

Supplementary Information for
'Minimal Reaction Schemes for Pattern Formation'
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Fraser R. Waters^{1,2*}, Christian A. Yates^{1,2}, Jonathan H. P. Dawes¹

¹Department of Mathematical Sciences, University of Bath; Bath, BA2 7AY, UK

²Centre for Mathematical Biology, University of Bath; Bath, BA2 7AY, UK

*Corresponding author. Email: frw22@bath.ac.uk

Here we provide the mathematical details supporting the arguments advanced in the main text. Section A comprises a systematic analysis of all qualitatively distinct classes of reaction schemes of four reactions for type II patterns. We identify all the classes that contain Turing-unstable reaction schemes. In the case of type I patterns (see main text), for each class we are able to derive the exact constraints on the stoichiometric product coefficients that determine if a reaction scheme is Turing-unstable; i.e. if it can exhibit a Turing instability for *some* choice of positive reaction rate and diffusivity parameter values. For the type II patterns, deriving such constraints is more taxing, so instead we demonstrate that the identified classes do contain some Turing-unstable reaction scheme(s) by providing examples with appropriate choices of reaction rate parameters.

In Section B we show that the dynamics of all the minimal reaction schemes for type I patterns are described by the same phase diagram (Figure 5 of the main text); given that the reaction schemes can exhibit a Turing instability for *some* choice of reaction rate and diffusivity parameter values, we derive the exact conditions on said values under which the Turing instability occurs. It transpires that the functional form of these conditions is the same for all the minimal reaction schemes. In Section C we present exact expressions for the stability boundaries identified in Section B – for each class of minimal reaction scheme for type I patterns – in terms of the stoichiometric product coefficients, as well as some numerical simulations of the type I minimal schemes in one spatial dimension. In Section D we repeat the analysis of Section B for some of the type II minimal schemes, and we show that each of the type II minimal schemes cannot exhibit bi-stability with multiple positive spatially homogeneous steady states. In Section E we give full details of the numerical simulations presented in the main text (Figures 6, 7, and 8).

A Minimal Schemes for Type-II Patterns

There are three options for the reactants of a fourth reaction (with a different reactant combination) in addition to the necessary $2U \rightarrow \dots$, $U + V \rightarrow \dots$, and $U \rightarrow \dots$ reactions for type-II patterns: \emptyset , V , or $2V$. These possibilities are analysed here in turn. Analysis of the minimal schemes for type-II patterns is less compact than that of those for type-I. Here we derive the necessary constraints on the signs of the stoichiometric effects, thus eliminating all qualitatively distinct reaction schemes that *cannot* yield Turing-unstable examples. Of particular note, our analysis here does not explicitly solve the steady state equations for u^* and v^* . In each instance, we suppose the existence of a positive homogeneous steady state and use this to derive necessary constraints on the stoichiometric coefficients. We do not derive sufficient conditions for the existence of a positive homogeneous steady state.

To demonstrate that the remaining qualitatively distinct classes *can indeed* yield Turing-unstable schemes, we provide example choices of the stoichiometric coefficients and reaction rate parameters.

It may be shown that these examples exhibit Turing instability for any values of the diffusivity ratio $\delta = \frac{D_v}{D_u}$ satisfying

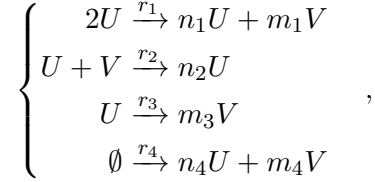
$$\sqrt{\frac{D_v}{D_u}} > \sqrt{\delta_c} = \frac{1}{J_{11}^*} \left(\sqrt{\det(J^*)} + \sqrt{-J_{12}^* J_{21}^*} \right). \quad (\text{A.1})$$

We reason, by continuity, that the same choice of stoichiometric coefficients admits a positive homogeneous steady state exhibiting a Turing instability for reaction rate parameters in some open neighbourhood of the example values given. Such examples are therefore sufficient to demonstrate that the qualitatively distinct classes of minimal reaction scheme shown are the only ones to yield Turing-unstable reaction schemes.

In total, we find 14 qualitatively distinct minimal reaction schemes satisfying the conditions for a Turing pattern instability of type II. These are presented in Figure 4 of the main text, and examples from each class are simulated numerically in Figure 7 of the main text.

A.1 Fourth reaction of zeroth order

If we choose the fourth reaction to be of zeroth order, then we may write our reaction scheme



with corresponding interaction terms

$$\begin{aligned} F(u, v) &= r_4 n_4 - r_3 u + r_1 (n_1 - 2) u^2 + r_2 (n_2 - 1) uv, \\ G(u, v) &= r_4 m_4 + r_3 m_3 u + r_1 m_1 u^2 - r_2 uv, \end{aligned}$$

where $n_1 > 2$ and the other stoichiometric product coefficients are yet to be determined. Supposing the existence of a positive steady state, the Jacobian J^* is given by

$$\begin{aligned} J^* &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_2(n_2 - 1)u^* \\ r_3 m_3 + 2r_1 m_1 u^* - r_2 v^* & -r_2 u^* \end{pmatrix} \\ &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_2(n_2 - 1)u^* \\ \frac{1}{u^*}(r_1 m_1 (u^*)^2 - r_4 m_4) & -r_2 u^* \end{pmatrix}, \end{aligned}$$

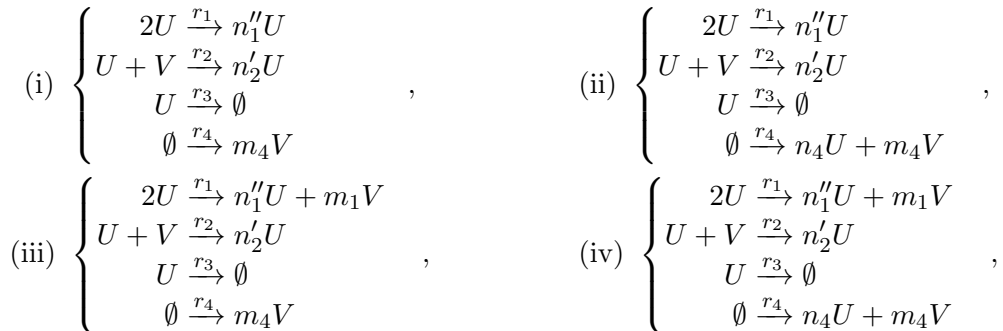
thus the conditions $J_{12}^* > 0$ and $J_{21}^* < 0$ respectively require that $n_2 > 1$ and $m_4 > 0$. Now attempting to solve for a steady state, we find that

$$r_4(n_4 + (n_2 - 1)m_4) + r_3((n_2 - 1)m_3 - 1)u^* + r_1((n_1 - 2) + m_1(n_2 - 1))(u^*)^2 = 0.$$

Since the coefficient of $(u^*)^2$ and the constant coefficient are necessarily positive, for this quadratic equation to admit a positive root u^* , we must have

$$(n_2 - 1)m_3 - 1 < 0 \iff m_3 < \frac{1}{n_2 - 1} \iff m_3 = 0.$$

This leaves four qualitatively distinct possibilities:



where $n_1'' > 2$, $m_1 > 0$, $n_2' > 1$, $n_4 > 0$, and $m_4 > 0$. Each case yields Turing-unstable reaction schemes if the stoichiometric product coefficients are chosen appropriately, for example:

