The '0–1 test for chaos' and strange nonchaotic attractors

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The '0–1 test for chaos' due to Gottwald & Melbourne has been proved by them to distinguish robustly between periodic and low-dimensional deterministic chaotic dynamics. In this paper we apply the '0–1 test' to a model 2D map exhibiting a transition between quasiperiodic dynamics and a strange nonchaotic attractor (SNA). The detection of a such a transition is a non-trivial numerical task since the Lyapunov exponent remains negative. We find that the '0–1 test' successfully detects the transition, and we propose a simple modification of the standard implementation of the '0–1 test' which considerably improves its ability to distinguish between the strange nonchaotic and purely quasiperiodic regimes.

Our results indicate the practical usefulness of the '0–1 test' for quasiperiodically-forced systems, and they show that the test offers a simple diagnostic method which is independent of any detailed knowledge of the underlying dynamics.

Keywords: power spectrum, deterministic chaos, quasiperiodicity, time series analysis

A key challenge in applied dynamical systems is the development of techniques to understand the internal dynamics of a nonlinear system, given only its observed outputs. Most commonly one observes (in numerical or experimental work) a regularly-spaced time series of one or more variables, and for several decades now, the field of nonlinear time series analysis has attempted to produce ways of characterising complex time series, including, most importantly, the distinguishing of deterministic low-dimensional chaotic dynamics from stochastic motion [1]. Most diagnostic tools, such as the estimation of Lyapunov exponents, require substantial amounts of data, free of noise, in order to perform well. Recently, a much simpler test for the presence of deterministic chaos was proposed by Gottwald & Melbourne [2, 3]. Their '0–1 test for chaos' takes as input a time series of measurements, and returns a single scalar value usually in the range $[0, 1]$. Theoretically, in the case of an infinite amount of noise-free data, the test returns the value unity in the presence of deterministic chaos, and zero otherwise. In this paper we investigate the performance of the '0–1 test' when applied to a model system containing a strange nonchaotic attractor (SNA) and show that the test is straightforward to implement and performs extremely well.

Strange nonchaotic attractors [4] arise commonly in systems subjected to quasiperiodic forcing. Briefly speaking, a SNA has a complicated (fractal) structure but does not have a positive Lyapunov exponent - trajectories separate at rates that are slower than exponential [4]. Hence standard analyses, such as the determination of the largest Lyapunov exponent, fail to detect transitions between quasiperiodic dynamics and SNAs. Moreover, analytical methods such as the approximation of quasiperiodic forcing by a series of rational approximations, or the analysis of the 'phase sensitivity' of the dynamics [4] demands a knowledge of the underlying map or differential equations which is often not available. Our main result is, therefore, that the '0–1 test' succeeds, strikingly, in distinguishing quasiperiodic from SNA dynamics. Moreover, we propose a modification to the standard implementation of the test that improves its performance.

We generate the data that we use from the model 2D map proposed by Grebogi, Ott, Pelikan & Yorke [5] (commonly referred to as the GOPY model). The dynamics of the GOPY model is completely understood, in a rigorous sense; this is a great advantage in the evaluation of the performance of the diagnostic '0–1 test'. In particular, since SNAs arise naturally in systems subject to quasiperiodic forcing, a typical measurement time series arising from a system with an SNA will naturally contain a strong (numerically approximated) quasiperiodic component that hinders the detection of the SNA.

With this in mind we propose a refinement of the '0–1 test' that enables the transition from trivial to SNA dynamics to be more accurately established. Somewhat counterintuitively, this refinement is to add random noise to an intermediate statistic created in the test algorithm; this additive noise suppresses unwanted correlations that arise from the quasiperiodic part of the input signal. We emphasise that we are not measuring the performance of the test

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when applied to noisy original data; the results of [6] show that the '0–1 test' is at least as robust as other methods for the detection of deterministic chaos in a noisy time series. Rather, our noise is added as part of the test procedure itself.

Interestingly, the design of the '0–1' test is echoed in many earlier papers in the literature on SNAs, where properties of power spectra were used to characterise their dynamics.

This paper is organised as follows. We first briefly describe the implementation of the '0–1' test, following [6]. Then we describe the GOPY model [5] and summarise its behaviour. Our numerical results for the standard, and the improved, implementations of the '0–1 test' follow. We close with a short summary and conclusions.

