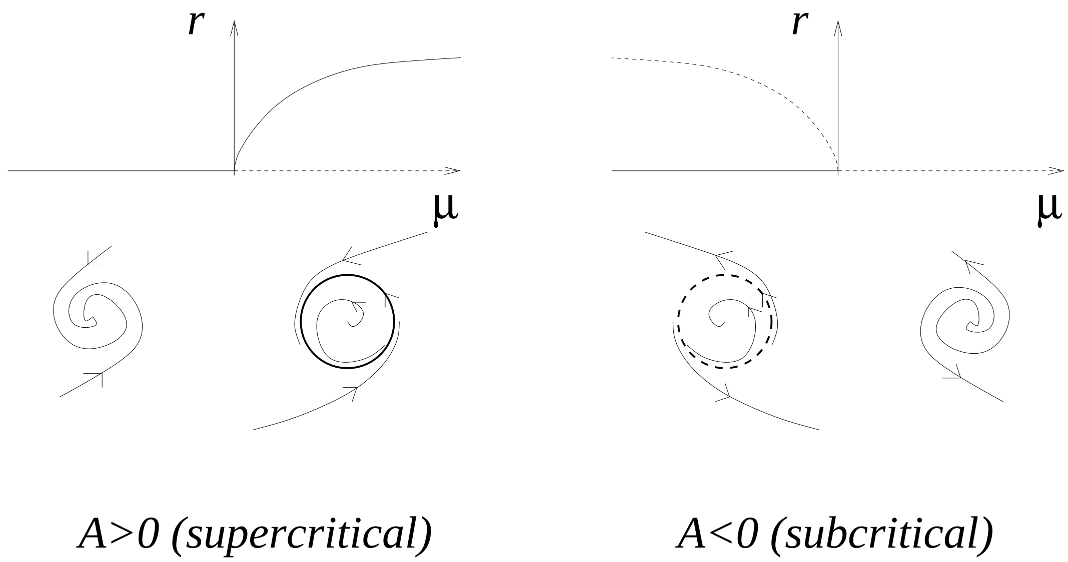
hence

$$J(0,0) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

Since the eigenvalues at the origin are $\mu \pm i\omega$ there is a bifurcation at $\mu = 0$. Change into polar coordinates $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$:

$$\begin{aligned}\dot{r} &= \mu r - Ar^3\\ \dot{\theta} &= \omega + Br^2\end{aligned}$$

so there is a periodic orbit when $r = \sqrt{\mu/A}$ since $\dot{r} = 0$ here, but $\dot{\theta} = \omega + \mu B/A$. This periodic orbit exists in $\mu > 0$ when $A > 0$ (and is stable), and exists in $\mu < 0$ when $A < 0$ (and is unstable). Compare this to the supercritical and subcritical cases of the pitchfork bifurcation.



Two theorems on periodic orbits in 2D

Theorem 3 (Poincaré–Bendixson) – *see Glendinning, section 5.8. If $\phi_t(x_0)$ enters and doesn't leave a closed, bounded domain $\mathcal{D} \subset \mathbb{R}^2$, i.e. $\phi_t(x_0) \in \mathcal{D}\ \forall\ t \geq T$ for some time T, and $\mathcal{D}$ contains no equilibria, then there is at least one periodic orbit inside $\mathcal{D}$.*

This theorem gives conditions for the existence of a p.o. (see Glendinning for the proof).

Theorem 4 (Dulac's criterion) – *see Glendinning, section 5.6. If there exists a continously differentiable real-valued function $\rho(x)$ such that $\nabla \cdot (\rho f) < 0$ on some simply-connected domain $\mathcal{D}$ (i.e. a domain with no holes) then the system of ODEs $\dot{x} = f(x)$ has no periodic orbits lying entirely within $\mathcal{D}$.*

This is a non-existence result, and the proof is a straightforward application of the divergence theorem, supposing that there were a periodic orbit in $\mathcal{D}$.

The idea of normal forms

Suppose $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}$, has an equilibrium point x^* and $J = Df|_{x=x^*} = 0$ when $\mu = \mu^*$ (the bifurcation point). If we also assume, as we have seen, the non-degeneracy conditions $f_\mu|_{(x^*,\mu^*)} \neq 0 \neq f_{xx}|_{(x^*,\mu^*)}$ then we expect to see a saddle-node bifurcation. Near the bifurcation point $\mu = \mu^*$ we therefore expect topological conjugacy of the dynamics with the dynamics of $\dot{z} = \lambda \pm z^2$ near $z = 0$, $\lambda = 0$ (z is real). Can we change coordinates $(x, \mu) \to (z, \lambda)$ explicitly and so simplify the problem? The coordinate transformations needed are

- a shift $(x^*, \mu^*) \to (0, 0)$

- near-identity transformations to remove terms at succesively higher orders. These don't really appear in 1D problems, but we'll go into them in more detail later on. Possibly we'll need rescalings of the variables also.