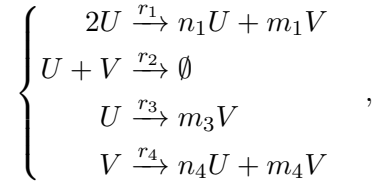
$$\begin{aligned}
\text{(i):} & \quad (n_1'', n_2', m_4) = (3, 2, 1), & (r_1, r_2, r_3, r_4) &= (0.75, 1.0, 0.8, 0.1), \\
\text{(ii):} & \quad (n_1'', n_2', n_4, m_4) = (3, 2, 1, 1), & (r_1, r_2, r_3, r_4) &= (0.75, 1.0, 0.8, 0.1), \\
\text{(iii):} & \quad (n_1'', m_1, n_2', m_4) = (3, 2, 2, 1), & (r_1, r_2, r_3, r_4) &= (0.15, 1.0, 0.85, 0.1), \\
\text{(iv):} & \quad (n_1'', m_1, n_2', n_4, m_4) = (3, 1, 2, 1, 2), & (r_1, r_2, r_3, r_4) &= (0.75, 1.0, 0.95, 0.05).
\end{aligned}$$

A.2 Fourth reaction with reactant V

As for the minimal schemes of type I, choosing the last reaction to have reactant combination V yields the most options for Turing-unstable reaction schemes. As in the main text, for convenience we divide these into three cases according to the qualitative type of the $U + V$ reaction.

A.2.1 Interspecific reaction removing both species ($U + V \rightarrow \emptyset$)

In the first case, we may write our reaction scheme as



with corresponding interaction terms

$$\begin{aligned}
F(u, v) &= -r_3u + r_4n_4v + r_1(n_1 - 2)u^2 - r_2uv, \\
G(u, v) &= r_3m_3u + r_4(m_4 - 1)v + r_1m_1u^2 - r_2uv,
\end{aligned}$$

where $n_1 > 2$ and the other stoichiometric product coefficients are yet to be determined. Supposing the existence of a positive steady state, we have

$$\begin{aligned}
J^* &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* - r_2v^* & r_4n_4 - r_2u^* \\ r_3m_3 + 2r_1m_1u^* - r_2v^* & r_4(m_4 - 1) - r_2u^* \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{u^*}(r_1(n_1 - 2)(u^*)^2 - r_4n_4v^*) & \frac{u^*}{v^*}(r_3 - r_1(n_1 - 2)u^*) \\ \frac{1}{u^*}(r_1m_1(u^*)^2 - r_4(m_4 - 1)v^*) & \frac{u^*}{v^*}(-r_3m_3 - r_1m_1u^*) \end{pmatrix},
\end{aligned}$$

and the conditions that $J_{12}^* > 0$ and $J_{21}^* < 0$ respectively require that $n_4 > 0$ and $m_4 > 1$. Further, to satisfy $J_{11}^* > 0 > J_{21}^*$ we must have

$$-r_3 + 2r_1(n_1 - 2)u^* > r_3m_3 + 2r_1m_1u^* \iff 2r_1((n_1 - 2) - m_1)u^* > r_3(m_3 + 1),$$

hence $n_1 - 2 > m_1$ and

$$u^* > \frac{r_3}{2r_1} \frac{m_3 + 1}{(n_1 - 2) - m_1}.$$

To satisfy $J_{12}^* > 0$ we must have

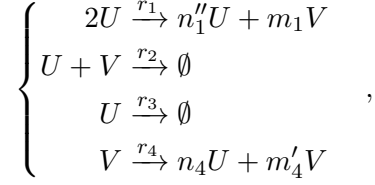
$$u^* < \frac{r_3}{r_1} \frac{1}{n_1 - 2},$$

and for these two bounds on u^* to be consistent, we must have

$$\frac{1}{2} \frac{m_3 + 1}{(n_1 - 2) - m_1} < \frac{1}{n_1 - 2} \iff (m_3 - 1)(n_1 - 2) < -2m_1.$$

For non-negative integers m_1 , m_3 , and $n_1 > 2$, this can only be satisfied if $m_3 = 0$. Finally, if $m_3 = 0$, the condition $J_{22} < 0$ requires that $m_1 > 0$ and we are reduced to only one qualitative possible reaction

scheme:



where $n_1'' > 2$, $m_1 > 0$, $n_4 > 0$, and $m_4' > 1$. One such Turing-unstable example is defined by the stoichiometric product coefficients

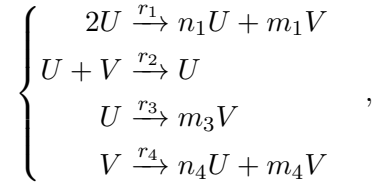
$$(n_1'', m_1, n_4, m_4') = (6, 1, 2, 2).$$

This example exhibits Turing instability if the reaction rates are equal to (or within some small open neighbourhood of)

$$(r_1, r_2, r_3, r_4) = (0.75, 1.0, 0.7, 0.125).$$

A.2.2 Interspecific reaction removing species V and preserving species U ($U + V \rightarrow U$)

In the second case, we may write our reaction scheme as



with corresponding interaction terms

$$\begin{aligned} F(u, v) &= -r_3u + r_4n_4v + r_1(n_1 - 2)u^2, \\ G(u, v) &= r_3m_3u + r_4(m_4 - 1)v + r_1m_1u^2 - r_2uv, \end{aligned}$$

where $n_1 > 2$ and the other stoichiometric product coefficients are yet to be determined. Supposing the existence of a positive steady state, we have

$$\begin{aligned} J^* &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* & r_4n_4 \\ r_3m_3 + 2r_1m_1u^* - r_2v^* & r_4(m_4 - 1) - r_2u^* \end{pmatrix} \\ &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* & r_4n_4 \\ \frac{1}{u^*}(r_1m_1u^* - r_4(m_4 - 1)v^*) & \frac{u^*}{v^*}(-r_3m_3 + r_1m_1u^*) \end{pmatrix}, \end{aligned}$$

and the conditions that $J_{12}^* > 0$ and $J_{21}^* < 0$ require respectively that $n_4 > 0$ and $m_4 > 1$. The condition that $J_{22}^* < 0$ requires that we cannot simultaneously have both $m_1 = 0$ and $m_3 = 0$. This leaves three qualitatively distinct possibilities:

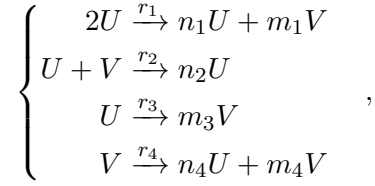
$$(i) \left\{ \begin{array}{l} 2U \xrightarrow{r_1} n_1''U \\ U + V \xrightarrow{r_2} U \\ U \xrightarrow{r_3} m_3V \\ V \xrightarrow{r_4} n_4U + m_4'V \end{array} \right. , \quad (ii) \left\{ \begin{array}{l} 2U \xrightarrow{r_1} n_1''U + m_1V \\ U + V \xrightarrow{r_2} U \\ U \xrightarrow{r_3} \emptyset \\ V \xrightarrow{r_4} n_4U + m_4'V \end{array} \right. , \quad (iii) \left\{ \begin{array}{l} 2U \xrightarrow{r_1} n_1''U + m_1V \\ U + V \xrightarrow{r_2} U \\ U \xrightarrow{r_3} m_3V \\ V \xrightarrow{r_4} n_4U + m_4'V \end{array} \right. ,$$

where $n_1'' > 2$, $m_1 > 0$, $m_3 > 0$, $n_4 > 0$, and $m_4' > 1$. Each case yields Turing-unstable reaction schemes if the stoichiometric product coefficients are chosen appropriately, for example:

$$\begin{aligned} (i): & \quad (n_1'', m_3, n_4, m_4') = (3, 1, 1, 2), & (r_1, r_2, r_3, r_4) &= (0.6, 1.0, 0.7, 0.2), \\ (ii): & \quad (n_1'', m_1, n_4, m_4') = (4, 1, 1, 2), & (r_1, r_2, r_3, r_4) &= (0.68, 1.0, 0.8, 0.2), \\ (iii): & \quad (n_1'', m_1, m_3, n_4, m_4') = (4, 1, 1, 1, 2), & (r_1, r_2, r_3, r_4) &= (0.61, 1.0, 0.85, 0.1). \end{aligned}$$