Implementation of the '0–1 test for chaos' is discussed in detail in [6]; see [3] for mathematical results that underpin the version used in here and in [6]. Briefly, the test takes as input a scalar time series of observations $\phi_1, \ldots, \phi_N$. We first fix a real parameter c and construct the Fourier transformed series $z_n = \sum_{j=1}^{n} \phi_j \mathrm{e}^{\mathrm{i}jc}$, $n = 1, \ldots, N$. Following [6] we then compute the smoothed mean square displacement

$$M_c(n) \;=\; \frac{1}{N-p} \sum_{j=1}^{N-p} |z_{j+n} - z_j|^2 - \langle \phi \rangle^2 \frac{1 - \cos nc}{1 - \cos c}, \tag{1}$$

where $\langle \phi \rangle \equiv \sum_{j=1}^{N} \phi_j / N$ denotes the average of the time series, and we take $n \leq p = N/10 \ll N$. We now wish to estimate the scaling of $M_c(n)$ with n: the subtraction of the final term in (1) removes a persistent artificial oscillation in $M_c(n)$ that complicates this next step. If the underlying dynamics of ϕ_j is deterministic chaos then we expect $M_c(n) \sim n$, i.e. z_n performs a Brownian motion in the complex plane. If the underlying dynamics is (quasi)periodic then we expect no drift, and $M_c(n)$ should not increase with n. The next step is numerically to estimate which of these cases arises. Xulvi–Brunet et al. [6] suggest computing a correlation coefficient to evaluate the strength of the linear growth, defining:

$$K_c \;=\; \frac{\mathrm{cov}(n, M_c(n))}{[\mathrm{cov}(n,n)\ \mathrm{cov}(M_c(n), M_c(n))]^{1/2}}, \tag{2}$$

where $\mathrm{cov}(x, y)$ denotes the usual covariance sum.

Awkward cases arise when c is a resonant frequency of the system: for such a c we expect $M_c(n) \sim n^2$. To avoid the effect of such resonances on the test, the above steps are repeated for n_c values of c equally spaced in the interval $[0.1, \pi/2]$ (there is always a strong resonance at $c = 0$ which we avoid). We then take the median value of K_c as the binary diagnostic value K of the existence, or not, of deterministic chaos. In the difficult case of quasiperiodic dynamics, resonant values of c will in theory be dense, and so may cause the test erroneously to return $K \approx 1$.

We propose a modification of the above algorithm, introducing an additive noise term into (1). We define $\tilde{M}_c(n) = M_c(n) + \sigma\eta_n$ where $\eta_n \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$ is a random variable distributed uniformly on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and the parameter σ is the noise level. The additive noise suppresses small-amplitude quadratic growth in $M_c(n)$ (i.e. cases when $M_c(n) \sim Vn^2$ but with $V \ll 1$) which nevertheless produce a high correlation K_c. In the case of quasiperiodic dynamics, many choices of c are near-resonant and, without the noise term, would lead to such quadratic growth resulting in high correlations and so to the '0–1 test' wrongly interpreting the dynamics as deterministic chaos. For the same reason, high correlations occur near resonances in the periodic case as well, but these are much more regular in appearance (see figure 2) and the resulting anomalous values of K_c are consistently eliminated by the selection of the median value of the distribution for K, as discussed in section 4 of [6].

We examine the performance of the '0–1 test' on data generated by a 2D map acting on $\mathbb{R} \times S^1$, the generalised GOPY model:

$$x_{j+1} \;=\; 2\alpha(\tanh x_j)\cos(2\pi\theta_j) + \epsilon\cos(2\pi(\theta_j + \beta)), \tag{3}$$

$$\theta_{j+1} \;=\; \theta_j + \omega \bmod 1, \tag{4}$$

where α, β and ϵ are non-negative parameters and usually $\omega = (\sqrt{5} - 1)/2$. We fix $\beta = 1/8$ in the remainder of this Letter. The case $\epsilon = 0$ corresponds to the original GOPY map. Calculation of the Lyapunov exponent for the invariant line $x = 0$ in the standard way yields $\lambda = \ln\alpha$, showing that $x = 0$ is attracting if $\alpha < 1$. If $\alpha > 1$ then it can be proved that an SNA exists; a typical numerical simulation is shown in figure 1(a). The attractor is termed 'strange nonchaotic' because (i) $\lambda < 0$ for typical orbits on the attractor, even when $\alpha > 1$, but (ii) the attractor is not differentiable [4]. A key observation establishing (ii) is to note that the set of points $(x, \theta) = (0, k\omega \pm 1/4 \bmod 1)$, $k \in \mathbb{N}$ is dense in the attractor.

