Localised Pattern Formation

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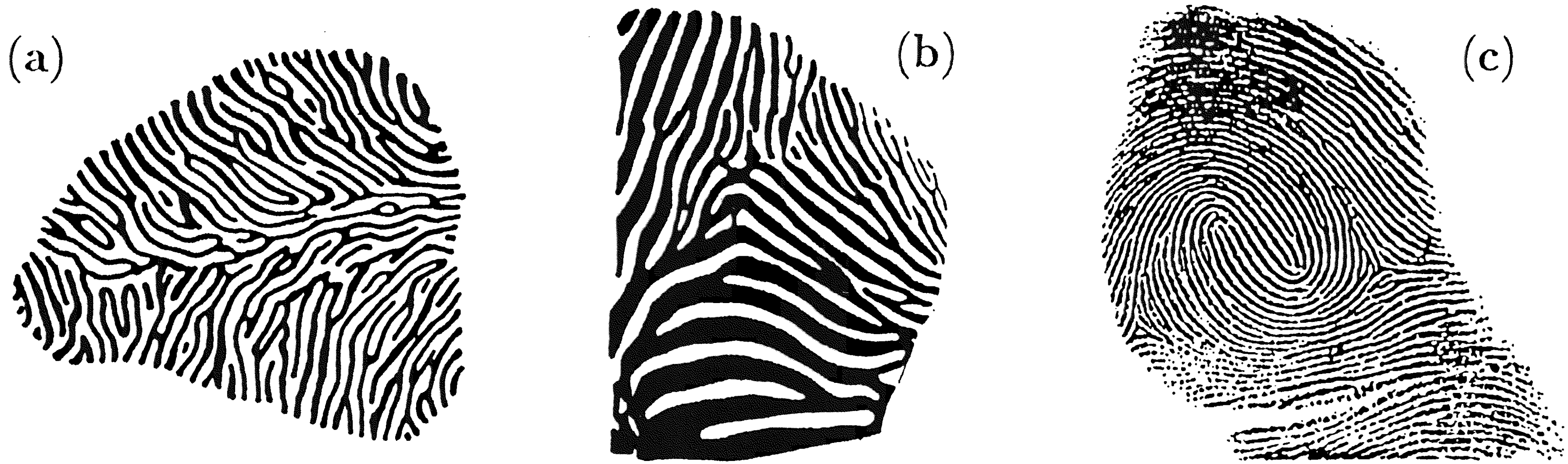
Outline

- Pattern formation
  - Natural examples
  - Laboratory experiments
- Localised patterns
  - Model PDE: Swift–Hohenberg equation
  - Reduction to a Ginzburg–Landau equation
  - ‘Homoclinic snaking’
- Large-scale modes
  - Magnetoconvection
  - Another Model PDE: Swift–Hohenberg equation + diffusion equation
  - Reduction to a nonlocal Ginzburg–Landau equation; ‘slanted snaking’
- Some conclusions

Pattern formation in nature



- Ocular dominance stripes in macaque monkey visual cortex
- flank of a Grevy's zebra
- fingerprint

J.D. Murray, *Mathematical Biology*. Volume II, p622. Springer (2004)

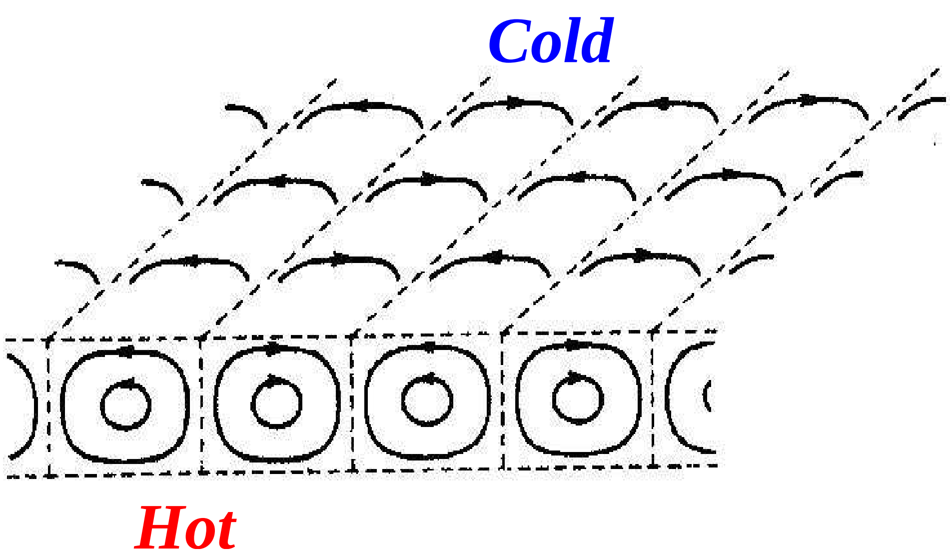
Pattern formation in nature

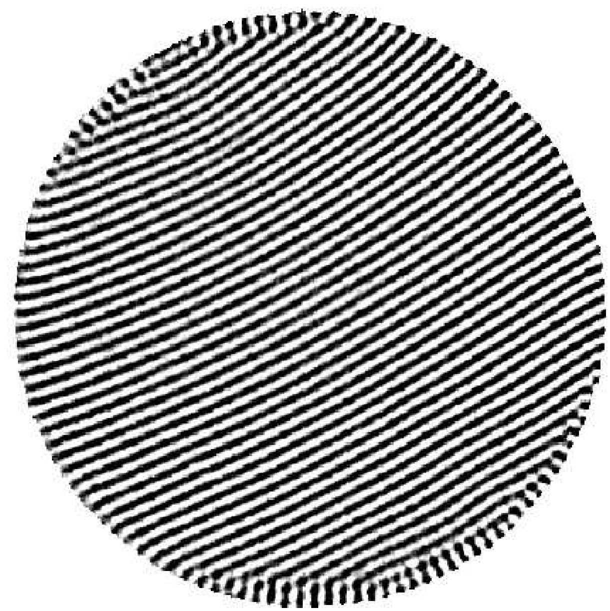
Emperor Angelfish (adult)
Pomocanthus imperator

Tinker's Butterfly
Chaetodon tinkeri

Pattern formation in the laboratory

Rayleigh–Bénard convection



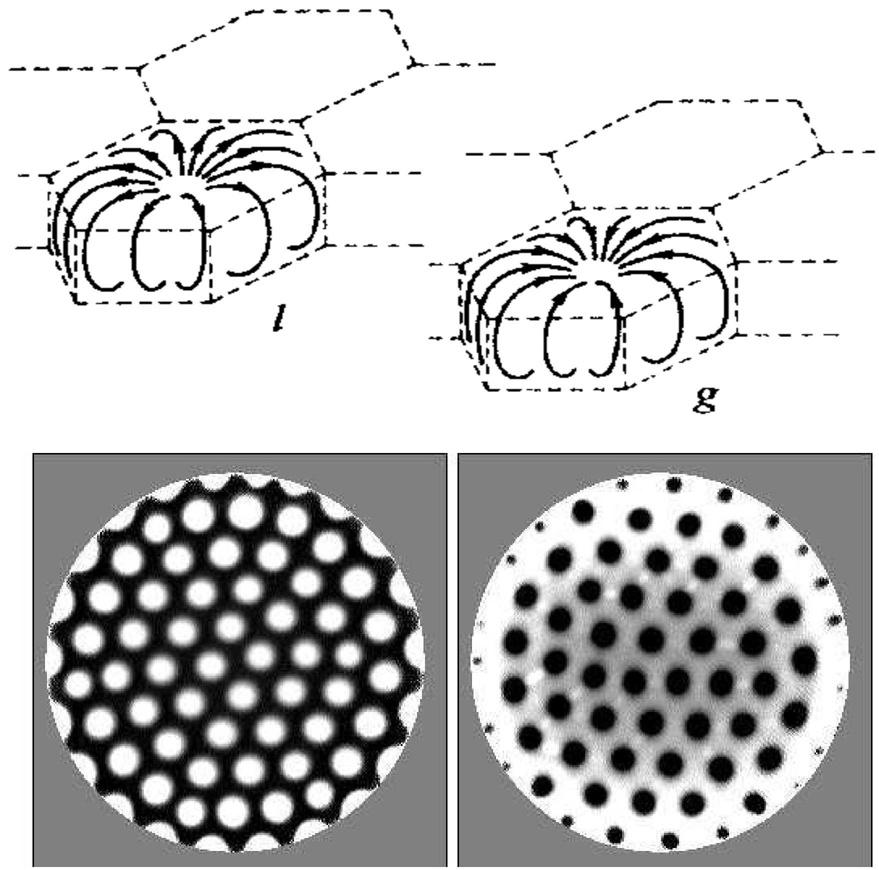


A.V. Getling *Rayleigh–Bénard Convection*. World Scientific (1999)

G. Ahlers (UCSB) - compressed gases, high accuracy

Pattern formation in the laboratory

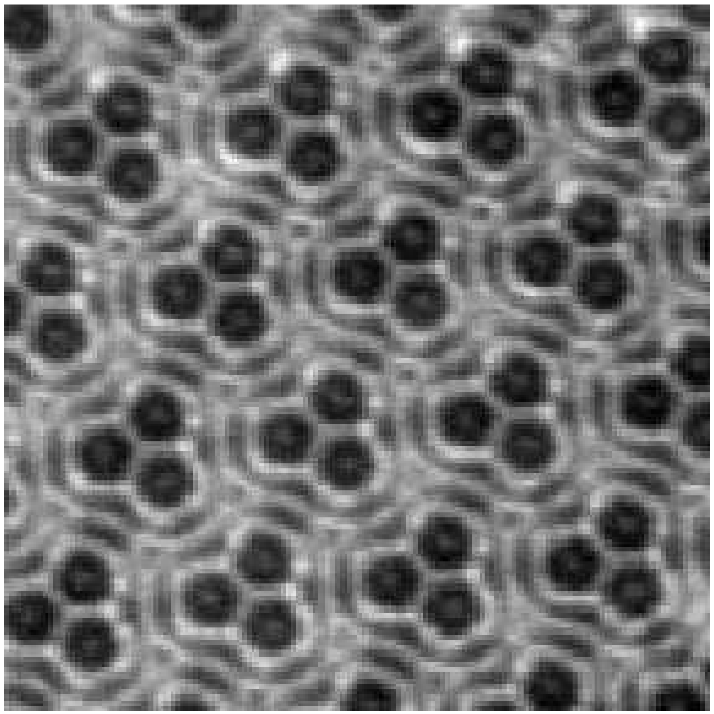
Rayleigh–Bénard convection



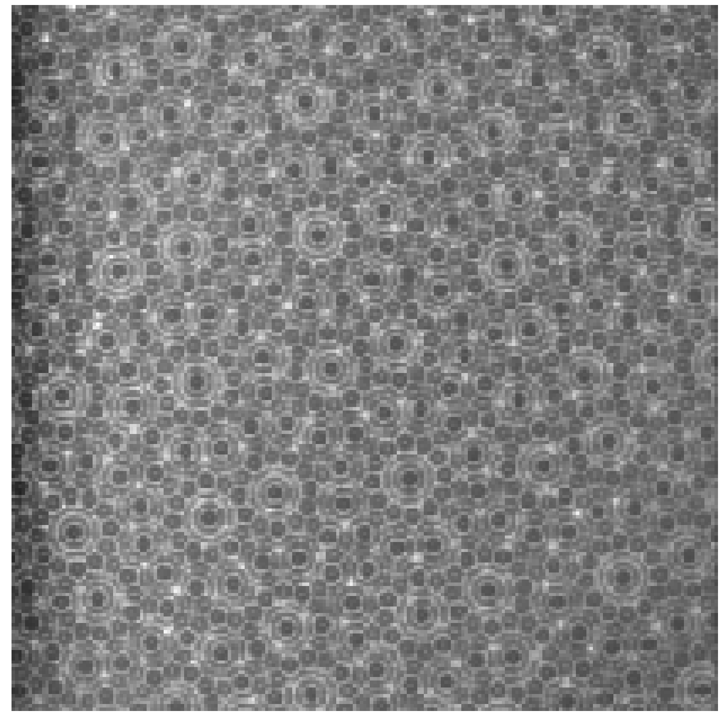
A.V. Getling *Rayleigh–Bénard Convection*. World Scientific (1999)
G. Ahlers (UCSB)

Pattern formation in the laboratory

Faraday waves (2-frequency forcing, harmonic response)



Superlattice (‘down’) triangles



12-fold quasiperiodic pattern

Kudrolli, Pier & Gollub *Physica* D **123**, 99 (1998)
Silber & Proctor *Phys. Rev. Lett.* **81**, 2450 (1998)

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Granular and viscoelastic Faraday experiments

Granular ‘oscillons’

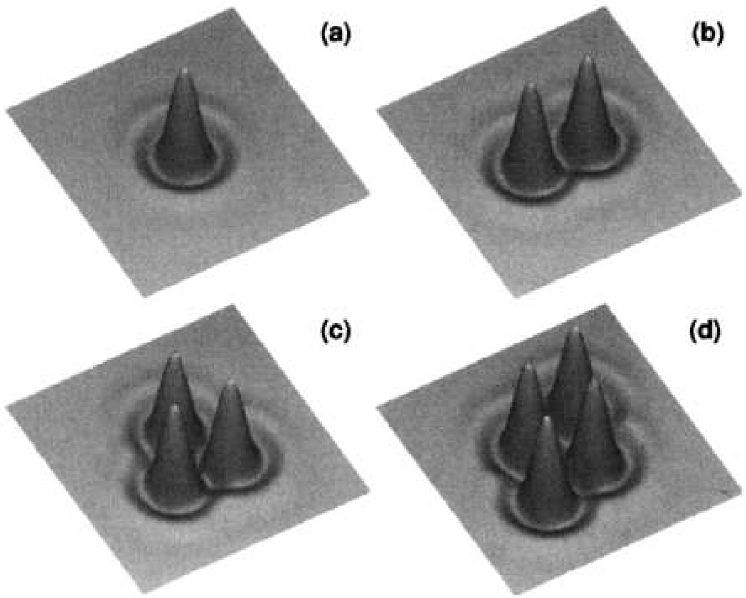
Viscoelastic ’holes’

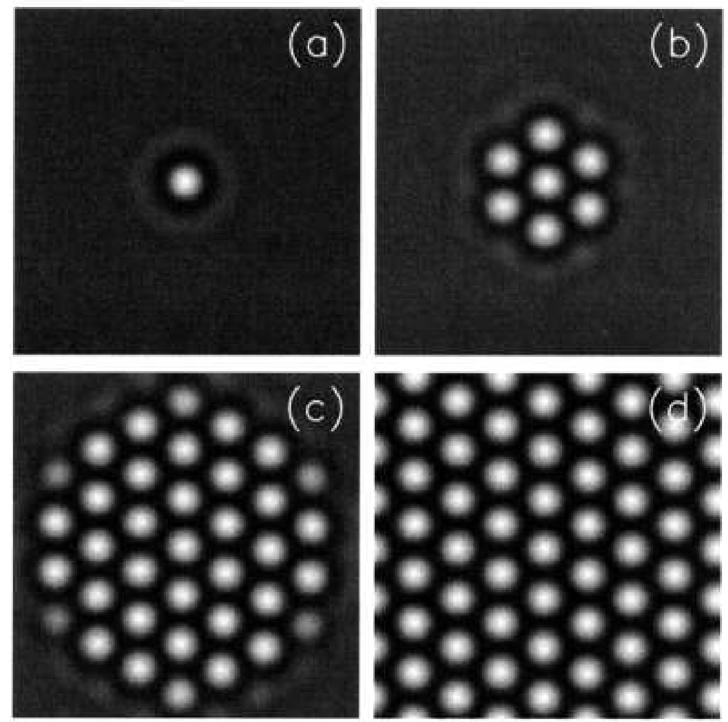
F. Merkt, R.D. Deegan, D. Goldman, E. Rericha, and H.L. Swinney, *Phys. Rev. Lett.* **98**, 184501 (2004)
P.B. Umbanhowar, F. Melo and H.L. Swinney, *Nature* **382**, 793 (1996)