A.2.3 Interspecific reaction removing species V and increasing species U ($U + V \rightarrow n'_2 U$)

In the third case, we may write our reaction scheme as



with corresponding interaction terms

$$\begin{aligned} F(u, v) &= -r_3 u + r_4 n_4 v + r_1 (n_1 - 2) u^2 + r_2 (n_2 - 1) uv, \\ G(u, v) &= r_3 m_3 u + r_4 (m_4 - 1) v + r_1 m_1 u^2 - r_2 uv, \end{aligned}$$

where $n_1 > 2$, $n_2 > 1$, and the other stoichiometric product coefficients are yet to be determined. Supposing the existence of a positive steady state, we have

$$\begin{aligned} J^* &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_4 n_4 + r_2(n_2 - 1)u^* \\ r_3 m_3 + 2r_1 m_1 u^* - r_2 v^* & r_4(m_4 - 1) - r_2 u^* \end{pmatrix} \\ &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_4 n_4 + r_2(n_2 - 1)u^* \\ \frac{1}{u^*}(r_1 m_1 u^* - r_4(m_4 - 1)v^*) & \frac{u^*}{v^*}(-r_3 m_3 - r_1 m_1 u^*) \end{pmatrix}, \end{aligned}$$

and the condition that $J_{21}^* < 0$ requires that $m_4 > 1$. Attempting to solve for a steady state, we find that

$$r_4(m_3 n_4 + (m_4 - 1))v^* + r_1((n_1 - 2)m_3 + m_1)(u^*)^2 + r_2((n_2 - 1)m_3 - 1)u^* v^* = 0.$$

Since the coefficients of v^* and $(u^*)^2$ are necessarily positive, for this equation to be satisfied at some positive steady state, we must have

$$(n_2 - 1)m_3 - 1 < 0 \iff m_3 < \frac{1}{n_2 - 1} \iff m_3 = 0.$$

The condition that $J_{22}^* < 0$ then requires that $m_1 > 0$, leaving two qualitatively distinct possibilities:

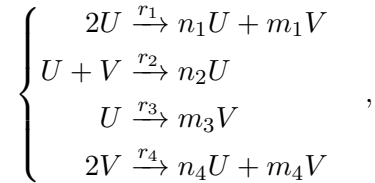


where $n''_1 > 2$, $m_1 > 0$, $n'_2 > 1$, $n_4 > 0$, and $m'_4 > 1$. Each case yields Turing-unstable reaction schemes if the stoichiometric product coefficients are chosen appropriately, for example:

$$\begin{aligned} (i): \quad & (n''_1, m_1, n'_2, m'_4) = (4, 2, 2, 2), \quad (r_1, r_2, r_3, r_4) = (0.15, 1.0, 0.7, 0.4), \\ (ii): \quad & (n''_1, m_1, n'_2, n_4, m'_4) = (4, 1, 2, 1, 2), \quad (r_1, r_2, r_3, r_4) = (0.15, 0.97, 0.8, 0.52). \end{aligned}$$

A.3 Fourth reaction with reactant combination $2V$

If we choose the fourth reaction to have reactant combination $2V$, then we may write our reaction scheme



with corresponding interaction terms

$$\begin{aligned} F(u, v) &= -r_3 u + r_1 (n_1 - 2) u^2 + r_2 (n_2 - 1) uv + r_4 n_4 v^2, \\ G(u, v) &= r_3 m_3 u + r_1 m_1 u^2 - r_2 uv + r_4 (m_4 - 2) v^2, \end{aligned}$$

where $n_1 > 2$ and the other stoichiometric product coefficients are yet to be determined. Supposing the existence of a positive steady state, the Jacobian J^* is given by

$$\begin{aligned} J^* &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_2(n_2 - 1)u^* + 2r_4n_4v^* \\ r_3m_3 + 2r_1m_1u^* - r_2v^* & -r_2u^* + 2r_4(m_4 - 2)v^* \end{pmatrix} \\ &= \begin{pmatrix} -r_3 + 2r_1(n_1 - 2)u^* + r_2(n_2 - 1)v^* & r_2(n_2 - 1)u^* + 2r_4n_4v^* \\ \frac{1}{u^*}(r_1m_1(u^*)^2 - r_4(m_4 - 2)(v^*)^2) & -r_2u^* + 2r_4(m_4 - 2)v^* \end{pmatrix}, \end{aligned}$$

thus the condition $J_{21}^* < 0$ requires that $m_4 > 2$. We also have that

$$u^* J_{21}^* + v^* J_{22}^* = r_3m_3u^* + 2r_1m_1(u^*)^2 - 2r_2u^*v^* + 2r_4(m_4 - 2)(v^*)^2 = -r_3m_3u^*,$$

hence simultaneous negativity of J_{12}^* and J_{22}^* requires $m_3 > 0$. Now attempting to solve for a steady state, we find that

$$r_1((n_1 - 2)m_3 + m_1)(u^*)^2 + r_2((n_2 - 1)m_3 - 1)u^*v^* + r_4(m_3n_4 + (m_4 - 2))(v^*)^2 = 0.$$

Since the coefficients of $(u^*)^2$ and $(v^*)^2$ are necessarily positive, for this equation to be satisfied at a positive steady state, we must have

$$(n_2 - 1)m_3 - 1 < 0 \iff n_2 - 1 < \frac{1}{m_3} \iff n_2 - 1 \leq 0,$$

i.e. if and only if $n_2 = 1$ or $n_2 = 0$. The condition $J_{12}^* > 0$ then requires that $n_4 > 0$, leaving four qualitatively distinct possibilities:

$$\begin{aligned} \text{(i)} \quad & \begin{cases} 2U \xrightarrow{r_1} n_1''U \\ U + V \xrightarrow{r_2} \emptyset \\ U \xrightarrow{r_3} m_3V \\ 2V \xrightarrow{r_4} n_4U + m_4''V \end{cases}, & \text{(ii)} \quad & \begin{cases} 2U \xrightarrow{r_1} n_1''U \\ U + V \xrightarrow{r_2} U \\ U \xrightarrow{r_3} m_3V \\ 2V \xrightarrow{r_4} n_4U + m_4''V \end{cases}, \\ \text{(iii)} \quad & \begin{cases} 2U \xrightarrow{r_1} n_1''U + m_1V \\ U + V \xrightarrow{r_2} \emptyset \\ U \xrightarrow{r_3} m_3V \\ 2V \xrightarrow{r_4} n_4U + m_4''V \end{cases}, & \text{(iv)} \quad & \begin{cases} 2U \xrightarrow{r_1} n_1''U + m_1V \\ U + V \xrightarrow{r_2} U \\ U \xrightarrow{r_3} m_3V \\ 2V \xrightarrow{r_4} n_4U + m_4''V \end{cases}, \end{aligned}$$

where $n_1'' > 2$, $m_1 > 0$, $m_3 > 0$, $n_4 > 0$, and $m_4'' > 2$. Each case yields Turing-unstable reaction schemes if the stoichiometric product coefficients are chosen appropriately, for example:

$$\begin{aligned} \text{(i):} \quad & (n_1'', m_3, n_4, m_4'') = (3, 1, 2, 3), & (r_1, r_2, r_3, r_4) &= (0.35, 0.995, 0.3, 0.94), \\ \text{(ii):} \quad & (n_1'', m_3, n_4, m_4'') = (3, 1, 2, 3), & (r_1, r_2, r_3, r_4) &= (0.35, 0.985, 0.3, 0.23), \\ \text{(iii):} \quad & (n_1'', m_1, m_3, n_4, m_4'') = (5, 1, 1, 2, 3), & (r_1, r_2, r_3, r_4) &= (0.32, 0.9993, 0.2, 0.26), \\ \text{(iv):} \quad & (n_1'', m_1, m_3, n_4, m_4'') = (4, 1, 1, 1, 3), & (r_1, r_2, r_3, r_4) &= (0.1, 0.993, 0.25, 0.41). \end{aligned}$$

B Linearised Analysis of Type-I

Here we show that the linearised dynamics of all the type-I minimal schemes are governed by the same two dimensionless parameter groups, and we derive the phase diagram in Figure 5 of the main text.