Say

$$\dot{x} = a\mu + b\mu x + cx^2 + d\mu^2 + O(3).$$

Write $\tilde{\mu} = a\mu$. Let $z = x + g(\tilde{\mu})$, then

$$\dot{z} = \tilde{\mu} + \frac{b}{a}\tilde{\mu}z - 2cgz - \frac{b}{a}\tilde{\mu}g + \frac{d}{a^2}\tilde{\mu}^2 + cz^2 + cg^2 + O(3).$$

We now choose $g(\tilde{\mu}) = \frac{b}{2ac}\tilde{\mu} + O(\tilde{\mu}^2)$ in order to eliminate the $\tilde{\mu}z$ term, so that

$$\dot{z} = \tilde{\mu} + \tilde{\mu}^2\left(\frac{d}{a^2} - \frac{b^2}{4a^2c}\right) + cz^2 + O(3).$$

Now introduce a (near-identity) change of coordinates to the bifurcation parameter: let

$$\lambda = \tilde{\mu} + \tilde{\mu}^2\left(\frac{d}{a^2} - \frac{b^2}{4a^2c}\right)$$

which has inverse

$$\tilde{\mu} = \lambda - \lambda^2\left(\frac{d}{a^2} - \frac{b^2}{4a^2c}\right) + O(\lambda^3),$$

by the IFT. So $\dot{z} = \lambda + cz^2 + O(3)$ has equivalent dynamics to the original ODE for x. Finally, rescale

$$\frac{d}{dt} \to |c|\frac{d}{d\tilde{t}}, \quad \lambda \to |c|\tilde{\lambda},$$

then, dropping the tildes,

$$\dot{z} = \lambda \pm z^2$$

at leading order (near the bifurcation point (x^*, μ^*)). So by these coordinate transformations we see that the saddle-node equation given earlier applies to all 1D bifurcation problems that have this bifurcation. With more work we could have removed (in this case all) terms at successively higher-orders by repeated use of these 'near-identity' transformations, as we'll see later.

1.4 Centre Manifolds

References: Guckenheimer & Holmes, chapter 3.2; Glendinning, chapter 8; Kuznetsov, chapter 5.

We treat bifurcations in higher-dimensional systems by separating the fast dynamics (associated with movement in the directions associated with eigenvalues with non-zero real parts) from the slow dynamics (evolution of the components in directions tangential to the eigenvectors with zero real parts). **The slow dynamics are important**.

Example:

$$\begin{aligned}\dot{u} &= \mu u - uv \\ \dot{v} &= -v + u^2\end{aligned}$$

there are equilibria at $(u, v) = (0, 0)$ and $(u, v) = (\pm\sqrt{\mu}, \mu)$.

$$
\begin{aligned}
J &= \begin{pmatrix} \mu - v & -u \\ 2u & -1 \end{pmatrix}, \\
\Rightarrow J(0,0) &= \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \\
\Rightarrow J(\pm\sqrt{\mu}, \mu) &= \begin{pmatrix} 0 & \mp\sqrt{\mu} \\ \pm 2\sqrt{\mu} & -1 \end{pmatrix}.
\end{aligned}
$$

So there is a steady-state bifurcation at $\mu = 0$. At the bifurcation point we have

$$
\begin{aligned}
\dot{u} &= -uv && \text{slow evolution} \\
\dot{v} &= -v + u^2 && \text{fast decay towards zero}
\end{aligned}
$$

So the linearised picture (sketch it!) is of exponentially fast decay in the v direction, and slow evolution in the u direction. $\mathbb{E}^s = < \binom{0}{1} >$, $\mathbb{E}^c = < \binom{1}{0} >$.

More generally, suppose $x = 0$ is a *non-hyperbolic* equilibrium point of $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and f smooth; i.e. there are eigenvalues of J with zero real part, and $\mathbb{E}^c$ is the space spanned by the corresponding eigenvectors. For notational convenience, let $\dim \mathbb{E}^c = n_0$, $\dim \mathbb{E}^s = n_-$, $\dim \mathbb{E}^u = n_+$.