For $\epsilon = 0$ it can be shown that the SNA exists for all $\alpha > 1$ [7]; if $\alpha > 1$ and $\epsilon > 0$ (referred to as 'model B' in [8]) it appears that the SNA no longer exists and is replaced by a smooth invariant curve on which the dynamics is quasiperiodic, see figure 1(b).

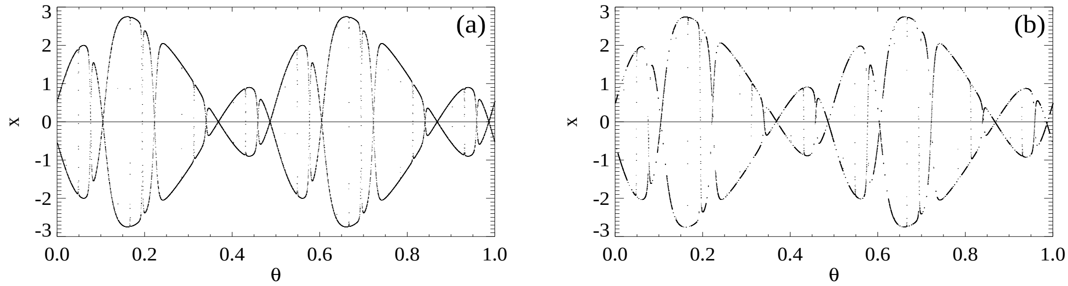

FIG. 1: The GOPY attractor for $\alpha = 1.5$ and (a) $\epsilon = 0.0$ for which the attractor is an SNA; (b) $\epsilon = 0.02$ for which the attractor is an smooth invariant curve $x(\theta)$, with quasiperiodic dynamics. $N = 10000$ iterates of (3) - (4).

We now turn to the problem of resonances and show that the 'modified 0–1 test' reduces the occurrence of high correlations due to small-amplitude correlations in $M_c(n)$ as c varies. We then discuss the behaviour of the test on data from (3) - (4) in different parameter regimes.

We take the linear combination $\phi_j = x_j/(4\alpha) + \theta_j/10$ of the GOPY map variables x_n and θ_n to form the time series for the test, anticipating that in practical situations a time series from a quasiperiodically-forced system will always contain a component of the quasiperiodic forcing. We have repeated the calculations for an equal-amplitude time series $\hat{\phi}_j = x_j/(4\alpha) + \theta_j$ with qualitatively no difference.

First we observe that, in the simple case where ω is rational, the '0–1 test' clearly signals the resonances between choices of c and the periodic component in the input signal; see figure 2 where the resonances at $c/\pi = k/10$, $1 \leq k \leq 5$ are clear.

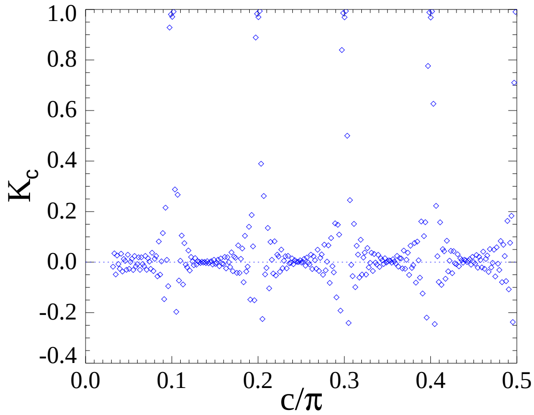

FIG. 2: '0–1' test results for a periodic input signal $\{\phi_j\}$: $\alpha = 1.1$, $\epsilon = 0$, $\omega = 0.3$, $N = 10000$, $n_c = 300$. Note the resonances at $c/\pi = k/10$, $k \in \mathbb{N}$.