Corresponding to the choice of third reactant combination \emptyset , U , or V , the mass-action PDE models for the type-I minimal schemes take one of three forms respectively:

(i) if the third reaction is of zeroth order

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_1 + a_4 u^2 + a_5 uv, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_1 + b_4 u^2 + b_5 uv; \end{aligned}$$

(ii) if the third reaction is of first order with reactant U

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_2 u + a_4 u^2 + a_5 uv, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_2 u + b_4 u^2 + b_5 uv;\end{aligned}$$

or (iii) if the third reaction is of first order with reactant V

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_3 v + a_4 u^2 + a_5 uv, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_3 v + b_4 u^2 + b_5 uv.\end{aligned}$$

Non-dimensionalising $\mathbf{x} = \xi \hat{\mathbf{x}}$, $t = \tau \hat{t}$, $(u, v) = (\sigma \hat{u}, \sigma \hat{v})$, and substituting in the forms for the coefficients $\{a_i, b_i\}$ in terms of $\{r_i, n_i, m_i\}$, if we set $\xi^2 = D_u \tau$ then the equations depend on the diffusivities D_u and D_v only through the ratio $\frac{D_v}{D_u}$. Similarly, setting $\sigma = \frac{1}{r_1 \tau}$, the equations depend on the reaction rates r_1 and r_2 only through the ratio $\frac{r_2}{r_1}$. Lastly, setting $\tau = \frac{1}{\sqrt{r_1 r_3}}$ in case (i) and $\tau = \frac{1}{r_3}$ in cases (ii) and (iii), we remove any other dependency on the reaction rate parameters. That is, we obtain the non-dimensionalised equations

(i):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + n_3 + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v}, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + m_3 + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v};\end{aligned}$$

(ii):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + (n_3 - 1) \hat{u} + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v}, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + m_3 \hat{u} + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v};\end{aligned}$$

and (iii):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + n_3 \hat{v} + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v}, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + (m_3 - 1) \hat{v} + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v},\end{aligned}$$

where $\delta = \frac{D_v}{D_u}$ and $\rho = \frac{r_2}{r_1}$. Considering the balance of terms in the steady state equations, we can deduce the steady state scalings:

(i):

$$(\hat{u}, \hat{v}) \sim (1, \rho^{-1}) \implies (u^*, v^*) \propto (\sigma, \sigma \rho^{-1}) = \left(\sqrt{\frac{r_3}{r_1}}, \frac{\sqrt{r_3 r_1}}{r_2} \right),$$

(ii):

$$(\hat{u}, \hat{v}) \sim (1, \rho^{-1}) \implies (u^*, v^*) \propto (\sigma, \sigma \rho^{-1}) = \left(\frac{r_3}{r_1}, \frac{r_3}{r_2} \right),$$

(iii):

$$(\hat{u}, \hat{v}) \sim (\rho^{-1}, \rho^{-2}) \implies (u^*, v^*) \propto (\sigma \rho^{-1}, \sigma \rho^{-2}) = \left(\frac{r_3}{r_2}, \frac{r_3 r_1}{r_2^2} \right)$$

(as indeed we observe in the derivation of the minimal type-I schemes), where each proportionality is up to some function of the integer stoichiometric product coefficients $\{n_i, m_i\}$. The Jacobian then scales as

$$(i): J^* \sim \begin{pmatrix} \sqrt{r_3 r_1} & \sqrt{\frac{r_3}{r_1} r_2} \\ \sqrt{r_3 r_1} & \sqrt{\frac{r_3}{r_1} r_2} \end{pmatrix}, \quad (ii): J^* \sim \begin{pmatrix} r_3 & \frac{r_3 r_2}{r_1} \\ r_3 & \frac{r_3 r_2}{r_1} \end{pmatrix}, \quad (iii): J^* \sim \begin{pmatrix} \frac{r_3 r_1}{r_2} & r_3 \\ \frac{r_3 r_1}{r_2} & r_3 \end{pmatrix},$$

where each entry is proportional (up to a differing function of $\{n_i, m_i\}$) to the given expression in the rate parameters. In each case, the determinant condition $\det(J^*) > 0$ and sign conditions on the entries of J^* become conditions on these functions of $\{n_i, m_i\}$ only (independent of the reaction rate parameters), while the trace condition $\text{tr}(J^*) < 0$ rearranges to give

$$\text{tr}(J^*) < 0 \iff \frac{-J_{22}^*}{J_{11}^*} > 1 \iff \frac{r_2}{r_1} > \rho_c(\mathbf{n}, \mathbf{m}),$$

where the exact form for ρ_c depends on the chosen reaction scheme. This gives the vertical phase boundary $\rho = \rho_c$ plotted in Figure 5 of the main text. For $\rho < \rho_c$, the homogeneous steady state is not linearly stable with respect to spatially uniform perturbations; for $\rho > \rho_c$ the homogeneous steady state is linearly stable to spatially uniform perturbations, but may or may not be linearly unstable with respect to non-uniform perturbations.

Finally, substituting the reaction rate parameter scalings for J_{11}^* , $\det(J^*)$, and $J_{12}^* J_{21}^*$ into the expression for $\sqrt{\delta_c}$, each case yields that

$$\sqrt{\delta_c} = \frac{1}{J_{11}^*} \left(\sqrt{\det(J^*)} + \sqrt{-J_{12}^* J_{21}^*} \right) \propto \sqrt{\frac{r_2}{r_1}},$$

where the proportionality is up to some function of $\mathbf{n} = (n_i)$ and $\mathbf{m} = (m_i)$, that is: for each minimal Turing-unstable reaction scheme of type I, we have

$$\delta_c = H(\mathbf{n}, \mathbf{m})\rho,$$

for some function H . This gives the diagonal phase boundary $\delta = \delta_c(\rho)$ plotted in Figure 5 of the main text. Supposing that $\rho > \rho_c$, then for $\delta < \delta_c(\rho)$ the homogeneous steady state is linearly stable with respect to all perturbations. If $\rho > \rho_c$ and $\delta > \delta_c(\rho)$, then the homogeneous steady state is linearly stable to spatial perturbations with sufficiently small or large wavenumbers, but is linearly unstable to perturbations with wavenumbers in some open interval $k \in (k_-, k_+)$.

C Stability Boundaries for Type-I

In the following two tables, we collate the expressions for ρ_c and $H = \delta_c \rho^{-1}$, which determine quantitatively the Turing instability boundaries, for each of the 11 classes of minimal type-I reaction schemes. As in Figure 3 of the main text, we label these classes alphabetically a. through to k.