We separate the centre from the stable and unstable directions: write $x = (u, v)$ where $u \in \mathbb{R}^{n_0}$, $v \in \mathbb{R}^{n_+ + n_-}$ by performing a linear change of coordinates that block-diagonalises the linear terms. Then the system of ODEs looks like

$$
\begin{aligned}
\dot{u} &= Au + f(u, v) \\
\dot{v} &= Bv + g(u, v)
\end{aligned}
$$

where the spectrum of A (its set of eigenvalues) lies on the imaginary axis, the spectrum of B is *off* the imaginary axis, and f and g contain only nonlinear terms.

The question now is, what can we say about the nonlinear behaviour? It turns out that there is a centre manifold (analogous in some ways to the stable and unstable manifolds defined earlier) that determines the dynamics near the non-hyperbolic equilibrium.

Theorem 5 (The centre manifold theorem for flows) *The statement of the theorem falls naturally into two parts.*

1. *There exists a locally defined centre manifold W^c_{loc} which is tangent to $\mathbb{E}^c$ at $x = 0$ and of the same dimension. It is not necessarily unique. Since $\mathbb{E}^c$ is the space $v = 0$, locally, the centre manifold can be written as a graph over $\mathbb{E}^c$:*

$$
W^c_{loc} = \{(u, v) : v = h(u)\}
$$

   where $h : \mathbb{R}^{n_0} \to \mathbb{R}^{n_+ + n_-}$ is smooth, and $h(u) \sim O(|u|^2)$.

2. *The ODEs*

$$
\begin{aligned}
\dot{u} &= Au + f(u, v) \\
\dot{v} &= Bv + g(u, v)
\end{aligned}
$$

are locally topologically equivalent to

$$\begin{aligned}\dot{u} &= Au + f(u, h(u)) \\ \dot{v} &= Bv.\end{aligned}$$

These equations are uncoupled, so the dynamics of the structurally unstable system $\dot{x} = f(x)$ *is essentially determined by the equation*

$$\dot{u} = Au + f(u, h(u)).$$

Remarks

- The $\dot{u}$ equation is a lower dimensional system, and so should be easier to understand. It describes the dynamics on the centre manifold.
- There is a similar theorem for maps.
- In the case $n_+ = 0$, existence of W^c_{loc} is proved by iterating $\mathbb{E}^c$ under the time–$t > 0$ map in an analogous way to the proof of the Stable Manifold Theorem. Various of the estimates required are more tricky than for the proof of the Stable Manifold Theorem, though. Here in general we do not have uniqueness of the centre manifold. For example integrate the system

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned}$$

explicitly. For further details, see J. Carr, *Applications of centre manifold theory.* Springer, 1981.

The graph of the centre manifold $h(u)$ is usually calculated by expanding f, g and h in Taylor series near $(u, v) = (0, 0)$ and by writing

$$\dot{v} = \frac{\partial h}{\partial u}\dot{u} = \frac{\partial h}{\partial u}(Au + f(u, h(u))) = Bh(u) + g(u, h(u))$$

and equating powers of u. [Note: this indicates where the non-uniqueness comes from in the example just above - in applications the non-uniqueness of W^c_{loc} is not a difficulty]. We'll illustrate this by working through the example we began earlier. We take $A = [0]$ and $B = [-1]$ (both 1×1 matrices), and set $f(u, v) = -uv$ and $g(u, v) = u^2$. We write

$$v = h(u) = Cu^2 + Du^3 + Eu^4 + \cdots$$

as there is no linear term since $\mathbb{E}^c$ is the u-axis (and so we know W^c_{loc} will be tangent to the u-axis at $(u, v) = (0, 0)$). Then, using the expressions for $\dot{v}$ we have

$$\begin{aligned}\dot{v} &= (2Cu + 3Du^2 + 4Eu^3)\dot{u} \equiv (2Cu + 3Du^2 + 4Eu^3)(-u)(Cu^2 + Du^3 + Eu^4) \\ &= -(Cu^2 + Du^3 + Eu^4) + u^2\end{aligned}$$

ignoring higher-order terms. Then, equating powers of u we find

$$\begin{array}{lll} u^2: & -C + 1 = 0 & C = 1 \\ u^3: & -D = 0 & D = 0 \\ u^4: & -E = -2 & E = 2 \end{array}$$