For the GOPY map with quasiperiodic forcing, $\omega = (\sqrt{5} - 1)/2$, in the trivial attractor regime, $\alpha = 0.9$, figure 3(b) indicates that the modified version of the test with $\sigma = 1.0$ is successful in eliminating near-resonance effects, returning a median value $K = 0.0225608$ which is a substantially better diagnostic than the value $K = 0.40683$ produced by the unmodified test (figure 3a).

Clearly, the addition of high-enough amplitude noise $\sigma \gg 1$ will eventually mask all linear correlations in $M_c(n)$: the optimal value for σ will vary from case to case. Although the effect of noise is always to reduce K_c, it appears that, at least in some cases, the reduction in K_c in the SNA regime is extremely small compared to its effect in the quasiperiodic regime; this leads to highly effective discrimination. Figure 4(b) shows the effect of noise ($\sigma = 1.0$) on data in the SNA regime, for parameters (other than α) as in figure 3. With $\sigma = 1.0$, the median value of K is $K = 0.957707$; the corresponding plot (figure 4a) without noise appears very similar (and has median $K = 0.971513$). One concludes that deterministic chaos is present.

Figure 5 shows the performance of the test in determining the bifurcation at which the invariant line $x = 0$ loses stability and the SNA is formed. A noise level of $\sigma = 1.0$ produces the clearest discrimination in this case. One

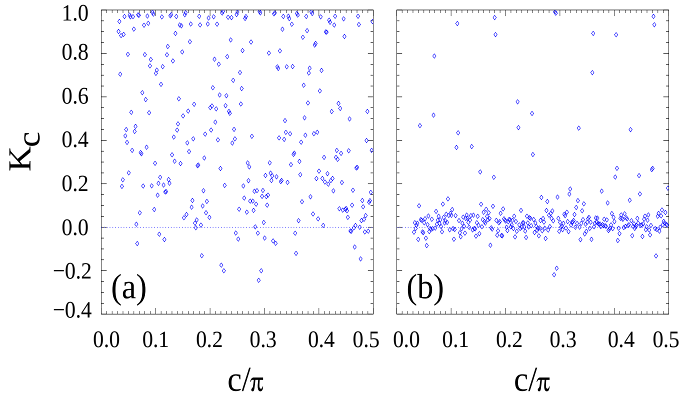

FIG. 3: '0–1' test results for a quasiperiodic input signal $\{\phi_j\}$: $\alpha = 0.9$, $\epsilon = 0$, $\omega = (\sqrt{5}-1)/2$, $N = 10000$, $n_c = 300$. (a) unmodified test, $\sigma = 0$, median $K = 0.407$ (3sf); (b) modified test, $\sigma = 1.0$, median $K = 0.0226$ (3sf).

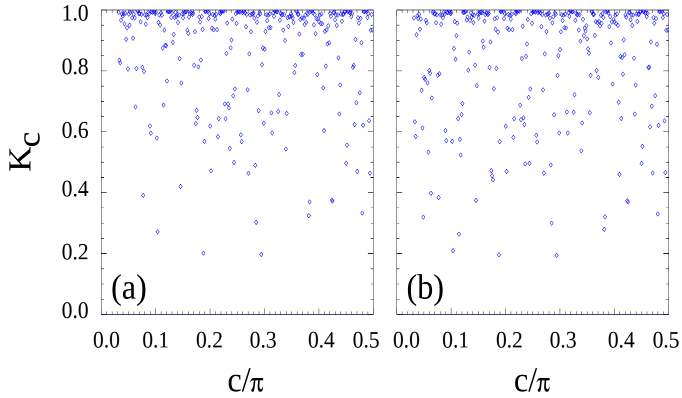

FIG. 4: '0–1' test results for the time series $\{\phi_j\}$ in the SNA regime: $\alpha = 1.5$, $\epsilon = 0$, $\omega = (\sqrt{5}-1)/2$, $N = 10000$, $n_c = 300$. (a) unmodified test, $\sigma = 0$, median $K = 0.972$ (3sf); (b) modified test, $\sigma = 1.0$, median $K = 0.958$ (3sf).