Class	$\rho_c(\mathbf{n}, \mathbf{m})$	$\sqrt{H(\mathbf{n}, \mathbf{m})}$
I.a	$2(n_1'' - 2) - m_1$	$\frac{1}{2(n_1'' - 2) - m_1} \left(\sqrt{2(m_1 - (n_1'' - 2))} + \sqrt{m_1} \right)$
I.b	$\frac{(2n_3 - m_3)(n_1'' - 2) - n_3 m_1}{n_3 - m_3}$	$\frac{\sqrt{n_3 - m_3}}{(2n_3 - m_3)(n_1'' - 2) - n_3 m_1} \left(\sqrt{2(n_3 - m_3)(m_1 - (n_1'' - 2))} + \sqrt{m_3(n_1'' - 2) + (n_3 - 2m_3)m_1} \right)$
I.c	$n_1'' - 2$	$\frac{1}{n_1'' - 2} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{m_1} \right)$
I.d	$n_1'' - 2$	$\frac{1}{n_1'' - 2} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{m_1} \right)$
I.e	$\frac{(n_1'' - 2)^2}{m_1}$	$\frac{1}{n_1'' - 2} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{2m_1 - (n_1'' - 2)} \right)$
I.f	$\frac{(n_3 m_1 + (n_1'' - 2))((2n_3 + 1)(n_1'' - 2) - n_3 m_1)}{(n_3 + 1)^2 m_1}$	$\frac{n_3 + 1}{(2n_3 + 1)(n_1'' - 2) - n_3 m_1} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{\frac{2(n_3 + 1)(n_1'' - 2)m_1}{n_3 m_1 + (n_1'' - 2)} - (n_1'' - 2)} \right)$
I.g	$2(n_1'' - 2) - m_1$	$\frac{1}{2(n_1'' - 2) - m_1} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{n_1'' - 2} \right)$
I.h	$\frac{(n_3 m_1 - (m_3' - 1)(n_1'' - 2))((2n_3 - (m_3' - 1))(n_1'' - 2) - n_3 m_1)}{(n_3 - (m_3' - 1))^2 m_1}$	$\frac{n_3 - (m_3' - 1)}{(2n_3 - (m_3' - 1))(n_1'' - 2) - n_3 m_1} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{\frac{2(n_3 - (m_3' - 1))(n_1'' - 2)m_1}{n_3 m_1 - (m_3' - 1)(n_1'' - 2)} - (n_1'' - 2)} \right)$
I.i	$\frac{(n_1'' - 2)^2}{m_1}$	$\frac{\sqrt{m_1}}{n_1'' - 2} (1 + \sqrt{2})$
I.j	$\frac{(n_1'' - 2)^2 - n_3^2 m_1^2}{m_1}$	$\frac{\sqrt{m_1}}{((n_1'' - 2) - n_3 m_1) \sqrt{(n_1'' - 2) + n_3 m_1}} \left(\sqrt{(n_1'' - 2) + n_3 m_1} + \sqrt{2(n_1'' - 2)} \right)$
I.k	$\frac{(n_1'' - 2)^2}{m_1}$	$\frac{1}{n_1'' - 2} \left(\sqrt{(n_1'' - 2)(m_2' - 1) + m_1} + \sqrt{(n_1'' - 2)(m_2' - 1) + 2m_1} \right)$

Table C.1: The critical reaction rate ratio ρ_c for linear stability of the homogeneous steady state (u^*, v^*) to spatially uniform perturbations, and the Turing instability threshold $\sqrt{H} = \sqrt{\delta_c \rho^{-1}}$ for each of the eleven type-I Turing-unstable classes of minimal reaction scheme as functions of the integer stoichiometric product coefficients. With reference to Figure 3 of the main text, the classes are labelled alphabetically down the columns: a-h column 1, i-j column 2, k column 3.

Class	$\sqrt{\delta_c(\rho_c)} = \sqrt{H(\mathbf{n}, \mathbf{m})} \sqrt{\rho_c(\mathbf{n}, \mathbf{m})}$	$\inf(\delta_c(\rho_c))$
I.a	$\frac{1}{\sqrt{2(n_1'' - 2) - m_1}} \left(\sqrt{2(m_1 - (n_1'' - 2))} + \sqrt{m_1} \right)$	1
I.b	$\frac{1}{\sqrt{(2n_3 - m_3)(n_1'' - 2) - n_3 m_1}} \left(\sqrt{2(n_3 - m_3)(m_1 - (n_1'' - 2))} + \sqrt{m_3(n_1'' - 2) + (n_3 - 2m_3)m_1} \right)$	1
I.c	$\frac{1}{\sqrt{n_1'' - 2}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{m_1} \right)$	1
I.d	$\frac{1}{\sqrt{n_1'' - 2}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{m_1} \right)$	1
I.e	$\frac{1}{\sqrt{m_1}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{2m_1 - (n_1'' - 2)} \right)$	1
I.f	$\frac{1}{\sqrt{m_1}} \frac{\sqrt{n_3 m_1 + (n_1'' - 2)}}{\sqrt{(2n_3 + 1)(n_1'' - 2) - n_3 m_1}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{\frac{2(n_3 + 1)(n_1'' - 2)m_1}{n_3 m_1 + (n_1'' - 2)} - (n_1'' - 2)} \right)$	1
I.g	$\frac{1}{\sqrt{2(n_1'' - 2) - m_1}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{n_1'' - 2} \right)$	1
I.h	$\frac{1}{\sqrt{m_1}} \frac{\sqrt{n_3 m_1 - (m_3' - 1)(n_1'' - 2)}}{\sqrt{(2n_3 - (m_3' - 1))(n_1'' - 2) - n_3 m_1}} \left(\sqrt{m_1 - (n_1'' - 2)} + \sqrt{\frac{2(n_3 - (m_3' - 1))(n_1'' - 2)m_1}{n_3 m_1 - (m_3' - 1)(n_1'' - 2)} - (n_1'' - 2)} \right)$	1
I.i	$1 + \sqrt{2}$	$3 + 2\sqrt{2}$
I.j	$\frac{1}{\sqrt{(n_1'' - 2) - n_3 m_1}} \left(\sqrt{(n_1'' - 2) + n_3 m_1} + \sqrt{2(n_1'' - 2)} \right)$	$3 + 2\sqrt{2}$
I.k	$\frac{1}{\sqrt{m_1}} \left(\sqrt{(n_1'' - 2)(m_2 - 1) + m_1} + \sqrt{(n_1'' - 2)(m_2 - 1) + 2m_1} \right)$	$3 + 2\sqrt{2}$

Table C.2: The minimal Turing instability threshold $\sqrt{\delta_c(\rho_c)} = \sqrt{H\rho_c}$ for each of the eleven type-I Turing-unstable classes of minimal reaction scheme, and the corresponding infima for $\delta_c(\rho_c)$ over the stoichiometric product coefficients satisfying the Turing instability constraints. With reference to Figure 3 of the main text, the classes are labelled alphabetically down the columns: a-h column 1, i-j column 2, k column 3.

D Non-dimensionalisation of Type-II

Here we repeat, for the type-II minimal schemes, the non-dimensionalisation arguments applied in Section B to the type-I minimal schemes. Deriving the phase boundaries in generality is complicated, but is done here for some specific cases of type-II reaction schemes.