MOVIE

Localised pattern formation

Nonlinear optics

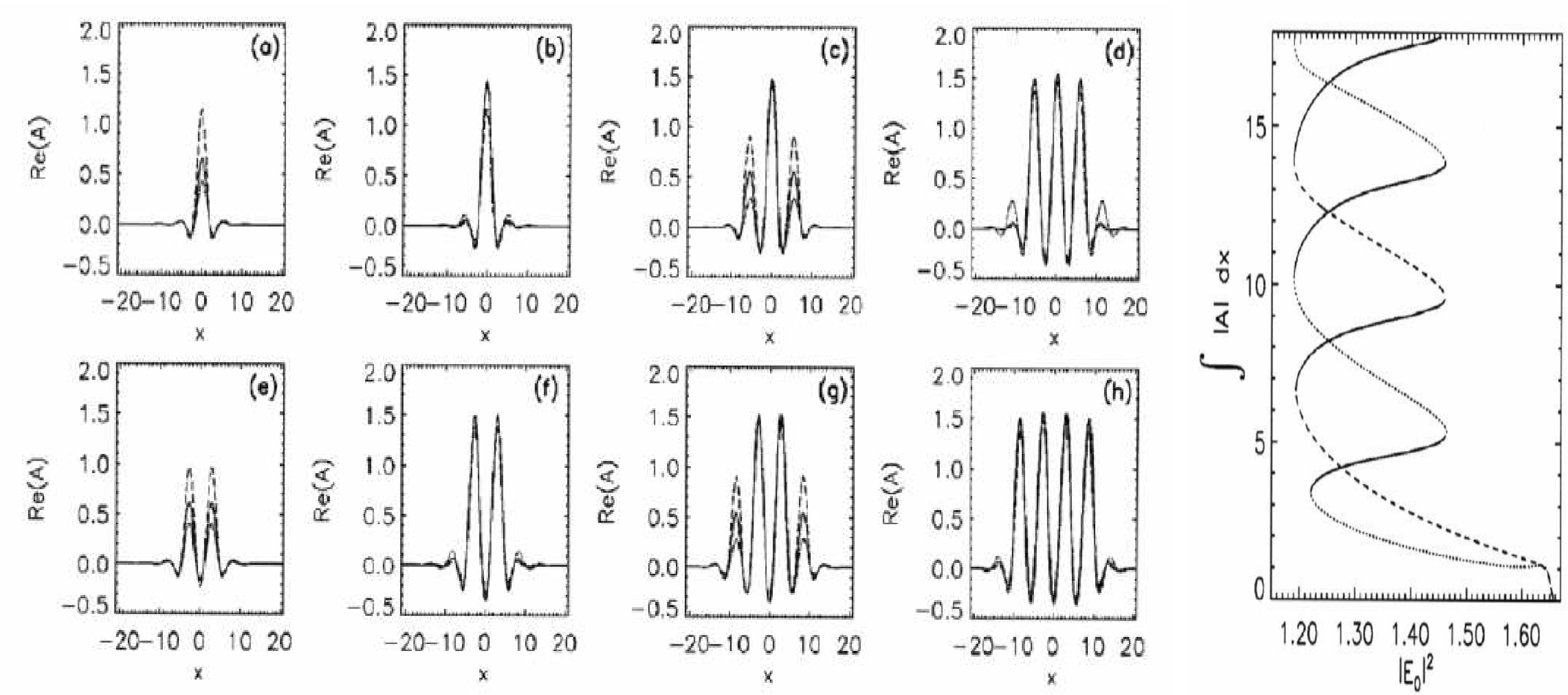




W. Firth (Strathclyde); J. M. McSloy et al. *Phys. Rev.* E **66**, 046606 (2002) / S. Residori (INLN)
N. Akhmediev and A. Ankiewicz (eds) *Dissipative Solitons*. Lect. Notes in Physics **661**. Springer, Berlin (2005)

Localised pattern formation

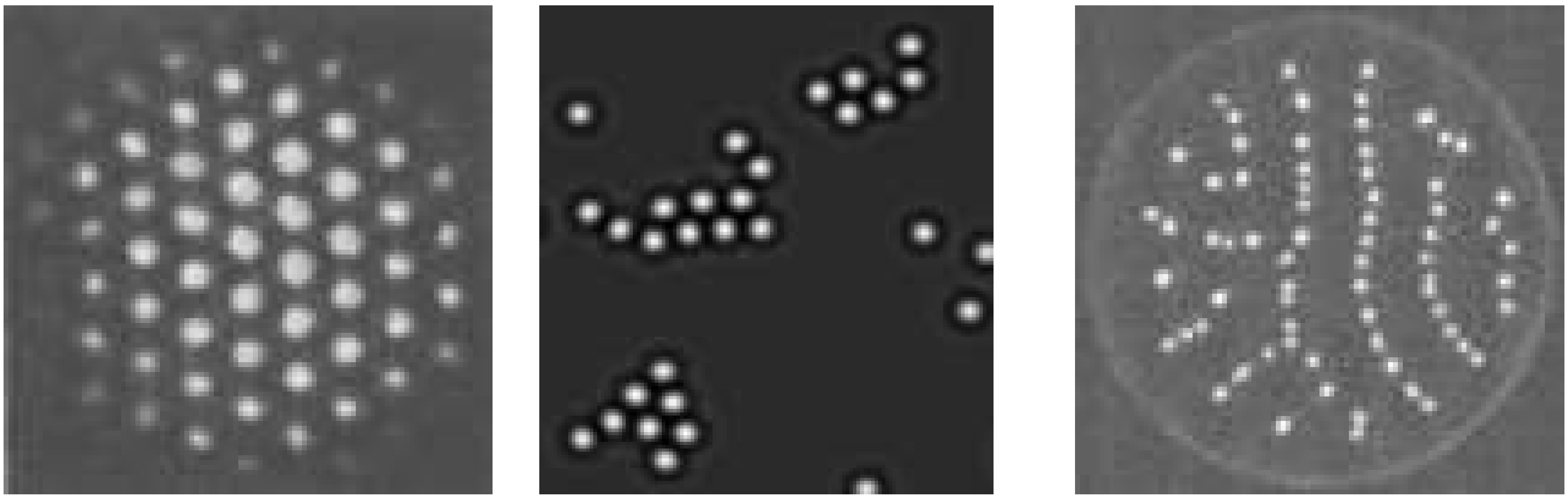
Nonlinear optics



J. M. McSloy et al. *Phys. Rev.* E **66**, 046606 (2002)

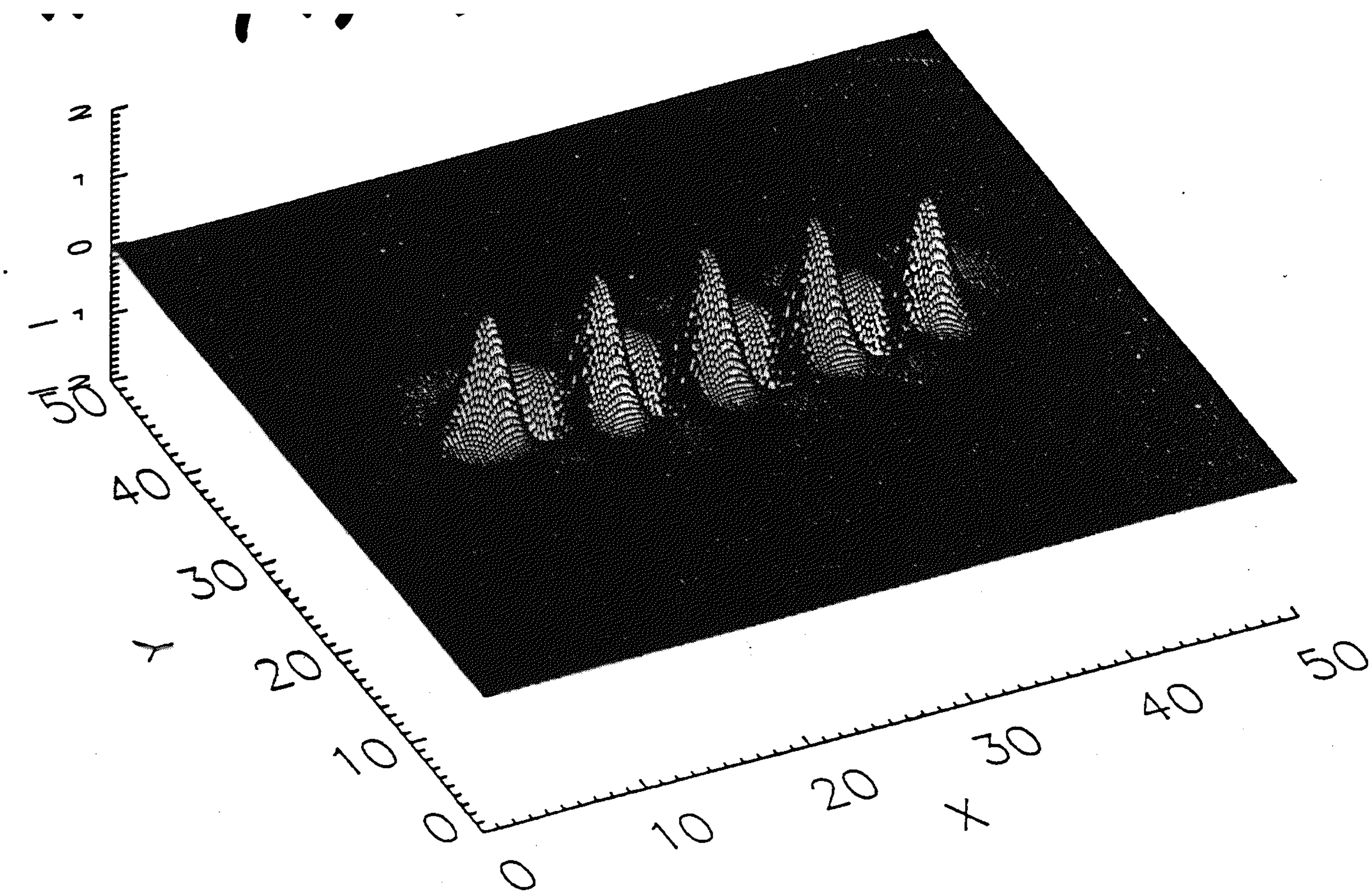
Localised pattern formation

Dielectric gas discharge



H.-G. Purwins (Münster)

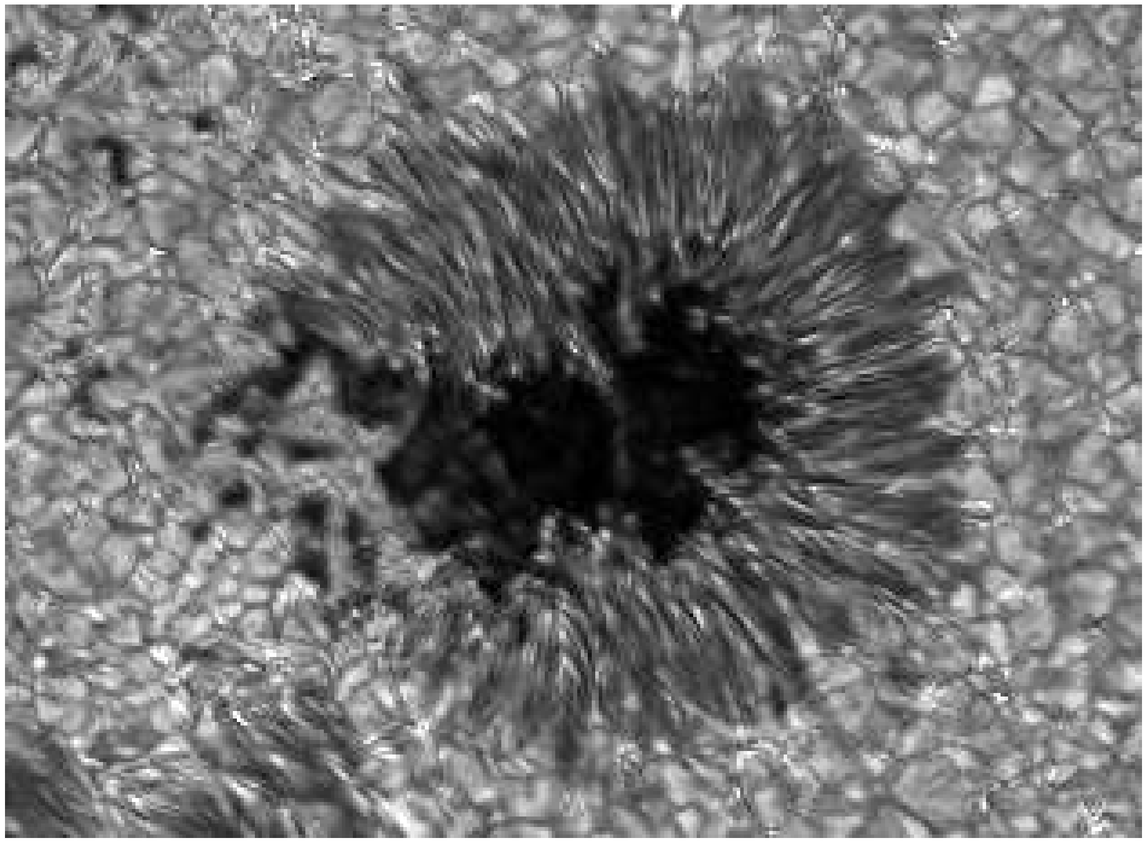
Localised stripes



H. Sakaguchi & H.R. Brand, *Physica* D **97**, 274–285 (1996)

Localised patterns - umbral dots

Bright points persist within the central umbral of a sunspot:



(AR 10786 imaged in the G Band, 8 June 2005)

Localised pattern formation

Numerical simulations: Rayleigh–Bénard convection with a vertical magnetic field

$R = 100000$, $Q = 1600$

$\sigma = 0.1$, $\zeta = 0.2$

stress-free, T fixed (lower)

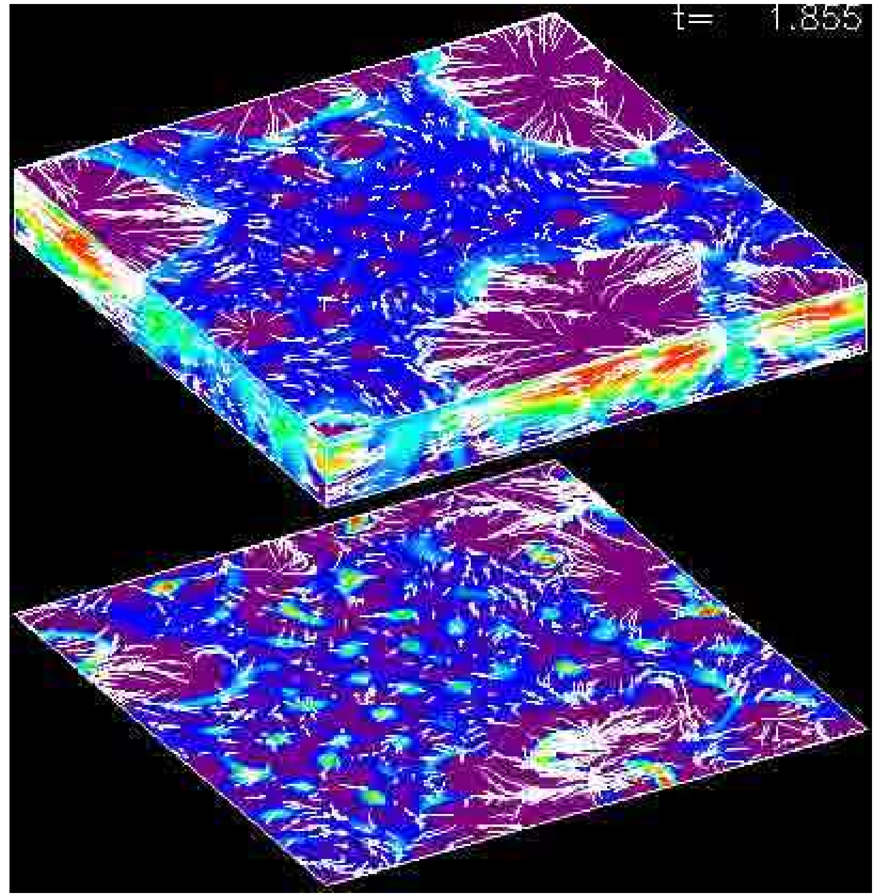
radiative b.c. (upper)