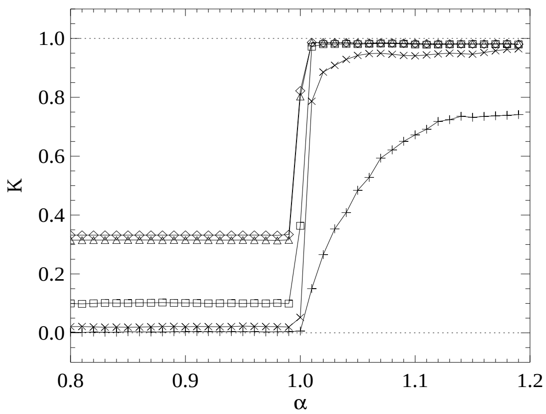

FIG. 5: '0–1' test results for input signal $\{\phi_j\}$: $\epsilon = 0$, $\omega = (\sqrt{5}-1)/2$, with varying noise level: $\sigma = 0.0$ ($\diamond$); $\sigma = 0.01$ ($\triangle$); $\sigma = 0.1$ ($\Box$); $\sigma = 1.0$ ($\times$); $\sigma = 10.0$ ($+$). Parameters: $N = 20000$, $n_c = 100$.

might expect that improved results would also be obtained by increasing N (assuming more data is available) but, as figure 6 shows, this is not straightforwardly the case. Our further numerical results (not shown here) indicate that the same is true for increases in n_c.

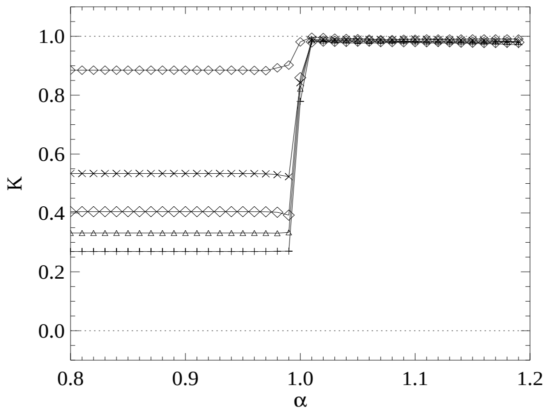

FIG. 6: '0–1' test results for input signal $\{\phi_j\}$: $\epsilon = 0$, $\omega = (\sqrt{5}-1)/2$, with varying time series length N and zero noise: $N = 1000$ ($\diamond$); $N = 5000$ ($\times$); $N = 10000$ ($\Diamond$); $N = 20000$ ($\triangle$); $N = 40000$ ($+$). Parameters: $\sigma = 0$, $n_c = 100$. Increasing N, within the range considered here, does not reduce K to zero in the quasiperiodic regime.

We now turn to the behaviour of the '0–1 test' for fixed $\alpha = 1.5$ and varying $\epsilon > 0$. Pikovsky & Feudel [9] argue, carefully, by considering the system dynamics when successive rational approximations to the quasiperiodic driving are considered, that the attractor is a smooth invariant curve for any $\epsilon > 0$, although it still has a wrinkled appearance for small ϵ, see figure 7. The wrinkled nature of the curve implies that numerical differences between the quasiperiodic and SNA regimes may be extremely small, and for decreasing ϵ in the range $0 < \epsilon < 0.4$ the torus becomes steadily more wrinkled while the dynamics remain quasiperiodic. The results of the '0–1 test' are shown in figure 8, on a log scale to show the behaviour at small ϵ more clearly. For $\epsilon > 0.4$ the smooth nature of the torus is apparent in the test results which converge rapidly to $K = 0$. As σ increases up to $\sigma = 10.0$ the value of K on the plateau region $0.03 < \epsilon < 0.2$ reduces rapidly, indicating that the motion for this range of ϵ is indeed quasiperiodic rather than SNA, while K remains closer to unity at $\epsilon = 0$. For $\sigma > 10.0$ the test results for $\epsilon = 0$ shown in figure 8 indicate that the noise degrades the performance of the '0–1 test' for all ϵ.