Corresponding to the choice of third reactant combination, the mass-action PDE models for the type-II minimal schemes take one of three forms:

(i):

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_1 + a_2 u + a_4 u^2 + a_5 uv, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_1 + b_2 u + b_4 u^2 + b_5 uv;\end{aligned}$$

(ii):

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_2 u + a_3 v + a_4 u^2 + a_5 uv, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_2 u + b_3 v + b_4 u^2 + b_5 uv;\end{aligned}$$

or (iii):

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + a_2 u + a_4 u^2 + a_5 uv + a_6 v^2, \\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + b_2 u + b_4 u^2 + b_5 uv + b_6 v^2.\end{aligned}$$

Non-dimensionalising $\mathbf{x} = \xi \hat{\mathbf{x}}$, $t = \tau \hat{t}$, $(u, v) = (\sigma \hat{u}, \sigma \hat{v})$, and substituting in the forms for the coefficients $\{a_i, b_i\}$ in terms of $\{r_i, n_i, m_i\}$, setting $\xi^2 = D_u \tau$ eliminates dependence on the diffusivities D_u and D_v except through their ratio $\frac{D_v}{D_u}$. Again setting $\sigma = \frac{1}{r_1 \tau}$, and choosing $\tau = \frac{1}{r_3}$, we obtain

(i):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + \frac{r_1 r_4}{r_3^2} n_4 + (n_3 - 1) \hat{u} + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v}, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + \frac{r_1 r_4}{r_3^2} m_4 + m_3 \hat{u} + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v};\end{aligned}$$

(ii):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + (n_3 - 1) \hat{u} + \frac{r_4}{r_3} n_4 \hat{v} + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v}, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + m_3 \hat{u} + \frac{r_4}{r_3} (m_4 - 1) \hat{v} + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v};\end{aligned}$$

or (iii):

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \hat{\nabla}^2 \hat{u} + (n_3 - 1) \hat{u} + (n_1 - 2) \hat{u}^2 + \rho (n_2 - 1) \hat{u} \hat{v} + \frac{r_4}{r_1} n_4 \hat{v}^2, \\ \frac{\partial \hat{v}}{\partial \hat{t}} &= \delta \hat{\nabla}^2 \hat{v} + m_3 \hat{u} + m_1 \hat{u}^2 + \rho (m_2 - 1) \hat{u} \hat{v} + \frac{r_4}{r_1} (m_4 - 2) \hat{v}^2,\end{aligned}$$

where $\delta = \frac{D_v}{D_u}$ and $\rho = \frac{r_2}{r_1}$. In each case we have a different second dimensionless group of reaction rate parameters:

$$(i): \varrho_1 = \frac{r_1 r_4}{r_3^2}, \quad (ii): \varrho_2 = \frac{r_4}{r_3}, \quad (iii): \varrho_3 = \frac{r_4}{r_1}.$$

The type-II minimal schemes are more difficult to analyse in generality than those of type I. Generally, the steady state concentrations and the entries of the steady state Jacobian are functions of both dimensionless groups of the reaction rate parameters, and these are not simple power law relations. Accordingly, it is more complicated to separate out the constraints on only the stoichiometric coefficients, under which the reaction rate parameters can be chosen in some open region so as to yield a Turing instability. The slight exception to this is case (i) above, where the steady state concentration u^* is independent of ρ . To illustrate some more general points about the type-II schemes, we now undertake a closer examination of the non-dimensionalised form (i), specifically for the minimal reaction schemes identified in Section A.1.

D.1 Turing analysis of minimal type-II schemes of form (i)

Reviewing the minimal schemes that take this non-dimensionalised form, we identify the following commonalities: $n_1 > 2$, $n_2 > 1$, $m_2 = n_3 = m_3 = 0$, $m_4 > 0$. Thus we consider the general non-dimensionalised PDE model

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \varrho_1 n_4 - u + (n_1 - 2)u^2 + \rho(n_2 - 1)uv, \\ \frac{\partial v}{\partial t} &= \delta \nabla^2 v + \varrho_1 m_4 + m_1 u^2 - \rho uv,\end{aligned}$$

where m_1 or n_4 may be zero. Solving the steady state equations, we find

$$((n_1 - 2) + m_1(n_2 - 1))(u^*)^2 - u^* + \varrho_1(n_4 + (n_2 - 1)m_4) = 0, \quad (\text{D.1})$$

hence under the condition that

$$\varrho_1 < \frac{1}{4(n_4 + (n_2 - 1)m_4)((n_1 - 2) + m_1(n_2 - 1))}, \quad (\text{D.2})$$

we have two positive spatially uniform steady states:

$$u^* = \frac{1 \pm \sqrt{1 - 4(n_4 + (n_2 - 1)m_4)((n_1 - 2) + m_1(n_2 - 1))\varrho_1}}{2((n_1 - 2) + m_1(n_2 - 1))}, \quad v^* = \frac{\varrho_1 m_4 + m_1(u^*)^2}{\rho u^*},$$

both of which satisfy $J_{12}^* > 0$ and $J_{22}^* < 0$. For any positive steady state, the Routh-Hurwitz linear stability condition $\det(J^*) > 0$ rearranges to give

$$u^* < \frac{1}{2((n_1 - 2) + m_1(n_2 - 1))},$$

and so we determine that only the smaller of the two roots of (D.1) can exhibit a Turing instability. Our homogeneous steady state is thus

$$u^* = \frac{1 - \sqrt{1 - 4(n_4 + (n_2 - 1)m_4)((n_1 - 2) + m_1(n_2 - 1))\varrho_1}}{2((n_1 - 2) + m_1(n_2 - 1))}, \quad v^* = \frac{\varrho_1 m_4 + m_1(u^*)^2}{\rho u^*}, \quad (\text{D.3})$$

and satisfies $\det(J^*) > 0$. Note that u^* is independent of ρ . The condition $J_{11}^* > 0$ is

$$(n_1 - 2)(u^*)^2 > \varrho_1 n_4.$$

If $n_4 = 0$, this is guaranteed for any positive u^* , since $n_1 > 2$. Otherwise, by substituting in the expression for our steady state u^* , we obtain the constraints

$$\varrho_1 > \frac{(n_1 - 2)n_4}{(2n_4(n_1 - 2) + m_1(n_2 - 1)n_4 + (n_1 - 2)(n_2 - 1)m_4)^2} \quad (\text{D.4})$$

and

$$\varrho_1 < \frac{n_1 - 2}{4n_4((n_1 - 2) + m_1(n_2 - 1))^2}. \quad (\text{D.5})$$

Similarly, the condition $J_{21} < 0$ is

$$m_1(u^*)^2 < \varrho_1 m_4,$$

and if $m_1 = 0$ then this is guaranteed, since $m_4 > 0$. If $m_1 > 0$, substituting in the expression for our steady state u^* we obtain that ϱ_1 must satisfy one of

$$\varrho_1 > \frac{m_1}{4m_4((n_1 - 2) + m_1(n_2 - 1))^2}, \quad (\text{D.6a})$$

or

$$\varrho_1 < \frac{m_1 m_4}{((n_1 - 2)m_4 + 2m_1(n_2 - 1)m_4 + m_1 n_4)^2}. \quad (\text{D.6b})$$

Lastly, the linear stability condition $\text{tr}(J^*) < 0$ rearranges to give

$$\rho > \frac{(n_1 - 2)(u^*)^2 - \varrho_1 n_4}{(u^*)^2}. \quad (\text{D.7})$$

Since u^* is independent of ρ , this constraint is a lower bound on ρ which will vary with ϱ_1 .