$8 \times 8 \times 1$ stratified layer

density contrast approx 11.

blue = strong field

purple = weak field



A.M. Rucklidge, N.O. Weiss, D.P. Brownjohn, P.C. Matthews & M.R.E. Proctor, *J. Fluid Mech.* **419**, 283–323 (2000)

Simple models for pattern formation

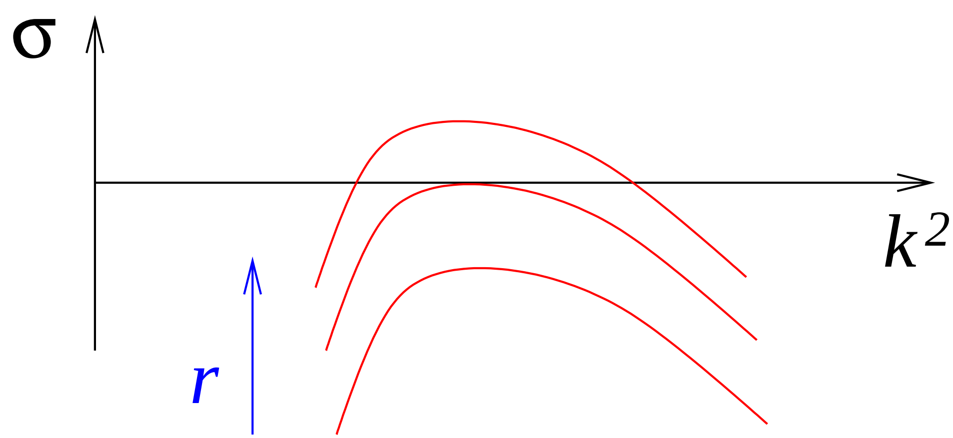
Suppose system state described by scalar variable $u(x,t)$, $x \in \mathbb{R}$.

'Turing instability' from trivial state to patterned state occurs.

Assume

- translational symmetry $x \to x + \delta$
- reflectional symmetry $x \to -x$
- unbounded domain: $-\infty < x < \infty$
  replaced with periodic boundaries (PBC) in practice

Eigenfunctions: plane waves $\mathrm{e}^{\mathrm{i}kx}$; steady state instability at $|k| = 1$:



Growth rate: $\sigma = r - (1 - k^2)^2$

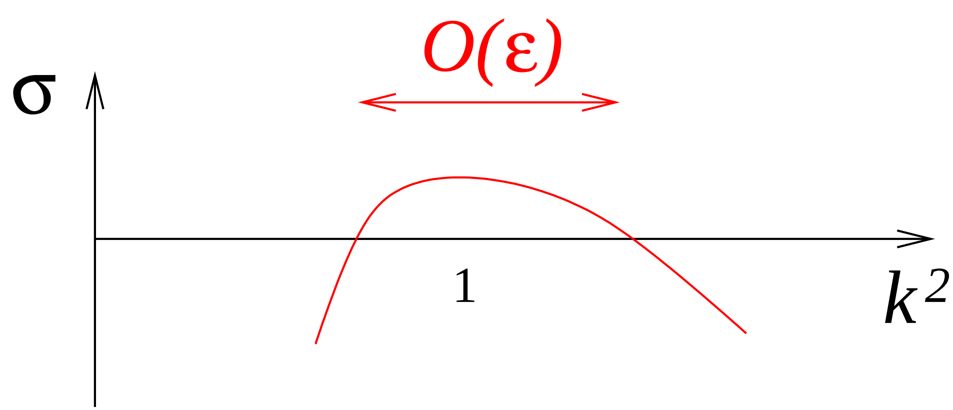
Swift–Hohenberg equation

$$\partial_t u = [r - (1 + \partial_{xx}^2)^2]u + N(u)$$

Nonlinearities:

- $-u^3$ – supercritical (‘forwards’) bifurcation
- $+su^2 - u^3$, $+su^3 - u^5$ – subcritical (‘backwards’) bifurcations

Near bifurcation point, $r = \varepsilon^2\mu$, there is a band of unstable modes:



- on $\mathbb{R}$ no centre manifold reduction is formally possible (BAD)
- but on a finite domain with PBC we have only a discrete set of modes and dynamics on centre manifold is finite-dimensional (GOOD)

Ginzburg–Landau approach

Suppose the pattern-forming instability is weakly subcritical and odd-symmetric: $w \to -w$.

Model equation: $w_t = [r - (1 + \partial_{xx}^2)^2]w + sw^3 - w^5$

Asymptotic scalings for a weakly subcritical bifurcation:

$$w(x,t) \quad = \quad \varepsilon\left(A(X,T)\mathrm{e}^{\mathrm{i}x} + c.c.\right) + \varepsilon^2 w_2 + \cdots$$

$s = \varepsilon^2 \hat{s} \qquad X = \varepsilon^2 x, \qquad T = \varepsilon^4 t, \qquad r = \varepsilon^4 \hat{r}$

At $O(\varepsilon^5)$ we deduce

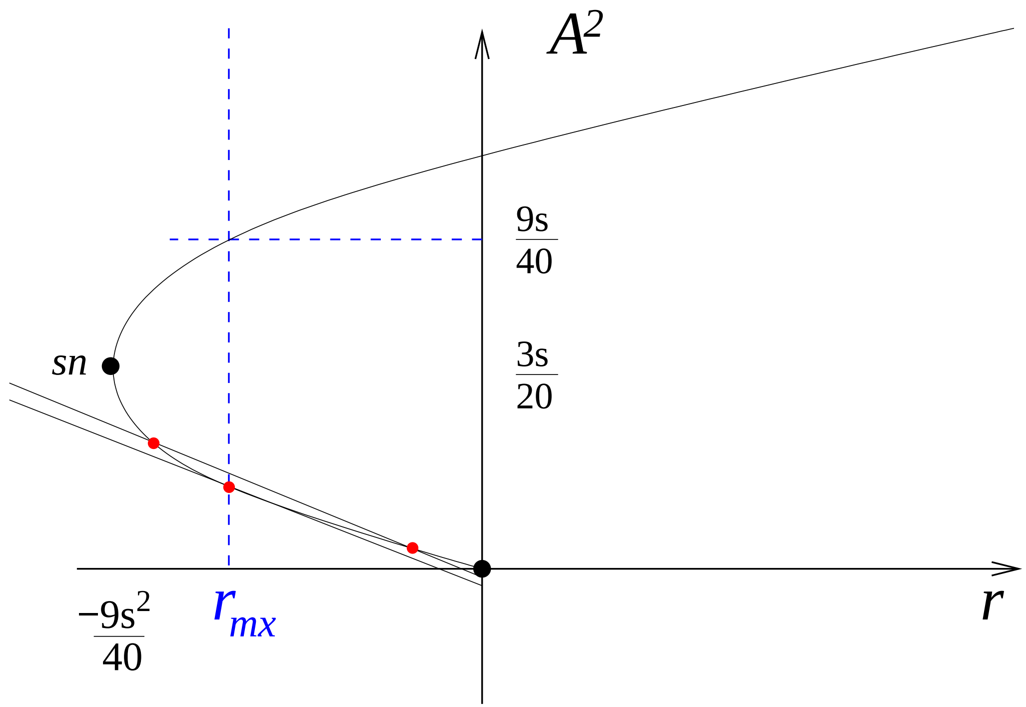
$$\boxed{A_T = \hat{r}A + 4A_{XX} + 3\hat{s}A|A|^2 - 10A|A|^4}$$

Now drop hats, and examine instability to perturbations $\mathrm{e}^{\mathrm{i}\ell X}$.

Modulational instability

Domain size $L = 2\pi/\ell$. No instability if $L < 8\pi\sqrt{10}/(3s)$.

In a large domain, instability occurs within $O(\ell^2)$ of $r = 0$ and below $r = r_{sn}$:

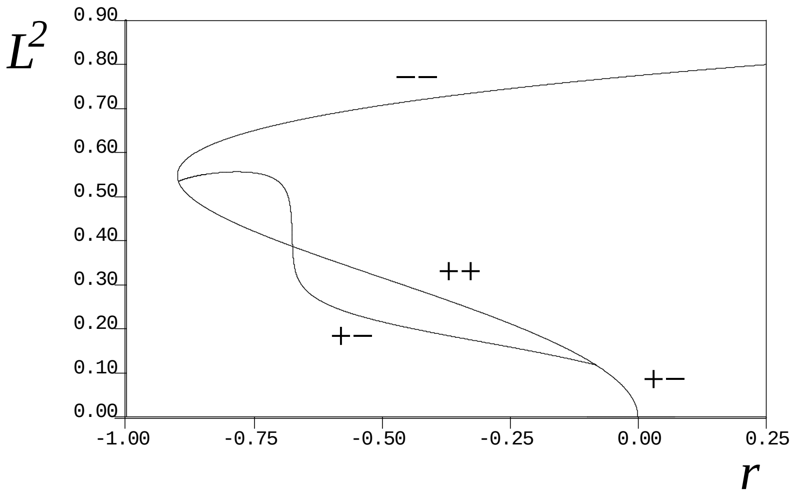


First integral: $E = \frac{r}{2}A^2 + 2A_X^2 + \frac{3s}{4}A^4 - \frac{5}{3}A^6$

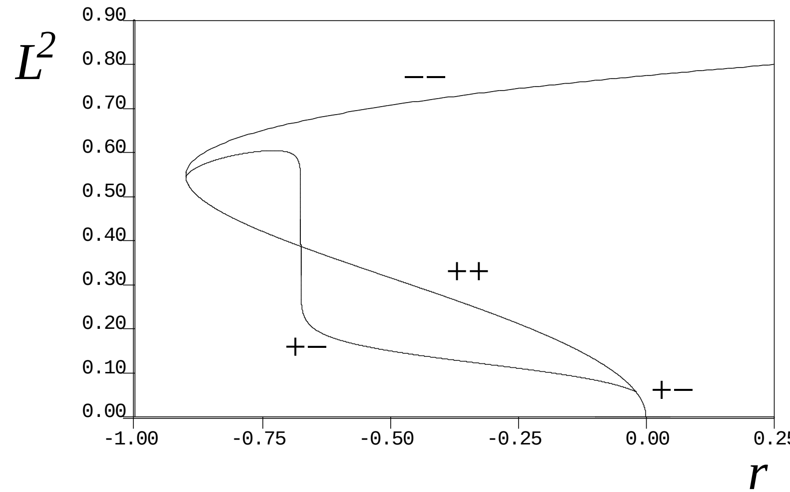
'Maxwell point': at $r = r_{mx}$: $E|_{A=0} = E|_{A=A_0^+}$ - stationary fronts exist.

Modulational instability

Branch of modulated pattern connects these two bifurcation points:



$L = 10\pi$

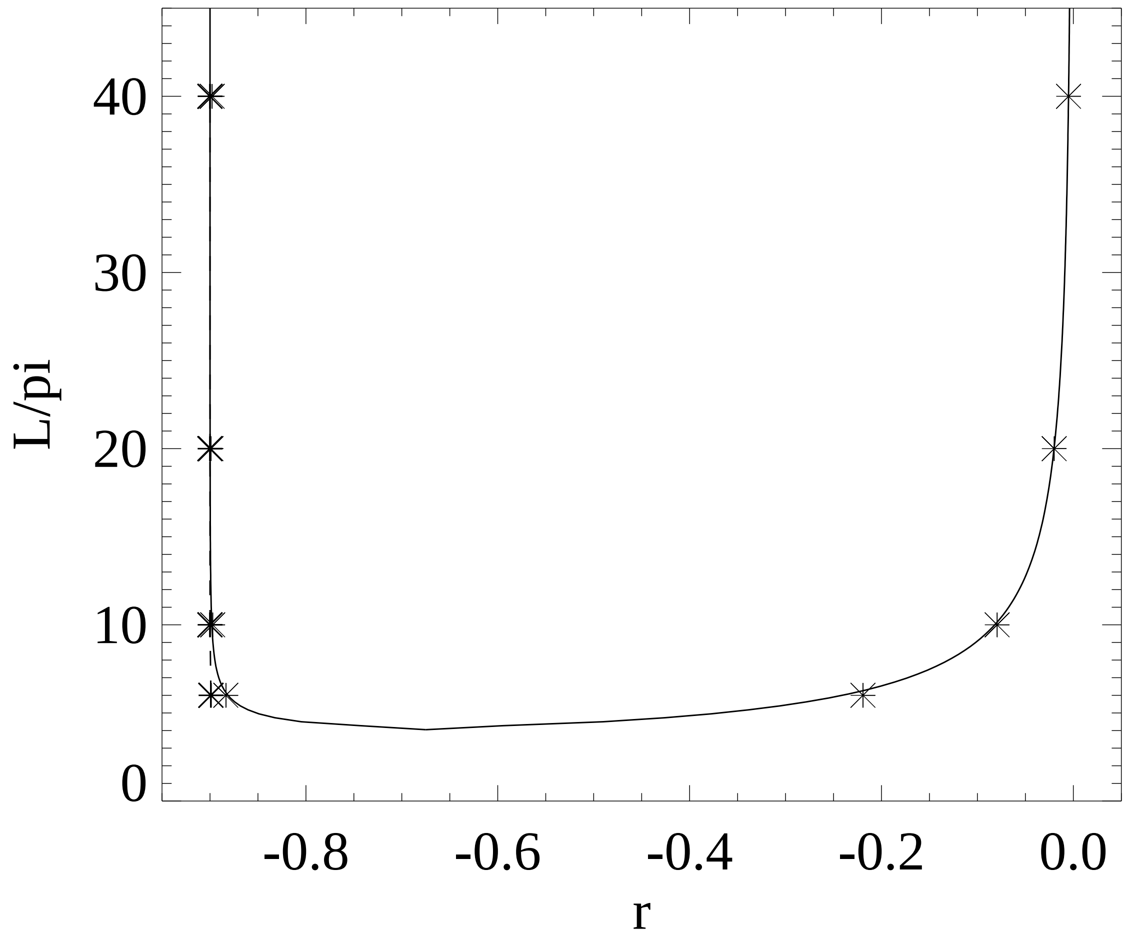


$L = 20\pi$

'Maxwell point': $E|_{A=0} = E|_{A=A_0^+}$ when $r = -0.675$.