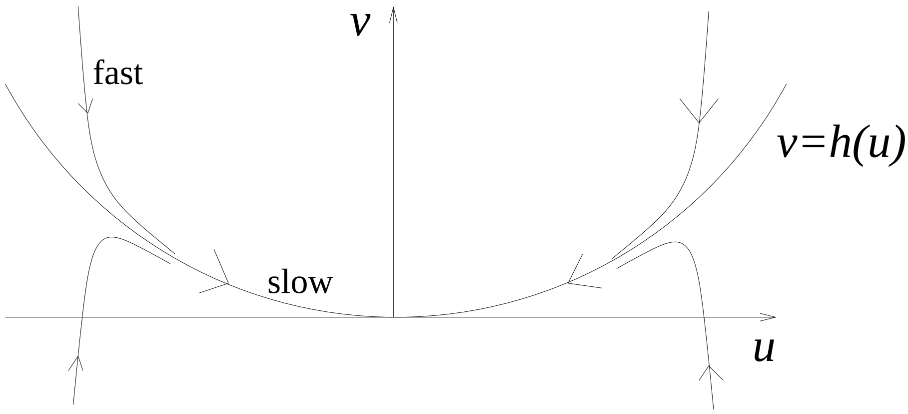
Hence

$$h(u) = u^2 + 2u^4 + \cdots$$

is the equation of the centre manifold (locally). The dynamics on W^c_{loc} are given by

$$\dot{u} = -u^3 - 2u^5 + \cdots$$

to leading order. So $u = 0$ is still a non-hyperbolic equilibrium point, but now we know that it is nonlinearly stable. Trajectories look something like:



Parameter dependence can be included by considering the extended system

$$\begin{aligned}\dot{\mu} &= 0\\ \dot{x} &= f(x, \mu)\end{aligned}$$

in $\mathbb{R}^{n+1}$, with $n_0 + 1$ eigenvalues on the imaginary axis, i.e. μ is now treated as an additional 'slow' variable. Note that we not automatically preserve the block-diagonal structure of the linear terms that we assumed previously. This is not a problem in practice - in general you have just to include possible linear terms in μ in the Taylor series expansion. Treating our example system in this way we have

$$\begin{aligned}\dot{\mu} &= 0\\ \dot{u} &= \mu u - uv\\ \dot{v} &= -v + u^2\end{aligned}$$

note that the μu term is now a nonlinear one. So

$$A = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \quad f(\mu, u, v) = \begin{pmatrix} 0\\ \mu u - uv\end{pmatrix}$$

while $B = (-1)$ and $g(\mu, u, v) = u^2$ as before. To compute the extended centre manifold we write

$$v = h(u, \mu) = Cu^2 + Du\mu + E\mu^2 + \cdots$$

and substitute in the two expressions for $\dot{v}$ as before to obtain

$$\begin{aligned}\dot{v} &= (2C + D\mu)\dot{u} + (Du + 2E\mu)\dot{\mu}\\ &\equiv (2Cu + D\mu)(\mu u - u(Cu^2 + Du\mu + E\mu^2))\\ &= -(Cu^2 + Du\mu + E\mu^2) + u^2.\end{aligned}$$

Again we match terms in powers of u and μ:

$$\begin{array}{lll} u^2: & -C+1=0 & C=1 \\ u\mu: & -D=0 & D=0 \\ \mu^2: & -E=0 & E=0 \end{array}$$

Hence $v = u^2 + O(3)$ to leading order; the equation for the centre manifold is unchanged. But, for the dynamics on the centre manifold we have

$$\dot{u} = \mu u - u^3 + O(4)$$

i.e. a pitchfork bifurcation occurs as μ passes thorough zero.

In this way we have reduced the dimension of the bifurcation problem by eliminating 'fast' directions and shown that it is one of our previously-studied 1D examples.

Summary of the general outline

We now have a strategy for identifying bifurcations in a set of ODEs $\dot{x} = f(x, \mu)$ in $\mathbb{R}^n$, with $\mu \in \mathbb{R}$:

1. Identify equilibria and bifurcation values of μ.
2. Near these bifurcation points, shift the origin of x and μ and perform a linear change of coordinates to get a system in the form

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{u} &= Au + f(\mu, u, v) \\ \dot{v} &= Bv + g(\mu, u, v) \end{aligned}$$

   where $u \in \mathbb{R}^{n_0}$.
3. Compute the (extended) centre manifold $v = h(u, \mu)$ by Taylor expanding, and hence derive a reduced set of ODEs on the centre manifold:

$$\dot{u} = Au + f(\mu, u, h(u, \mu))$$

4. Perform (near-identity) normal form transformations to simplify this reduced equation as much as possible.
5. Analyse properties of the resulting normal form, knowing that the results apply (locally) to the original system in $\mathbb{R}^n$.