In summary, the region in $(\varrho_1, \rho, \delta)$ parameter space in which we observe a Turing instability is most restricted in the ϱ_1 -axis. For existence of a positive steady state, ϱ_1 must satisfy (D.2). If $n_4 \neq 0$ then ϱ_1 must also satisfy both (D.4) and (D.5); if $m_1 \neq 0$ then ϱ_1 must also satisfy either (D.6a) or (D.6b). Provided these constraints are met, then the Turing instability region is semi-infinite in the ρ -axis, since the only constraint on ρ is the ϱ_1 -dependent lower bound (D.7). As for the type-I minimal schemes, the region is semi-infinite in the δ -axis, now with lower bound $\delta > \delta_c(\varrho_1, \rho)$ given by (A.1).

As noted, the other two forms of the type-II minimal schemes are more difficult to analyse in generality. However, for any given set of stoichiometric product coefficients, the analysis to determine whether a reaction scheme can support a Turing pattern – and if so, to determine under what conditions on the reaction rate parameters – is standard. Solving the steady-state equations can be done exactly. For any positive steady state, it then remains to check the stability criteria $\text{tr}(J^*) < 0$, $\det(J^*) > 0$, and the type-II sign pattern criteria $J_{11}^*, J_{12}^* > 0$ and $J_{21}^*, J_{22}^* < 0$. In general, the parameter regimes where Turing patterns occur (if at all) are highly constrained and fine tuning of the reaction rate parameters is required.

D.2 Uniqueness of positive stable homogeneous equilibria in minimal type-II schemes

We show here for each of the Turing-unstable minimal schemes for type-II patterns that if a positive homogeneous steady state (u^*, v^*) exists and is linearly stable (to spatially uniform perturbations), then any other positive homogeneous steady state is linearly unstable. There may however be linearly stable steady states where one or both of u^*, v^* are zero. We show our result by considering in turn each of the three (dimensional) forms as listed above. The relevant deduction for schemes of form (i) is already made in the above linearised analysis, however it is repeated here in a more succinct way that is then applied similarly to the other two forms. For what follows, it will be useful to define the cross differences

$$c_{i,j} := b_i a_j - a_i b_j,$$

and note that $c_{i,j} = -c_{j,i}$.

D.2.1 Form (i)

For type-II minimal schemes of this form, the reaction polynomials are

$$\begin{aligned} F(u, v) &= a_1 + a_2 u + a_4 u^2 + a_5 uv, \\ G(u, v) &= b_1 + b_2 u + b_4 u^2 + b_5 uv. \end{aligned}$$

We note that if a steady state exists with $u^* \neq 0$, then v^* is uniquely defined in terms of u^* . The Jacobian evaluated at any steady state (u^*, v^*) is given by

$$J^* = \begin{pmatrix} a_2 + 2a_4u^* + a_5v^* & a_5u^* \\ b_2 + 2b_4u^* + b_5v^* & b_5u^* \end{pmatrix},$$

and using the steady state equations, we find that

$$\det(J^*) = 2c_{5,4}(u^*)^2 - c_{2,5}u^*.$$

Hence the linear stability condition $\det(J^*) > 0$ implies that $u^* \neq 0$ and further that $2c_{5,4}u^* - c_{2,5} > 0$ for any positive linearly stable steady state. We also have that

$$0 = b_5F(u^*, v^*) - a_5G(u^*, v^*) = c_{5,1} - c_{2,5}u^* + c_{5,4}(u^*)^2.$$

For there to be two positive steady states (u^*, v^*) , there must be two distinct positive roots u^* to this equation. If that is the case, then $c_{5,4} \neq 0$ and $(c_{2,5})^2 \neq 4c_{5,4}c_{5,1}$ and the roots are given by

$$u_{\pm}^* := \frac{c_{2,5} \pm \sqrt{(c_{2,5})^2 - 4c_{5,4}c_{5,1}}}{2c_{5,4}},$$

which yields

$$2c_{5,4}u_{\pm}^* - c_{2,5} = \pm \sqrt{(c_{2,5})^2 - 4c_{5,4}c_{5,1}}.$$

Thus only one of u_+^*, u_-^* satisfies $2c_{5,4}u^* - c_{2,5} > 0$, and we deduce that there can be at most one positive linearly stable homogeneous steady state (u^*, v^*) .

D.2.2 Form (ii)

For minimal schemes of this form, the reaction polynomials are

$$\begin{aligned} F(u, v) &= a_2u + a_3v + a_4u^2 + a_5uv, \\ G(u, v) &= b_2u + b_3v + b_4u^2 + b_5uv. \end{aligned}$$

We note again that if a steady state exists with $u^* \neq 0$, then v^* is uniquely defined in terms of u^* . The Jacobian evaluated at any steady state (u^*, v^*) is given by

$$J^* = \begin{pmatrix} a_2 + 2a_4u^* + a_5v^* & a_3 + a_5u^* \\ b_2 + 2b_4u^* + b_5v^* & b_3 + b_5u^* \end{pmatrix},$$

and using the steady state equations, we find that for any positive steady state

$$\det(J^*) = 2c_{5,4}(u^*)^2 - (c_{4,3} - c_{5,2})u^*.$$

Hence $2c_{5,4}u^* - (c_{4,3} - c_{5,2}) > 0$ for any positive linearly stable steady state. We also have that

$$0 = b_4F(u^*, v^*) - a_4G(u^*, v^*) = c_{4,2}u^* + c_{4,3}v^* + c_{4,5}u^*v^*,$$

and

$$0 = b_5F(u^*, v^*) - a_5G(u^*, v^*) = c_{5,2}u^* + c_{5,3}v^* + c_{5,4}(u^*)^2.$$

For there to be two positive steady states (u^*, v^*) , there must be two distinct positive solutions (u^*, v^*) to these simultaneous equations. If that is the case, then $c_{5,3} \neq 0$ and the solutions correspond to two distinct positive real roots u^* of

$$0 = c_{5,3}c_{4,2} - c_{4,3}c_{5,2} - (c_{4,3} - c_{5,2})c_{5,4}u^* + (c_{5,4})^2(u^*)^2.$$

If this equation does have two positive real roots then these are given by

$$u_{\pm}^* = \frac{(c_{4,3} - c_{5,2}) \pm \sqrt{(c_{4,3} + c_{5,2})^2 - 4c_{5,3}c_{4,2}}}{2c_{5,4}},$$

which yields

$$2c_{5,4}u_{\pm}^* - (c_{4,3} - c_{5,2}) = \pm \sqrt{(c_{4,3} + c_{5,2})^2 - 4c_{5,3}c_{4,2}}.$$

Thus only one of u_+^*, u_-^* satisfies $2c_{5,4}u^* - (c_{4,3} - c_{5,2}) > 0$, and we deduce that there can be at most one positive linearly stable homogeneous steady state (u^*, v^*) .