Modulational instability in a finite domain

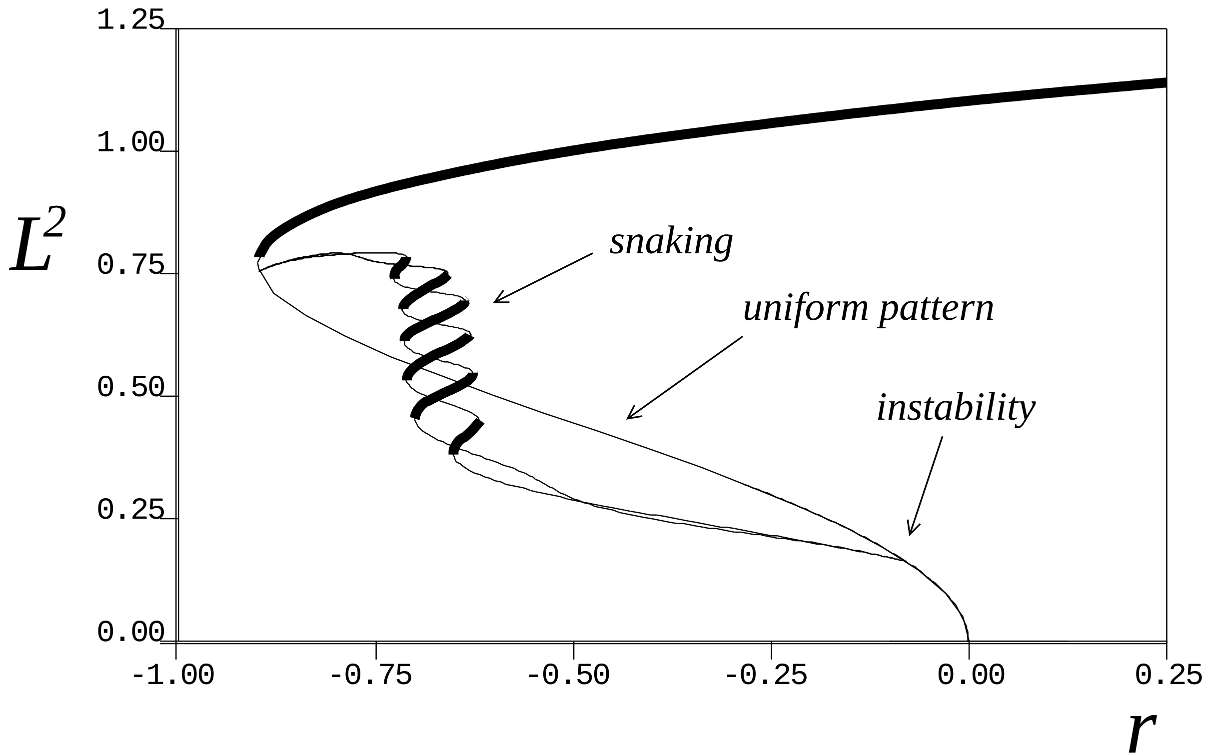
Ginzburg–Landau approach agrees well with solving Swift–Hohenberg equation $(*)$. Finite domain $0 \leq x \leq L$, fixing $s = 2.0$:



‘Snaking’ in a finite domain

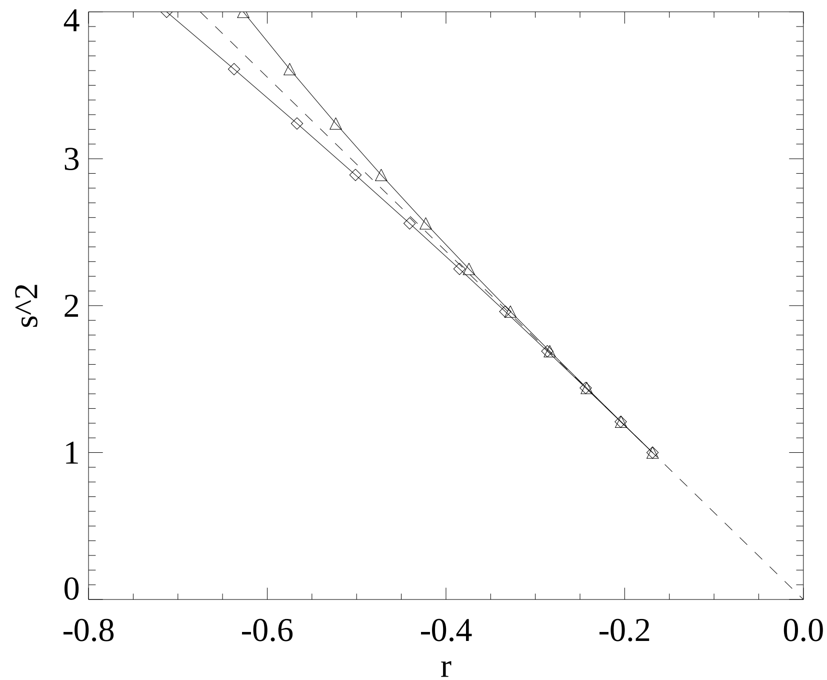
Return to Swift–Hohenberg equation: $w_t = [r - (1 + \partial_{xx}^2)^2]w + sw^3 - w^5$.

Fix $s = 2.0$ and domain size $L = 10\pi$ (PBC).



Snaking region’ is exponentially small in s

$$w_t = [r - (1 + \partial^2_{xx})^2]w + sw^3 - w^5$$

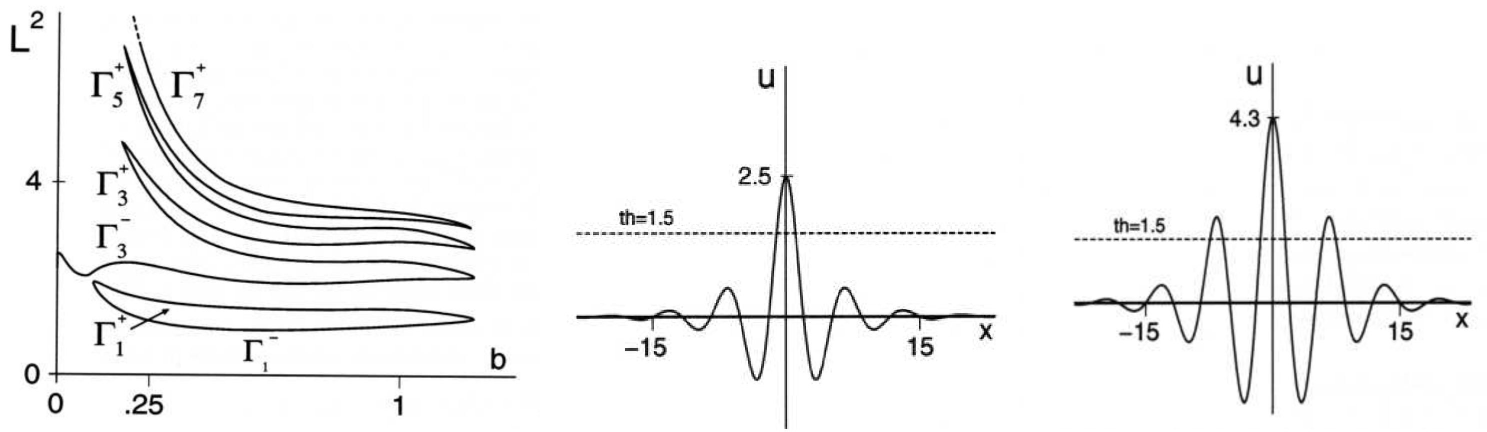


As in Sakaguchi & Brand *Physica* D **97**, 274 (1996)

- many authors have glossed over details
- quadratic-cubic case done by G. Kozyreff & S.J. Chapman *Phys. Rev. Lett.* **97**, 044502 (2006) – ...and the next talk

Homoclinic snaking: examples

- (i) Neuroscience:



$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} w(x,y) f(u(y,t))dy$$

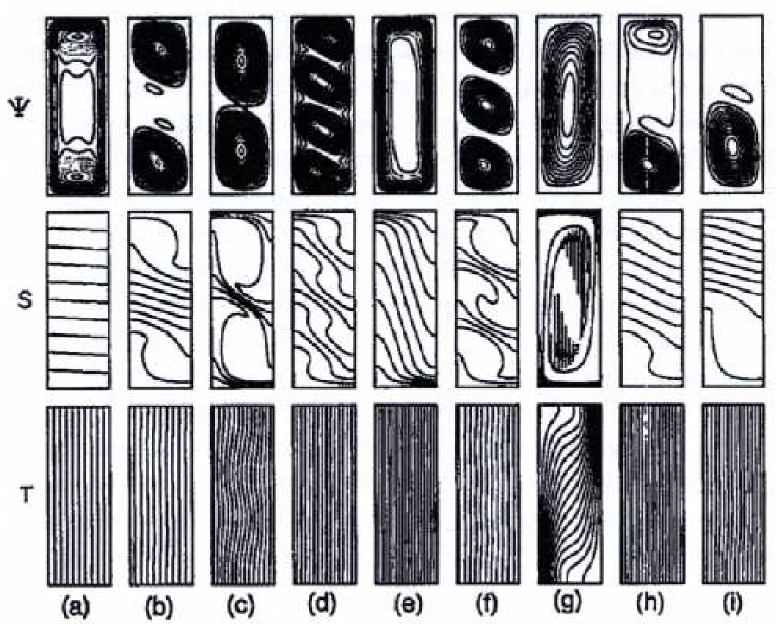
$$w(x) = e^{-b|x|}(b\sin|x| + \cos x), \qquad f(u) = 2e^{-r/(u-h)^2} H(u-h)$$

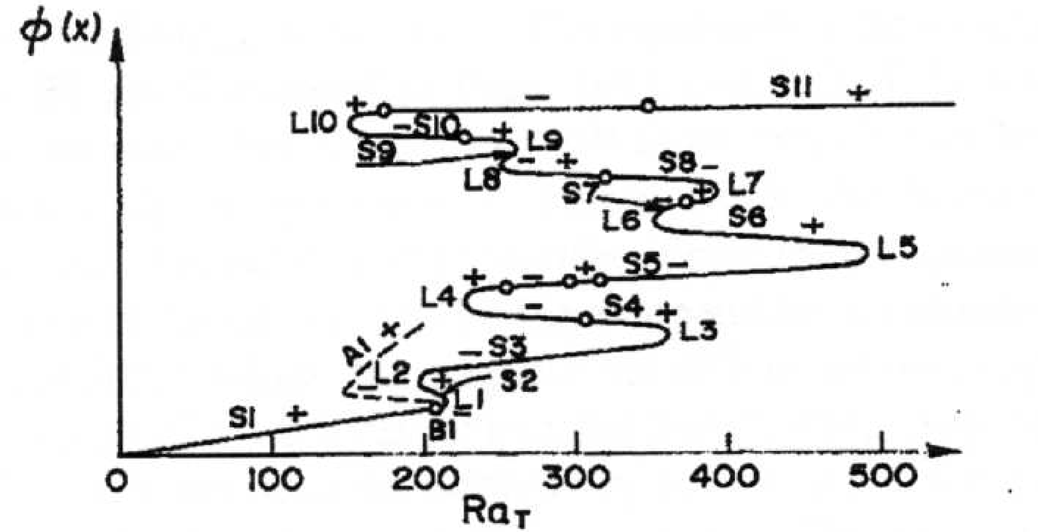
$$\Rightarrow \mathcal{F}[u_{xxxx} - 2(b^2-1)u_{xx} + (b^2+1)^2 u - 4b(b^2+1)f(u)] = 0$$

C.R. Laing, W.C. Troy, B. Gutkin & G.B. Ermentrout *SIAM J. Appl. Math.* **63**, 62 (2002)

Homoclinic snaking: examples

- (ii) Double-diffusive, sidewall-heated, convection

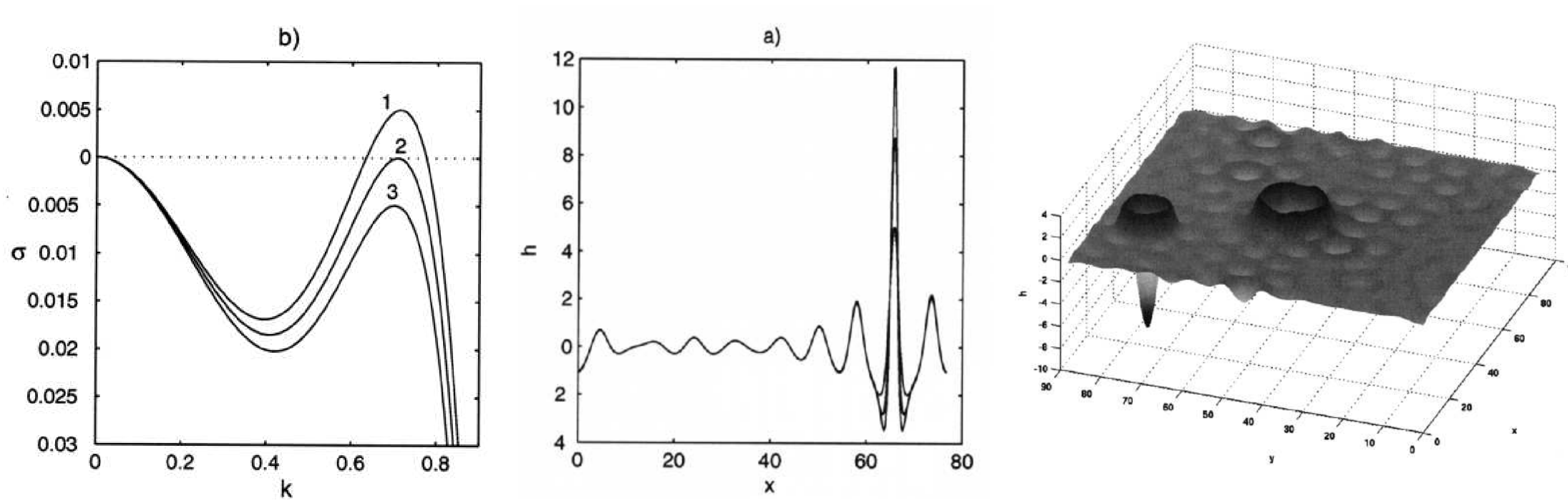




N. Tsitverblit & E. Kit, *Phys. Fluids* **5**, 1062 (1993)

Homoclinic snaking: examples

- (iii) Thin films; height $h(x, y, t)$

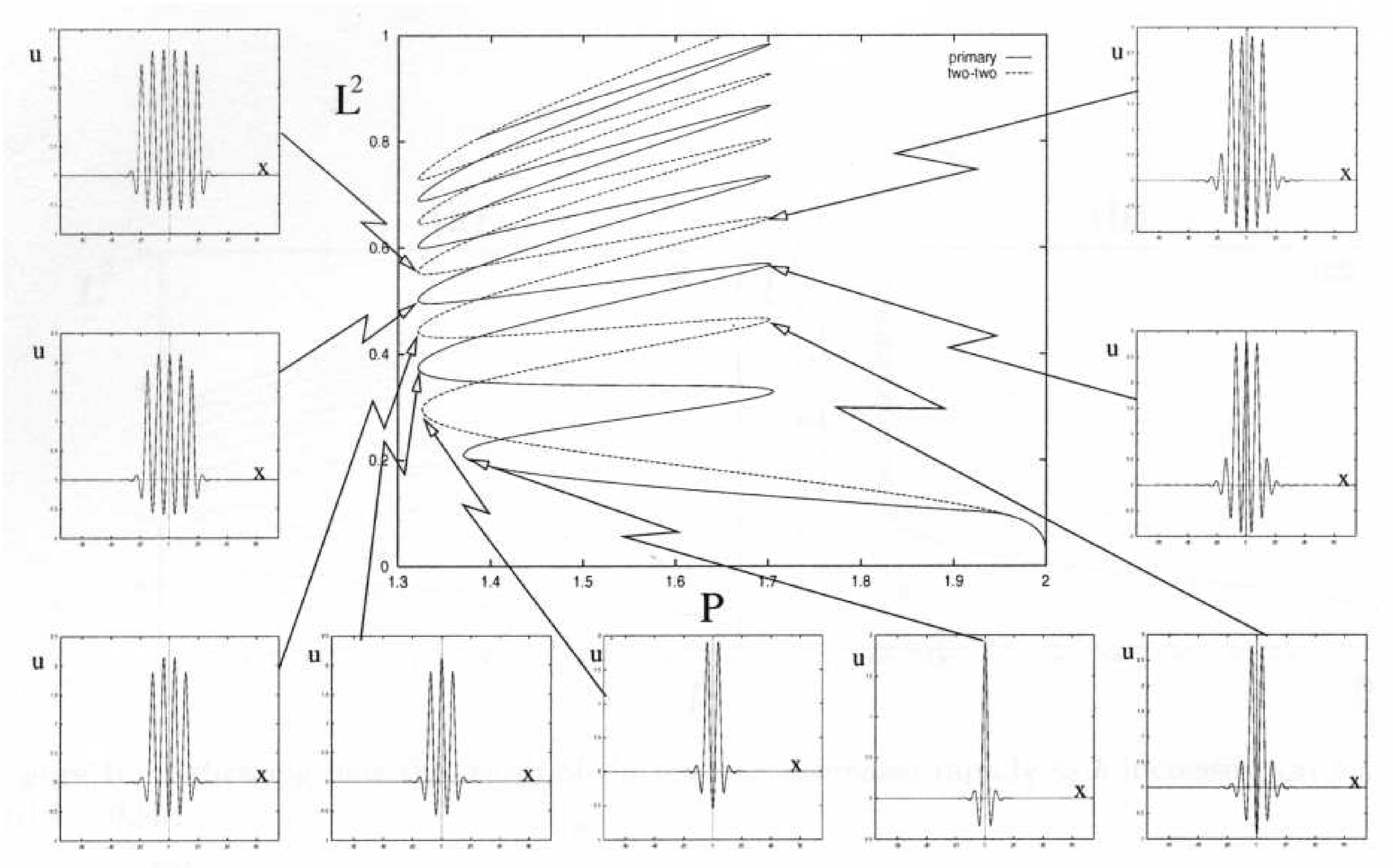


- Large-scale mode (i.e. dispersion relation is not the same as for S–H)
- localised solutions (incidental problem here: blow-up in finite time)

$$h_t = g\nabla^2 h + \nabla^4 h + \nabla^6 h + \nabla^2[h\nabla^2 h + p(\nabla h)^2 + qh^2]$$