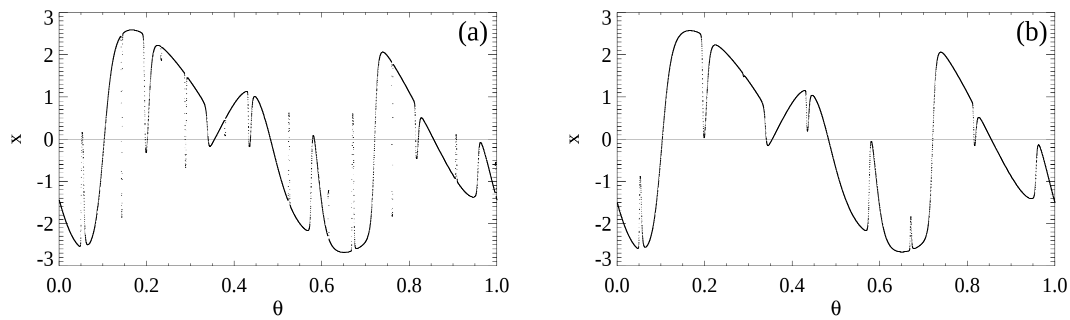

FIG. 7: GOPY attractor for $\alpha = 1.5$; (a) $\epsilon = 0.37$ and (b) $\epsilon = 0.4$ showing the abrupt disappearance of the remaining wrinkles in the invariant curve at $\epsilon \approx 0.4$. $N = 10000$ iterates of (3) - (4).

In summary, in this paper we have demonstrated that the straightforward and computationally cheap '0–1 test for chaos' is a robust diagnostic algorithm for determining the presence of deterministic but 'strange nonchaotic' behaviour and distinguishing it from quasiperiodic motion, even in cases where numerical pictures of the attractor appear identical. The '0–1 test' detects transitions that occur without Lyapunov exponents turning positive; this is a qualitative distinction between this test and traditional tests for chaotic dynamics. The modification of the '0–1 test' by the addition of noise improves the ability of the test to distinguish quasiperiodicity from strange nonchaotic attractors; such quantitative improvements are particularly valuable for clarifying cases where obtaining a longer time series is difficult or impossible.

Since the test may be applied to time series without reference to a detailed knowledge of the structure of the underlying system (except that it is assumed to be deterministic), it has clear advantages, in terms of applicability,

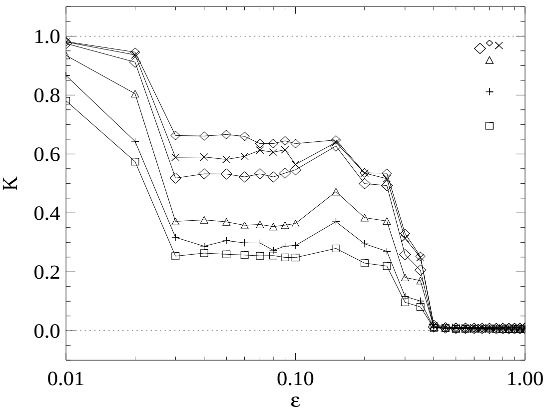

FIG. 8: '0–1 test' results for the GOPY model (3) - (4) for $\alpha = 1.5$, $\beta = 0.125$ and increasing noise level: $\sigma = 0.0$ ($\diamond$); $\sigma = 1.0$ ($\times$); $\sigma = 3.0$ ($\Diamond$); $\sigma = 10.0$ ($\triangle$); $\sigma = 20.0$ ($+$); $\sigma = 30.0$ ($\square$). Symbols on right hand side indicate K values for $\epsilon = 0$. Parameters: $N = 40000$, $n_c = 100$.

over other methods for detecting SNAs (for example in continuous time systems) which involve replacing quasiperiodic forcing with rational approximations. In this paper we used the GOPY map simply for illustrative purposes. In future work we intend to address two point. Firstly, to investigate the scaling laws of $M_c(n)$ for data derived from an SNA since the statistics of the variable $\{x_j\}$ are well-known to have complicated power-law exponents: the 'singular continuous' spectrum [8, 10]. Secondly, we intend to report in more detail on the statistical behaviour of the '0–1 test' and its use for the detection of chaotic dynamics in continuous-time systems, for example chaotic advection and mixing in periodically-forced fluid flows [11–13].

In conclusion, the algorithm we have described and tested provides a valuable numerical diagnostic tool for nonlinear time series analysis and, given its broad applicability, we anticipate that it will prove useful in many scientific fields.

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