D.2.3 Form (iii)

For minimal schemes of this form, the reaction polynomials are

$$\begin{aligned} F(u, v) &= a_2u + a_4u^2 + a_5uv + a_6v^2, \\ G(u, v) &= b_2u + b_4u^2 + b_5uv + b_6v^2. \end{aligned}$$

We note that if a positive steady state exists then u^* is uniquely defined in terms of the ratio $\frac{v^*}{u^*}$. The Jacobian evaluated at any steady state (u^*, v^*) is given by

$$J^* = \begin{pmatrix} a_2 + 2a_4u^* + a_5v^* & a_5u^* + 2a_6v^* \\ b_2 + 2b_4u^* + b_5v^* & b_5u^* + 2b_6v^* \end{pmatrix},$$

and using the steady state equations, we find that for any positive steady state

$$\det(J^*) = 2c_{2,6}v^* - c_{5,2}u^*.$$

Hence $2c_{2,6}\frac{v^*}{u^*} - c_{5,2} > 0$ for any positive linearly stable steady state. We also have that

$$0 = b_2F(u^*, v^*) - a_2G(u^*, v^*) = c_{2,4}(u^*)^2 + c_{2,5}u^*v^* + c_{2,6}(v^*)^2,$$

and so for a positive steady state we must have

$$0 = c_{2,4} - c_{5,2}\frac{v^*}{u^*} + c_{2,6}\left(\frac{v^*}{u^*}\right)^2.$$

For there to be two positive steady states (u^*, v^*) , there must be two distinct positive solutions $\frac{v^*}{u^*}$ to this quadratic equation. If that is the case, then $c_{2,6} \neq 0$ and the roots are given by

$$\left(\frac{v^*}{u^*}\right)_{\pm} = \frac{c_{5,2} \pm \sqrt{(c_{5,2})^2 - 4c_{2,6}c_{2,4}}}{2c_{2,6}},$$

which yields

$$2c_{2,6}\left(\frac{v^*}{u^*}\right)_{\pm} - c_{5,2} = \pm\sqrt{(c_{5,2})^2 - 4c_{2,6}c_{2,4}}.$$

Thus only one of $\left(\frac{v^*}{u^*}\right)_{+}$, $\left(\frac{v^*}{u^*}\right)_{-}$ satisfies $2c_{2,6}\frac{v^*}{u^*} - c_{5,2} > 0$, and we deduce that there can be at most one positive linearly stable homogeneous steady state (u^*, v^*) .

E Parameter values for numerical simulations

Here we tabulate, in Tables E.1, E.2, and E.3 respectively, the parameter values used for the numerical simulations displayed in Figures 6, 7, and 8 of the main text. All simulations were run in MATLAB using a modified fourth-order Runge-Kutta-type exponential time differencing scheme (mETD4RK)¹ which has been shown to be a suitable choice for stiff PDE problems².

For the saturating type-I example in Figure 8, the initial conditions used were

$$u(x, 0) = u^* + 10^{-6} \cos\left(3 \cdot \frac{2\pi}{L}x\right), \quad v(x, 0) = v^* + 10^{-6} \cos\left(3 \cdot \frac{2\pi}{L}x\right).$$

For the simulations in Figures 6 and 7, the initial conditions were the linearly stable homogeneous steady state (u^*, v^*) perturbed with small spatial noise:

$$u(x_i, 0) = (1 + 10^{-4}\xi_i)u^*, \quad v(x_i, 0) = (1 + 10^{-4}\eta_i)v^*, \quad \xi_i, \eta_i \stackrel{\text{i.i.d.}}{\sim} \text{Unif}(-0.5, 0.5).$$

¹S. M. Cox, P. C. Matthews, Exponential Time Differencing for Stiff Systems. *Journal of Computational Physics* **176**, 430–455 (2002)

²A.-K. Kassam, L. N. Trefethen, Fourth-order time-stepping for stiff PDEs. *SIAM Journal of Scientific Computing* **26**, 1214–1233 (2005)

Class	Stoichiometric coefficients	Rate constant r_2	Diffusivity D_v	Final time T
I.a	$(n_1'', n_3, m_1) = (4, 1, 3)$	1.2	15	60
I.b	$(n_1'', n_3, m_1, m_3) = (4, 3, 3, 1)$	0.6	25	60
I.c	$(n_1'', n_3', m_1) = (3, 2, 2)$	1.2	10	30
I.d	$(n_1'', n_3', m_1, m_3) = (3, 3, 2, 1)$	1.2	10	30
I.e	$(n_1'', m_1) = (3, 2)$	0.6	6	30
I.f	$(n_1'', n_3, m_1) = (3, 1, 2)$	0.5	15	10
I.g	$(n_1'', n_3, m_1) = (4, 1, 3)$	1.2	10	10
I.h	$(n_1'', n_3, m_1, m_3') = (4, 3, 3, 2)$	0.7	20	5
I.i	$(n_1'', m_1) = (3, 1)$	1.2	10	30
I.j	$(n_1'', n_3, m_1) = (4, 1, 1)$	3.6	20	60
I.k	$(n_1'', m_1, m_2') = (3, 1, 2)$	1.2	15	90

Table E.1: Parameter values chosen for simulations shown in Figure 6. All simulations used $r_1 = r_3 = D_u = 1$, a fixed domain length $L = 32$ with a uniform spatial mesh of $2^9 = 512$ nodes, and a fixed time step of size $h_t = 2^{-15}$. As a proxy for ρ , the rate constant r_2 was chosen to be approximately $1.2\rho_c$. As a proxy for δ , the diffusivity D_v was chosen to be approximately $1.2\delta_c(\rho) = 1.44\delta_c(\rho_c)$.

Class	Stoichiometric coefficients	rate constants				Diffusivity D_v	Final time T
		r_1	r_2	r_3	r_4		
II.a	$(n_1'', n_2', m_1, m_4) = (3, 2, 2, 1)$	0.6	1.0	0.85	0.1	20	100
II.b	$(n_1'', n_2', n_4, m_1, m_4) = (3, 2, 1, 1, 2)$	0.75	1.0	0.95	0.05	50	200
II.c	$(n_1'', n_2', m_4) = (3, 2, 1)$	0.75	1.0	0.8	0.1	50	200
II.d	$(n_1'', n_2', n_4, m_4) = (3, 2, 1, 1)$	0.75	1.0	0.78	0.1	50	100
II.e	$(n_1'', n_4, m_1, m_3, m_4') = (4, 1, 1, 1, 2)$	0.61	1.0	0.85	0.1	100	200
II.f	$(n_1'', n_4, m_1, m_3, m_4'') = (4, 1, 1, 1, 3)$	0.1	0.992	0.25	0.41	100	400
II.g	$(n_1'', n_4, m_3, m_4') = (3, 1, 1, 2)$	0.6	1.0	0.7	0.2	40	100
II.h	$(n_1'', n_4, m_3, m_4'') = (3, 2, 1, 3)$	0.35	0.985	0.3	0.23	80	400
II.i	$(n_1'', n_2', n_4, m_1, m_4') = (4, 2, 1, 1, 2)$	0.15	0.97	0.8	0.52	80	300
II.j	$(n_1'', n_2', m_1, m_4') = (4, 2, 2, 2)$	0.15	1.0	0.7	0.4	5	200
II.k	$(n_1'', n_4, m_1, m_4') = (6, 2, 1, 2)$	0.75	1.0	0.7	0.125	20	200
II.l	$(n_1'', n_4, m_1, m_4'') = (4, 1, 1, 2)$	0.68	1.0	0.8	0.2	60	200
II.m	$(n_1'', n_4, m_1, m_3, m_4'') = (5, 2, 1, 1, 3)$	0.32	0.9993	0.2	0.26	60	400
II.n	$(n_1'', n_4, m_3, m_4'') = (3, 2, 1, 3)$	0.35	0.995	0.3	0.94	60	100

Table E.2: Parameter values chosen for simulations shown in Figure 7. All simulations used $D_u = 1$, a fixed domain length $L = 128$ with a uniform spatial mesh of $2^9 = 512$ nodes, and a fixed time step of size $h_t = 2^{-15}$.

Class	stoichiometric coefficients	rate constants			diffusivity D_v	domain length L	final time T
		r_1	r_2	r_3			
I.a	$(n_1'', n_3, m_1) = (4, 1, 3)$	1	1.2	1	15	28	200

Table E.3: This simulation used $D_u = 1$, a uniform spatial mesh of $2^9 = 512$ nodes, and a fixed time step of size $h_t = 2^{-15}$. The initial conditions were the linearly stable homogeneous steady state (u^*, v^*) , plus a small cosine perturbation with wavenumber k maximising the linearised temporal growth rate of the perturbation.