A.A. Golovin, S.H. Davis & P.W. Voorhees *Phys. Rev.* E **68**, 056203 (2003)

Homoclinic snaking



$$0 = u + Pu_{xx} + u_{xxxx} - u^2 + 0.3u^3$$

P.D. Woods and A.R. Champneys *Physica* D **129**, 147 (1999)

Spatial dynamics

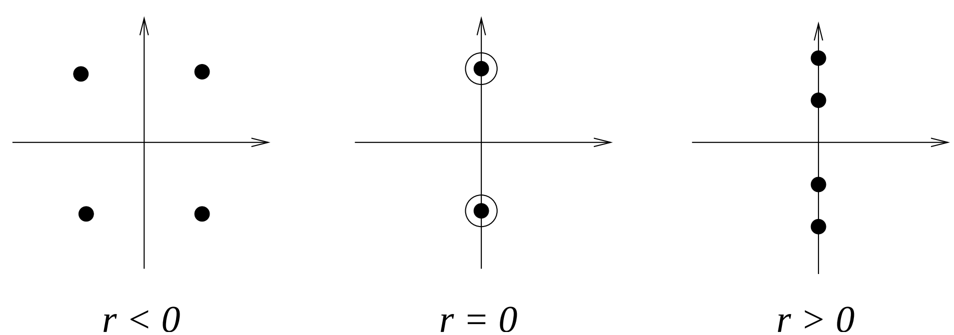
Think of x as 'time' variable and look for steady states:

$$0 = (r-1)u - 2u_{xx} - u_{xxxx} - N(u)$$

4D reversible dynamical system: $v = u_x$, $w = u_{xx}$, $z = u_{xxx}$

$$\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r-1 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} - N(u, v, w, z)$$

Bifurcation parameter $\mu \propto r$. Eigenvalues at bifurcation point $r = 0$ are $\pm \mathrm{i}$, twice each:



'Hamiltonian–Hopf' bifurcation

Spatial dynamics: normal form

- change variables to put linear part in Jordan normal form.
- then use (near-identity) nonlinear transformations to tidy up nonlinear terms order by order
- normal form symmetry: nonlinear part of the ODEs up to order N commute with $\exp(sL^T)$ (Elphick et al. 1987)

New variables: $A, B \in \mathbb{C}$. Linear part is now $L \equiv \begin{pmatrix} \mathrm{i}\omega & I \\ 0 & \mathrm{i}\omega \end{pmatrix}$

Normal form is

$$\begin{aligned} \frac{dA}{dx} &= \mathrm{i}\omega A + B + \mathrm{i}AP\left(\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + R_A \\ \frac{dB}{dx} &= \mathrm{i}\omega B + \mathrm{i}BP\left((\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + AQ\left(\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + R_B \end{aligned}$$

G. Iooss & M.C. Pérouème *J. Diff. Eq.* **102**, (1993)

Spatial dynamics: normal form

$$\frac{dA}{dx} = \mathrm{i}\omega A + B + \mathrm{i}AP\left(\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + R_A$$

$$\frac{dB}{dx} = \mathrm{i}\omega B + \mathrm{i}BP\left((\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + AQ\left(\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)\right) + R_B$$

Take only lowest order terms in the polynomials P and Q:

$$P(\mu, v, w) = p_1\mu + p_2 v + p_3 w$$
$$Q(\mu, v, w) = -q_1\mu + q_2 v + q_3 w + q_4 v^2$$

p_j, q_j real constants. Can derive a single equation for the dynamics of $y = |A|^2$:

$$\frac{1}{4}\left(\frac{dy}{dx}\right)^2 = \frac{1}{3}q_4 y^4 + \frac{1}{2}q_2 y^3 + (q_3 K - q_1\mu)y^2 + Hy - K^2 \equiv -F(y)$$

'Ball rolling in a 1D potential $F(y)$.'

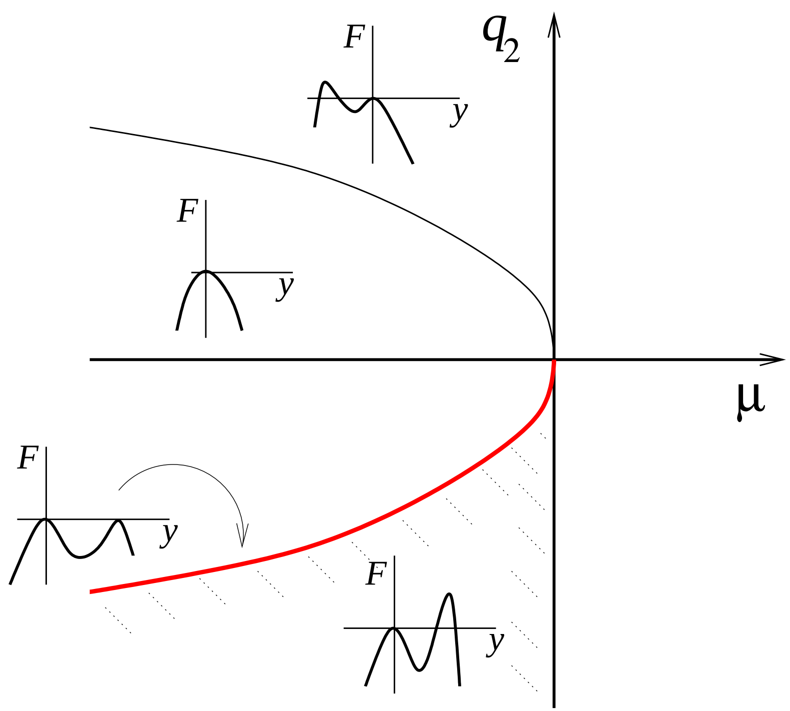
Spatial dynamics: normal form

q_2 is the coefficient of $A|A|^2$ in the dB/dx equation:

- $q_2 < 0$: subcritical bifurcation
- $q_2 > 0$: supercritical bifurcation

Potential ($y > 0$): $$F(y) = y^2 \left(q_1 \mu - \frac{1}{3} q_4 y^2 - \frac{1}{2} q_2 y \right)$$

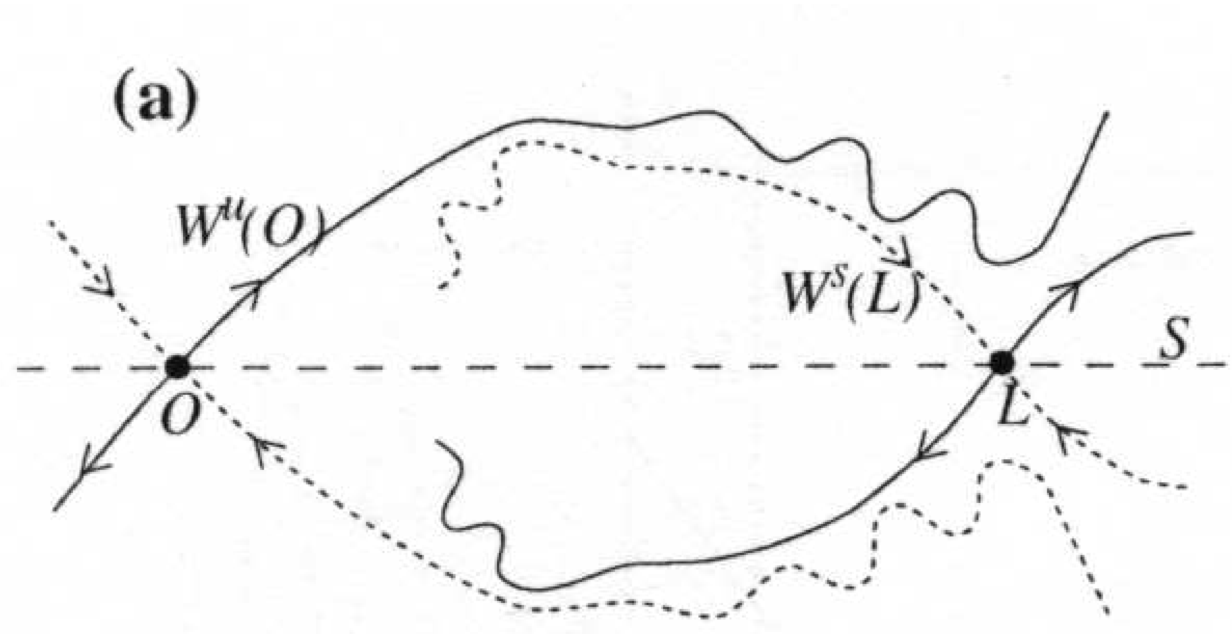
For $q_1, q_4 > 0$ the potential looks like



Red curve: heteroclinic connection between zero and the spatially periodic state - ‘Maxwell point’.

Homoclinic tangle

Interpret as dynamics of a map on a Poincaré section:



Homoclinic tangle

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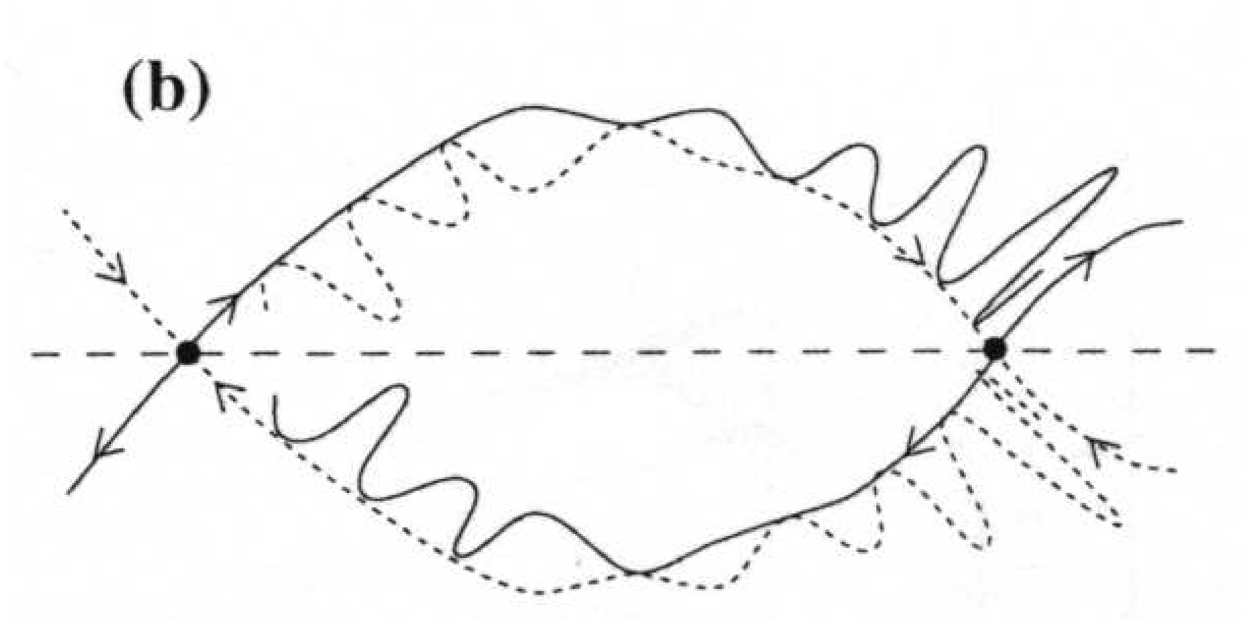

Homoclinic tangle

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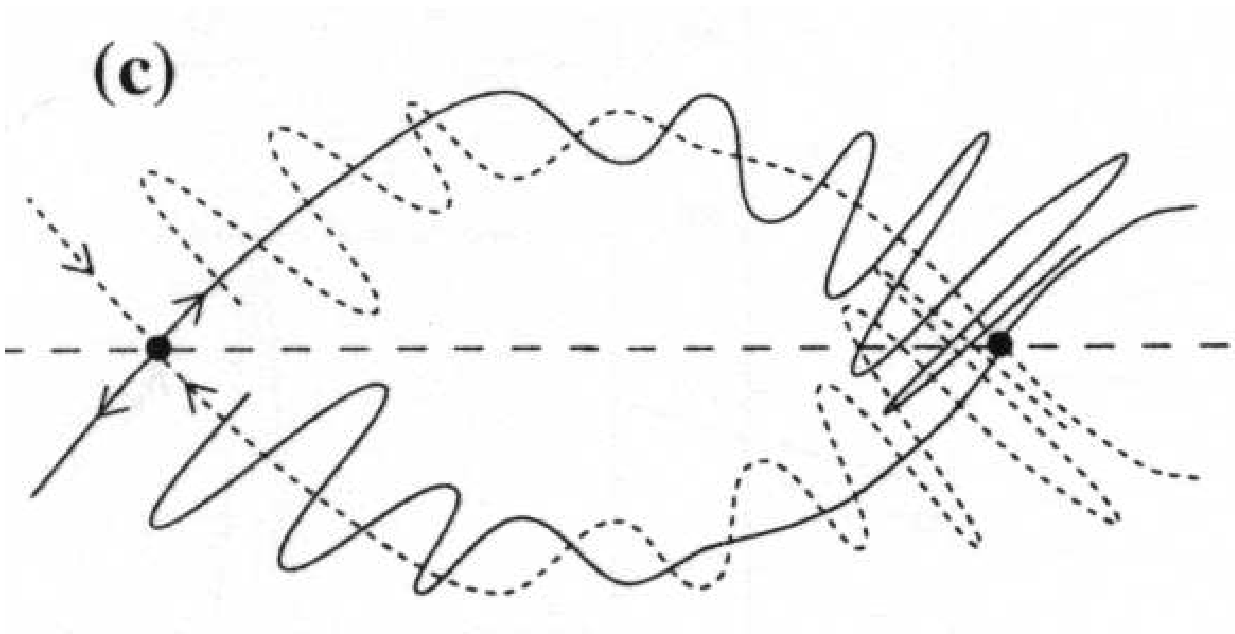

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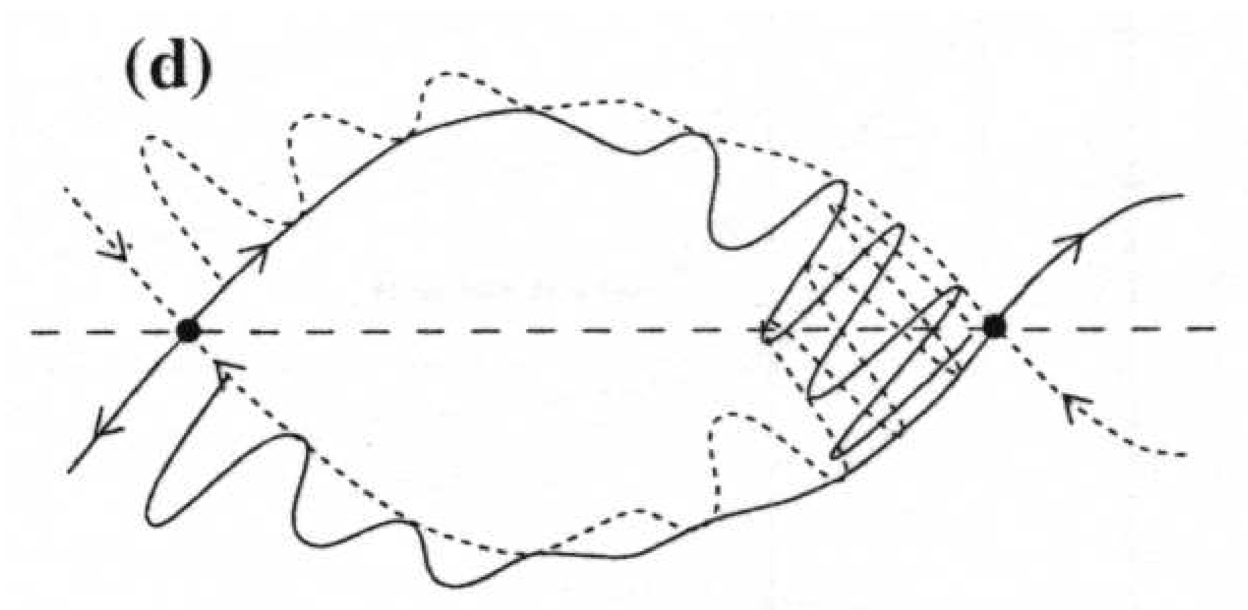

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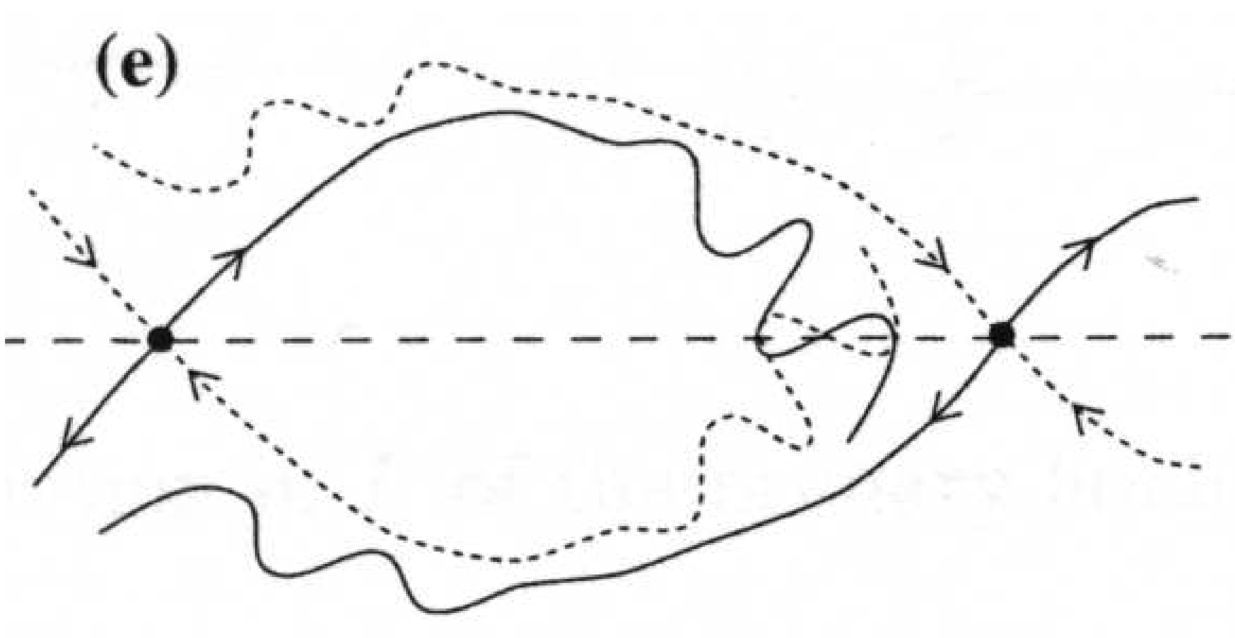

Homoclinic tangle

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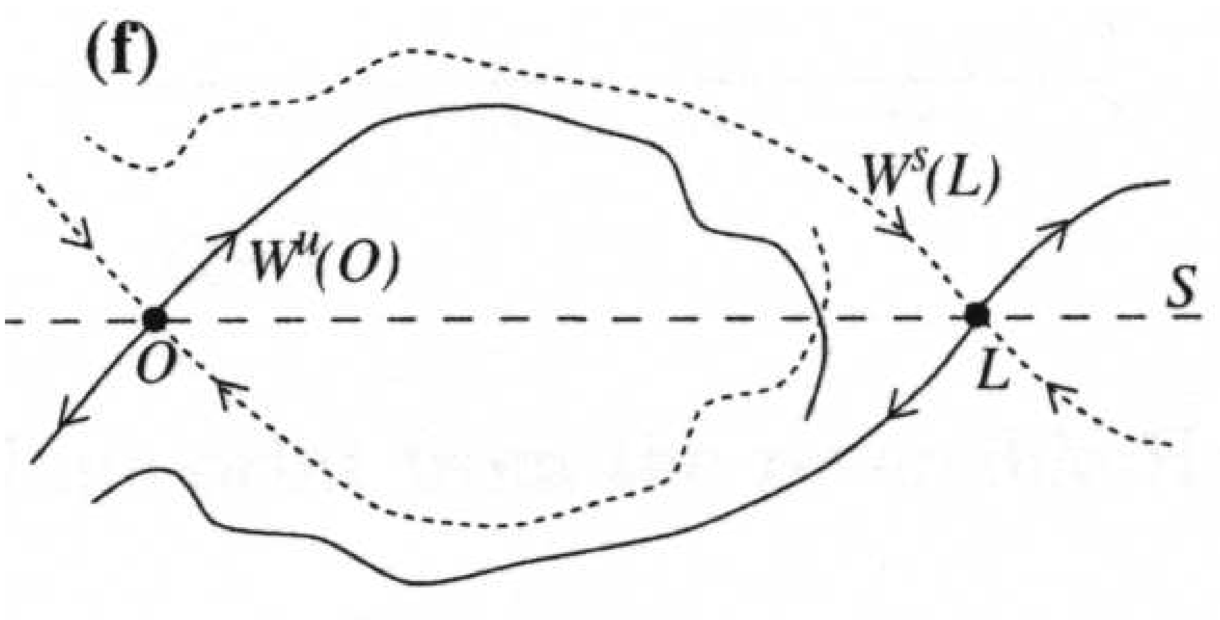
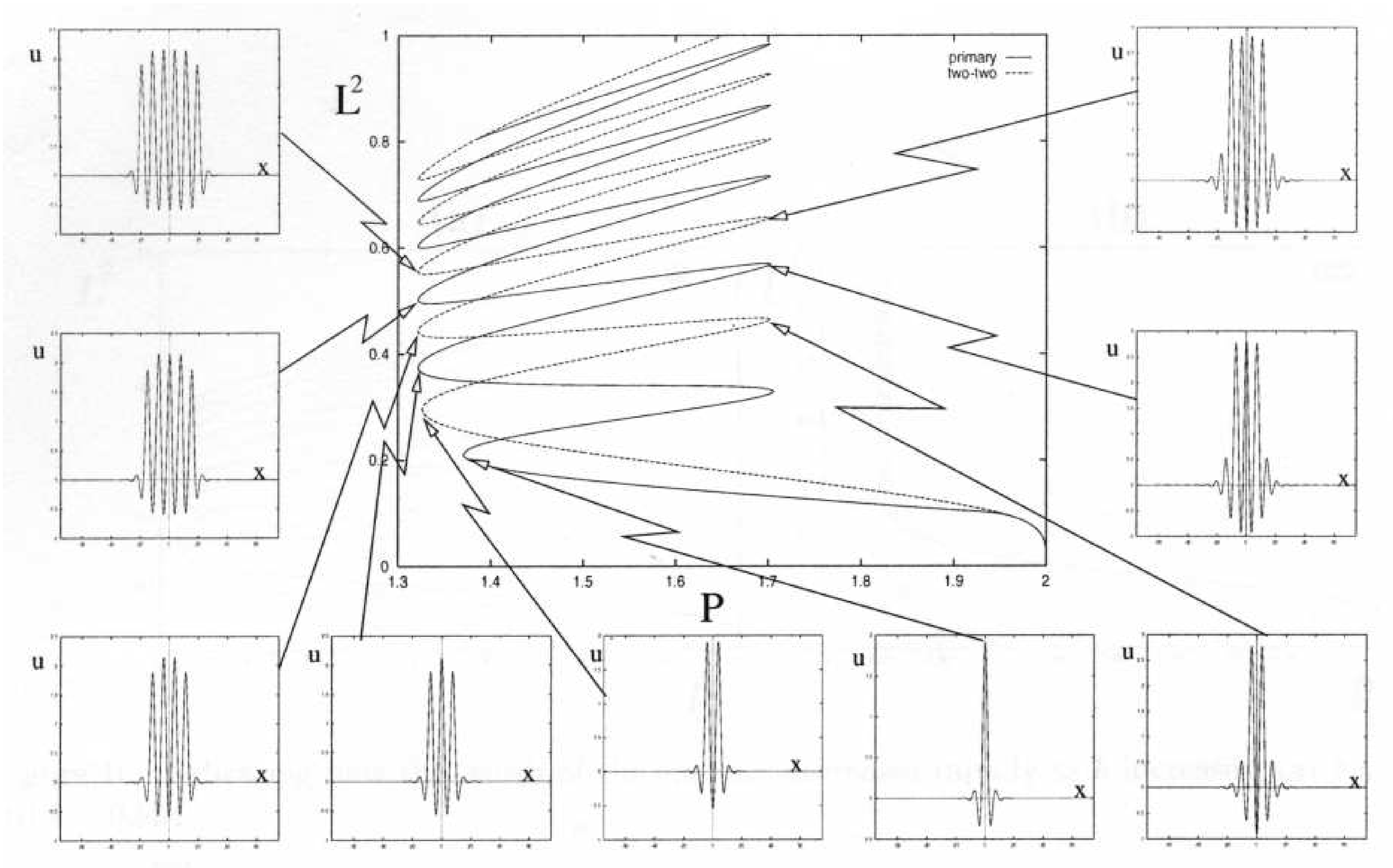


Homoclinic snaking

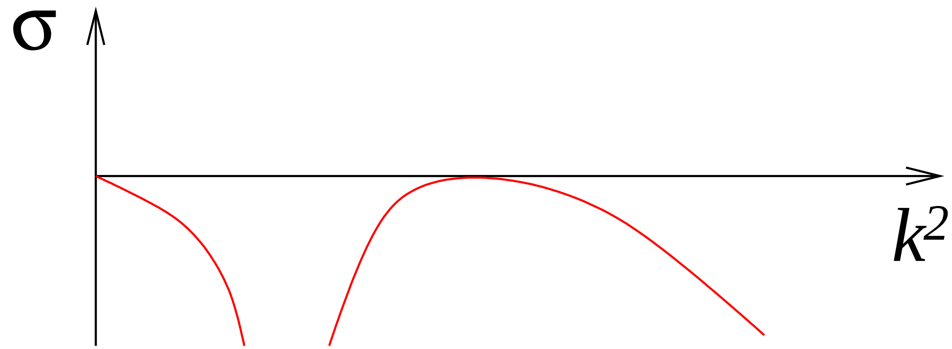


$$0 = u + Pu_{xx} + u_{xxxx} - u^2 + 0.3u^3$$

P.D. Woods and A.R. Champneys *Physica* D **129**, 147 (1999)

Large scale modes

Suppose that a neutral long-wavelength mode exists in addition to the pattern forming instability:



E.g. conservation law: $\partial_t \rho + \partial_x f(\rho) = 0$

+ suitable boundary conditions $\Rightarrow$ $\frac{d}{dt}\int_0^L u \, \mathrm{d}x = 0.$

In a reflection-symmetric problem: $x \to -x$, $u \to u$

$\Rightarrow$ linear terms in $\partial_x f(u)$ are $u_{xx}, u_{xxxx}, \ldots$

P.C. Matthews & S.M. Cox *Nonlinearity* **13**, 1293–1320 (2000)

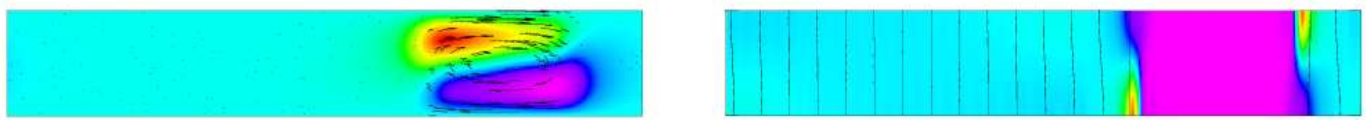
A.A. Golovin, A.A. Nepomnashchy & L.M. Pismen, *Phys. Fluids* **6**, 34 (1994)

N. Komarova & A.C. Newell, *J. Fluid Mech.* **415**, 285 (2000)

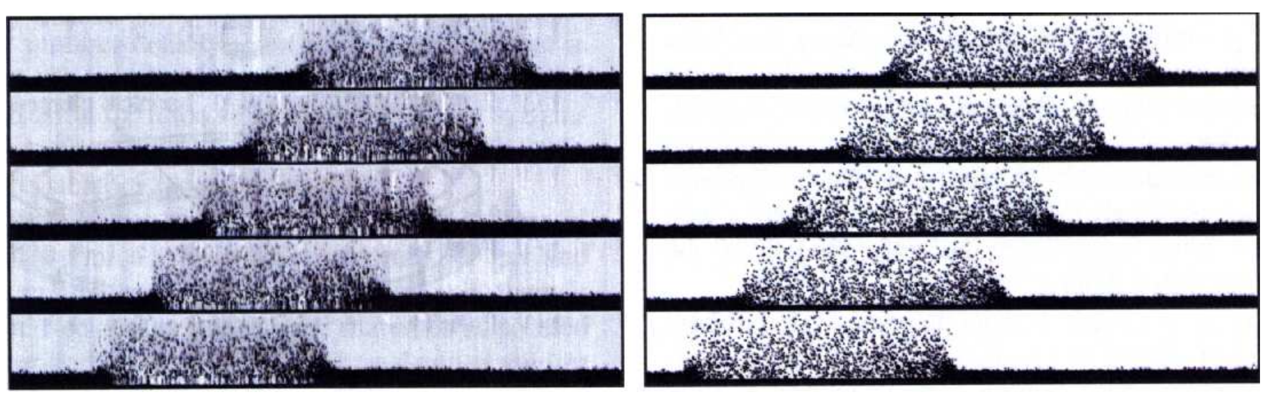
Large scale modes

Two particular physical situations with conserved quantities:

- Magnetoconvection with a vertical field:
  Conserved quantity is total flux of field through fluid layer: $F_B = \int B_z \, dx$



- Granular Faraday experiment
  Conserved quantity is total amount of material: $\int \rho \, dxdy$.



Left: Experiment, Right: molecular dynamics simulation

A. Götzendorfer, J. Kreft, C.A. Kruelle and I. Rehberg, Sublimation of a vibrated granular monolayer: coexistence of gas and solid *Phys. Rev. Lett.* **95**, 135704 (2005)

Weakly nonlinear theory

Spatial extension to include wavenumbers near k - might expect only the 'Ginzburg–Landau' term a_{XX} to be needed.
Introduce long length and time scales $X = \varepsilon x$, $T = \varepsilon^2 t$.
Matthews and Cox pointed out the need to include a large-scale mode $A_0(X,T)$ for the field:

$$\begin{aligned} \psi &= \varepsilon a(X,T)\mathrm{e}^{ikx}\sin\pi z + c.c. + O(\varepsilon^2) \\ \theta &= \varepsilon c_1 a(X,T)\mathrm{e}^{ikx}\sin\pi z + c.c. + O(\varepsilon^2) \\ A &= \varepsilon c_2 a(X,T)\mathrm{e}^{ikx}\cos\pi z + \varepsilon c_2 A_0(X,T) + c.c. + O(\varepsilon^2) \end{aligned}$$

Derive coupled amplitude equations:

$$\begin{aligned} a_T &= \mu a + a_{XX} - a|a|^2 - aA_{0X} \\ A_{0T} &= \zeta A_{0XX} + \pi(|a|^2)_X \end{aligned}$$

Coupling terms represent suppression and flux expulsion

Weakly nonlinear theory

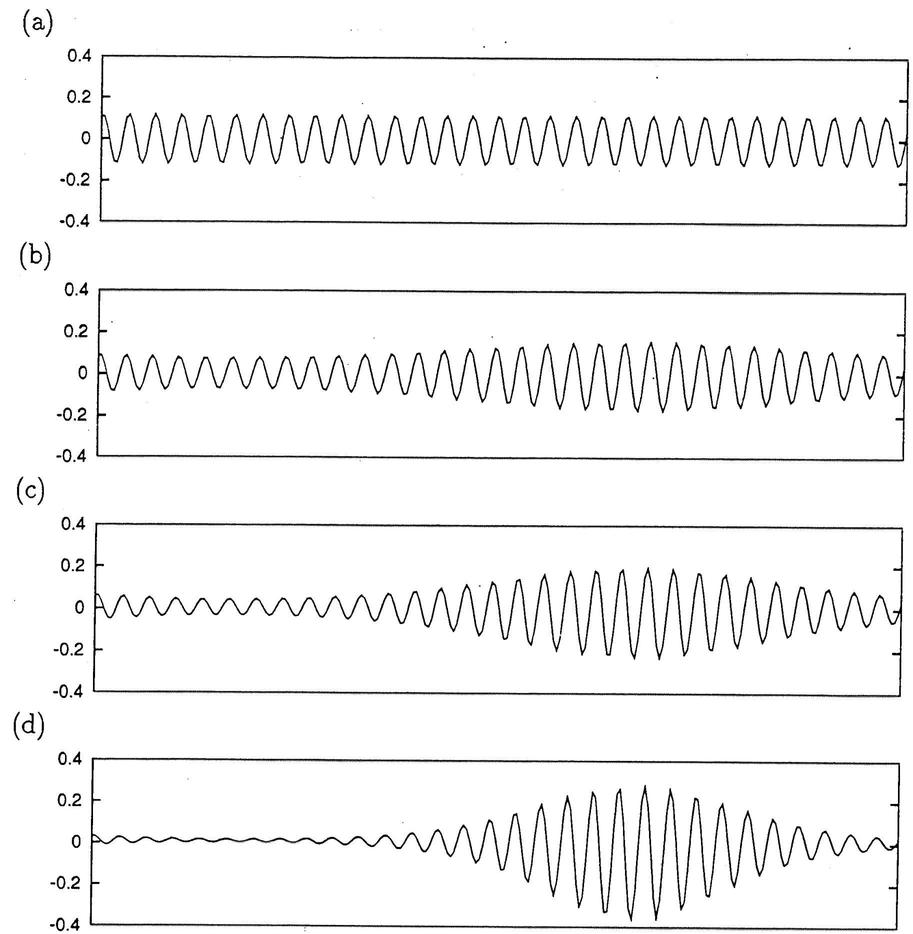
Constant-amplitude, steady rolls:

$a = \text{const}, \; A_{0X} = 0$

are unstable (at onset) to modulational disturbances if:

$\zeta^2 k^4 (\pi^2 + k^2) < \pi^2 (2k^2 - \pi^2)(k^2 + 3\pi^2)$

i.e. if ζ is sufficiently small, for fixed Q.



P.C. Matthews and S.M. Cox, *Nonlinearity* **13**, 1293–1320 (2000)

Model problem for magnetoconvection

J.H.P. Dawes, Localised pattern formation with a large-scale mode: slanted snaking. Preprint.

$$w_t = [r - (1 + \partial_{xx}^2)^2]w - w^3 - QB^2 w \quad (1)$$

$$B_t = \zeta B_{xx} + \frac{1}{\zeta}(w^2 B)_{xx} \quad (2)$$

Symmetries:

- $w \to -w$ (Boussinesq problem)
- $B \to -B$ (direction of magnetic field)

Parameters:

- r - reduced Rayleigh number $r = R/R_c$
- Q - Chandrasekhar number $\propto |B_0|^2$
- $\langle B \rangle = 1$ after nondimensionalising
- ζ - magnetic/thermal diffusivity ratio $\zeta = \eta/\kappa$

Remark: We could write $w = \varepsilon w_1 + \cdots$, $B = 1 + \varepsilon^2 B_2 + \cdots$, $X = \varepsilon x$, $T = \varepsilon^2 t$ – this was done by
P.C. Matthews & S.M. Cox *Nonlinearity* **13**, 1293–1320 (2000)

Model problem for magnetoconvection

Set $\partial_t \equiv 0$. Integrate (2) twice:

$$\zeta P \quad = \quad B\left(\zeta + \frac{w^2}{\zeta}\right)$$

where P is a constant.

Re-arrange and integrate over the domain $[0, L]$: $\left\langle \frac{P}{1+w^2/\zeta^2} \right\rangle = \langle B \rangle = 1$

Hence

$$\frac{1}{P} \quad = \quad \left\langle \frac{1}{1 + w^2/\zeta^2} \right\rangle$$

P measures the higher concentration of the large-scale mode in the region *outside* the localised pattern.

Substituting, we obtain

$$0 \quad = \quad [r - (1 + \partial_{xx}^2)^2]w - w^3 - \frac{QP^2 w}{(1 + w^2/\zeta^2)^2}$$

Ginzburg–Landau reduction

$$0 = [r - (1 + \partial_{xx}^2)^2]w - w^3 - \frac{QP^2 w}{(1 + w^2/\zeta^2)^2}$$

- Suppose $\zeta \ll 1$
- Introduce the long scales $X = \zeta x$, $T = \zeta^2 t$.
- Rescale: $Q = \zeta^2 q$ and $r = \zeta^2 \mu$.
- Expand: $w(x,t) = \zeta A(X,T) \sin x + O(\zeta^2)$, assuming $A(X,T)$ real.

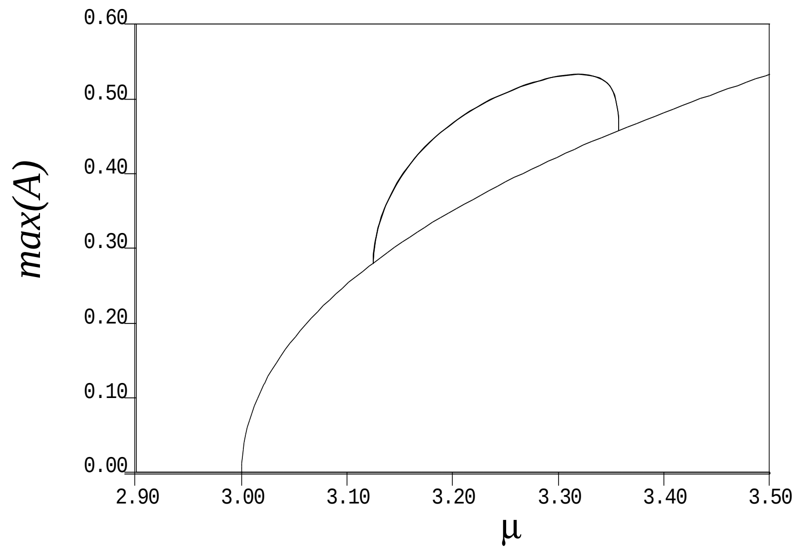
Interpret spatial average as over both $x \in [0, 2\pi]$ and X:

$$\frac{1}{P} = \left\langle \left\langle \frac{1}{1 + A^2 \sin^2 x} \right\rangle_x \right\rangle_X = \left\langle \frac{1}{\sqrt{1 + A^2}} \right\rangle_X$$

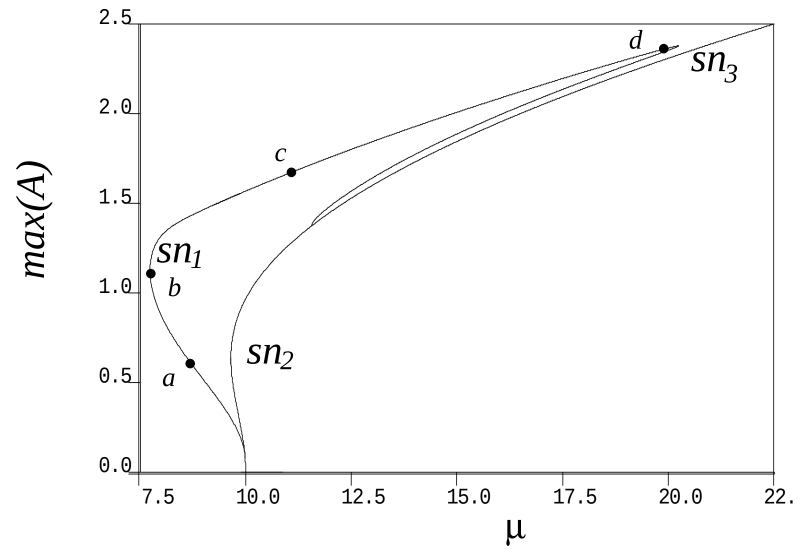
Extract solvability condition by multiplying by $\sin x$ and integrating over x:

$$0 = \mu A + 4A_{XX} - 3A^3 - \frac{qP^2 A}{(1 + A^2)^{3/2}}$$

Ginzburg–Landau reduction



$q = 3$



$q = 10$

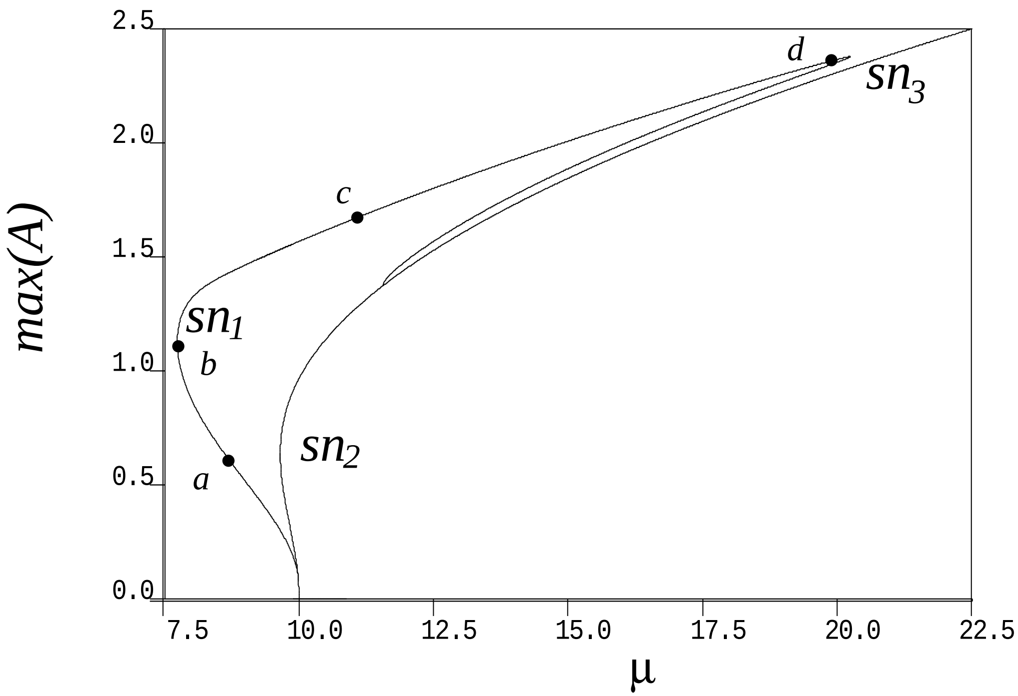
For large q:

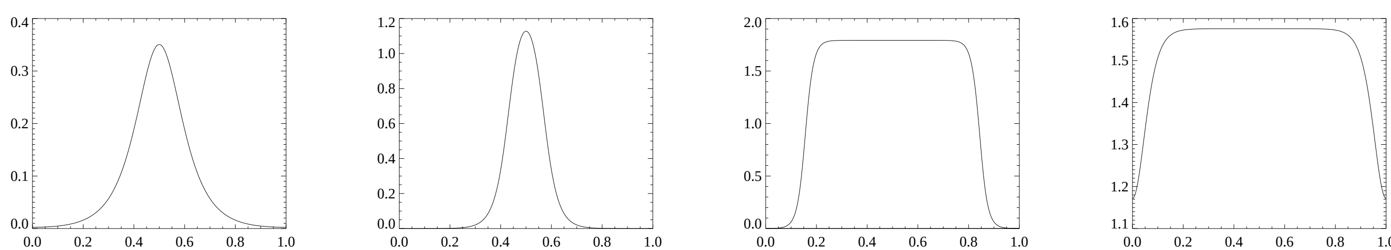
- localised branch has saddle-node far below uniform branch
- and second saddle-node bifurcation before it rejoins uniform branch

$c = 1$, $\varepsilon L = 10\pi$.

Modified Ginzburg–Landau reduction

$q = 10$, $c = 1$, $\varepsilon L = 10\pi$.

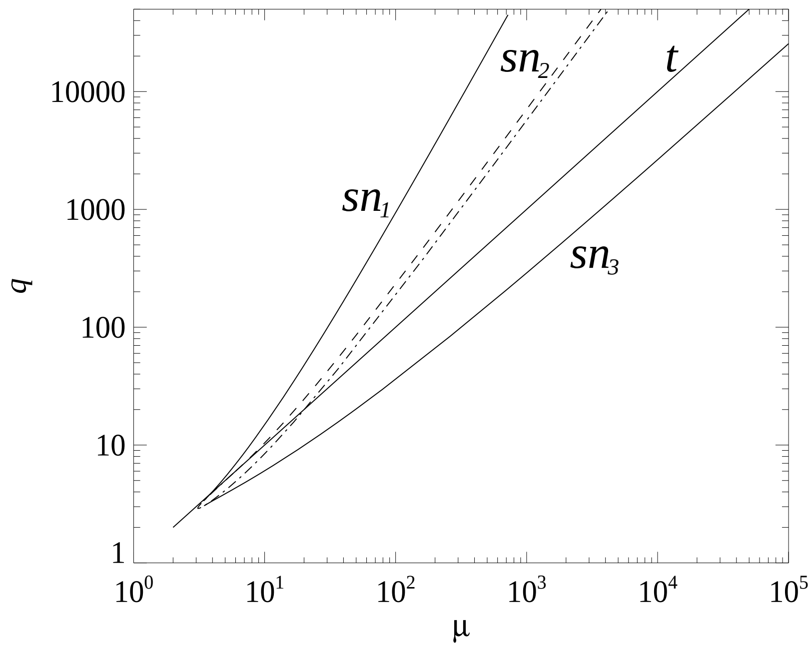


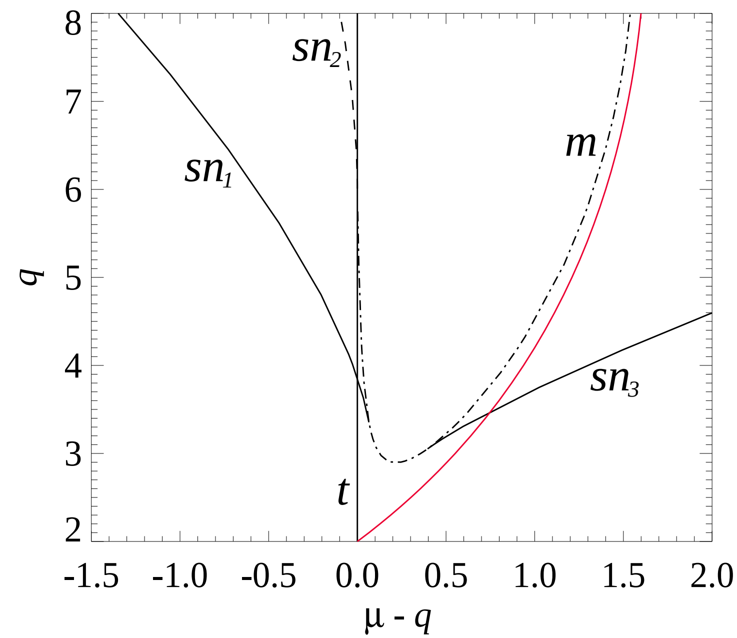


(a) (b) (c) (d)

Modified Ginzburg–Landau reduction

Bifurcation curves in the (μ, q) plane:

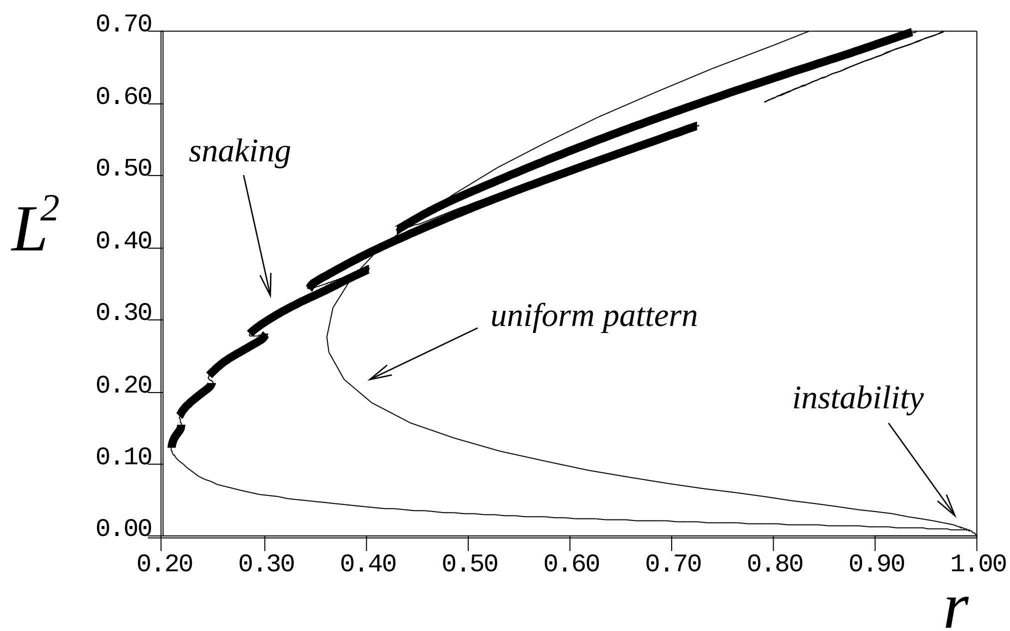




- Solid lines: saddle-nodes on the localised branch. Modulated states exist between sn_1 and sn_3.
- Dashed lines: sn_2 – saddle-node on the uniform branch. t – bifurcation from trivial state, at $\mu = q$.

Return to (w, B) equations

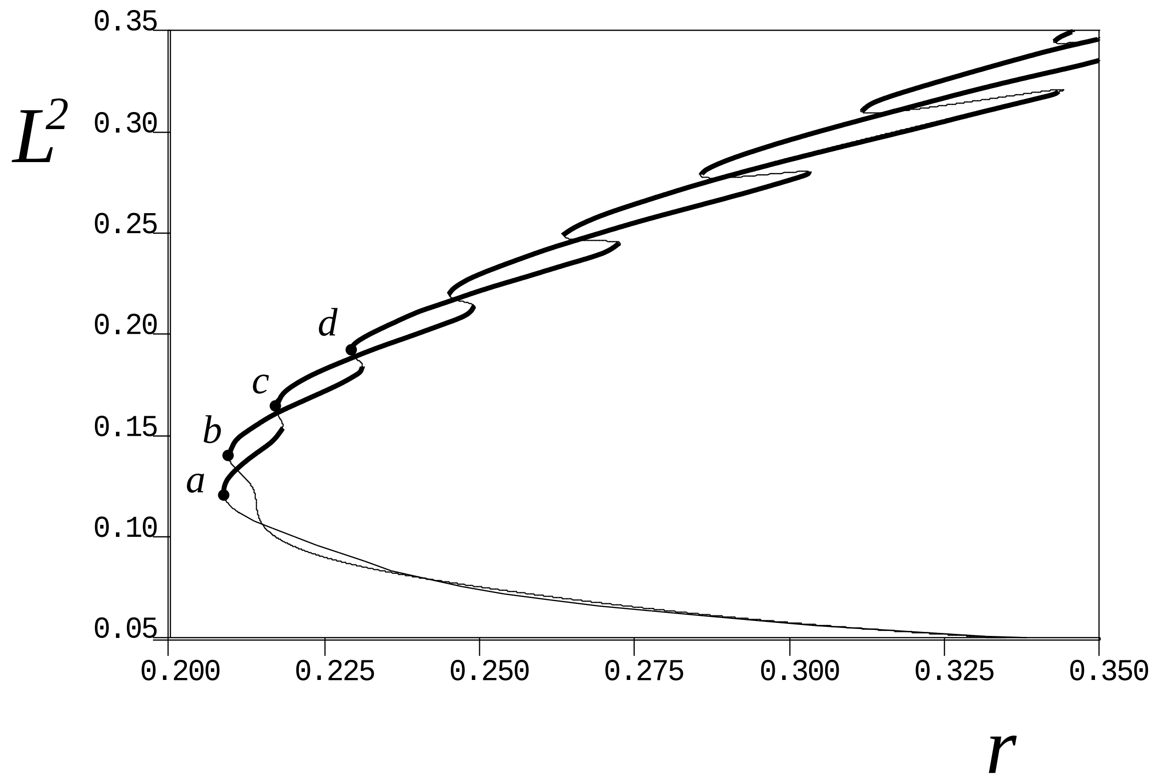
$$w_t = [r - (1 + \partial_{xx}^2)^2]w - w^3 - QB^2 w \qquad (1)$$
$$B_t = \zeta B_{xx} + \frac{1}{\zeta}(w^2 B)_{xx} \qquad (2)$$

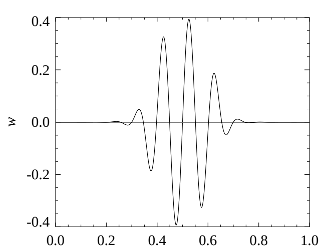


Slanted snaking

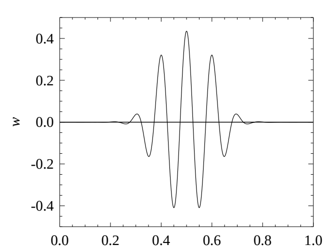
J.H.P. Dawes, Localised pattern formation with a large-scale mode: slanted snaking. Preprint.

Slanted snaking - details

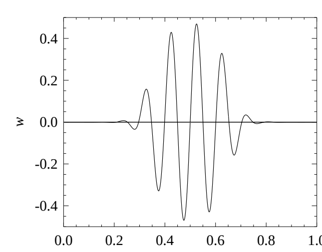




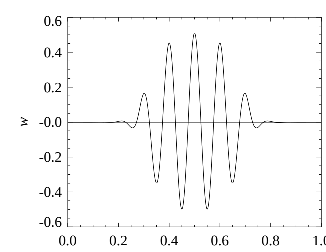
(a)



(b)



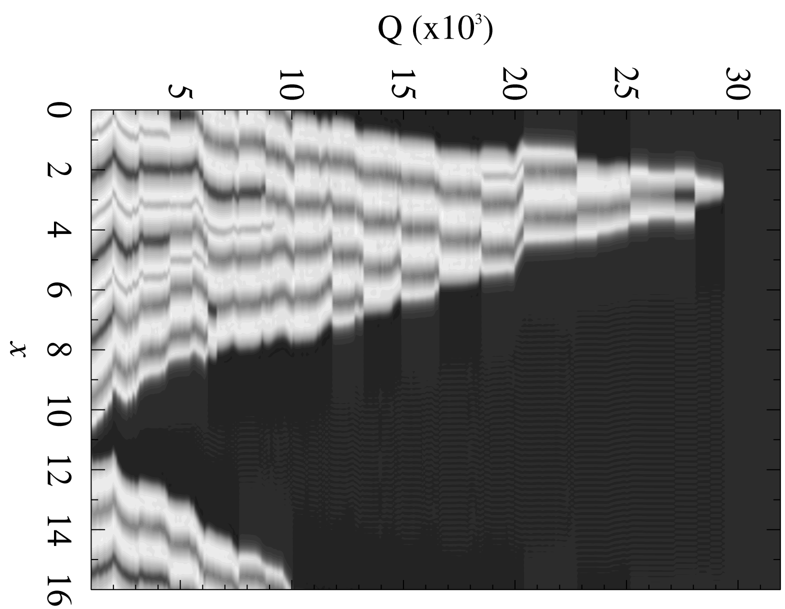
(c)



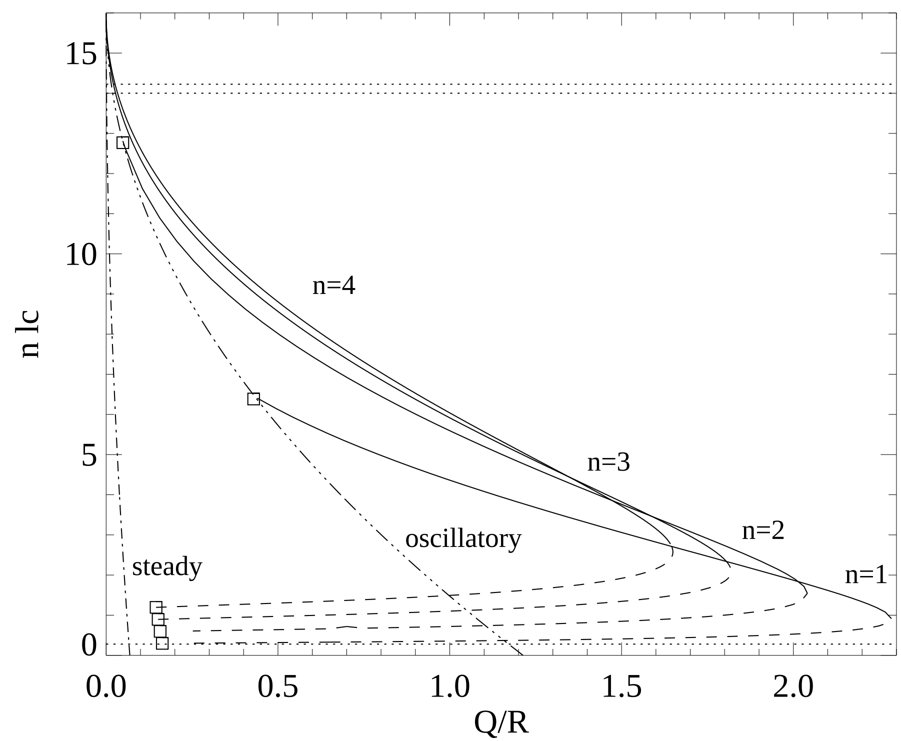
(d)

Locations of saddle-nodes

Full magnetoconvection equations:



Simple fluid-mechanical model:

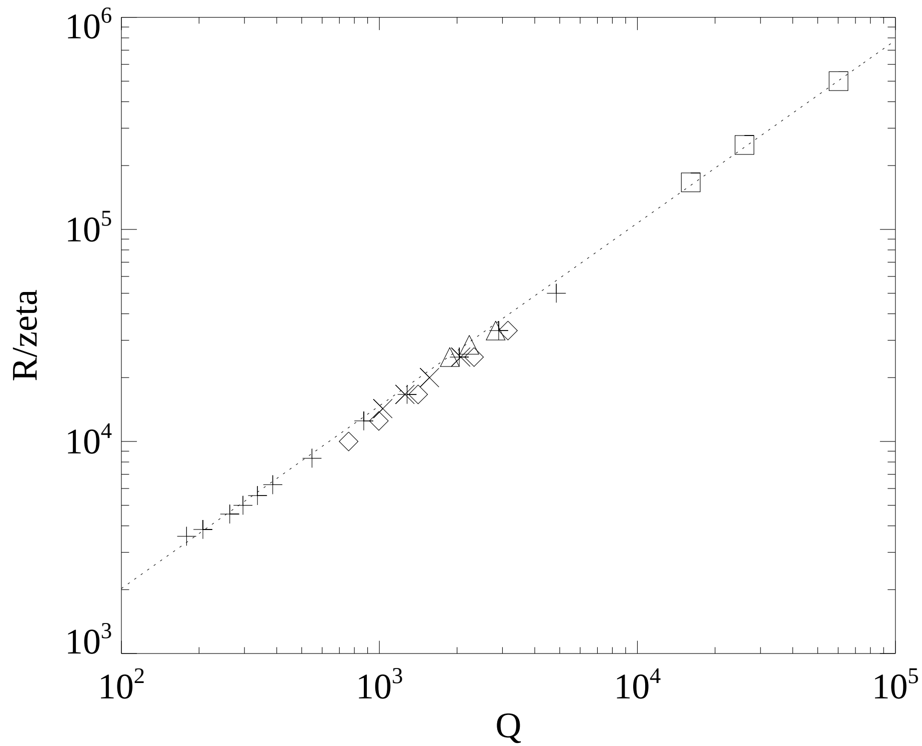


...an example of *slanted* snaking

J.H.P. Dawes, Localised convection cells in the presence of a vertical magnetic field. *J. Fluid Mech.* **570**, 385–406 (2007)

Navier–Stokes equations - scaling law

Location of last saddle-node bifurcation ($n = 1$):

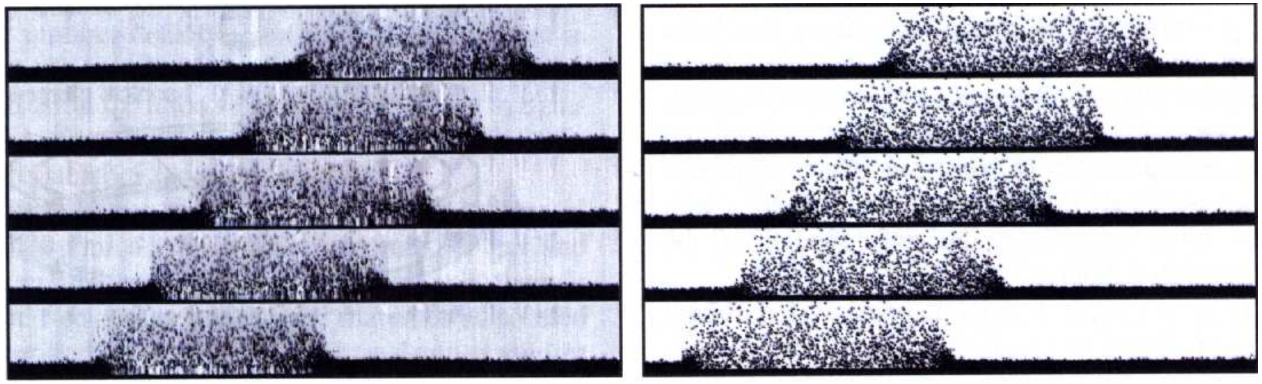


Symbol	R	L	
$+$	5000	6.0	truncated PDEs
$\times$	10000	6.0	Boussinesq eqns
$\triangle$	20000	6.0	Boussinesq eqns
$\square$	50000	6.0	truncated PDEs
$\diamond$	5000	10.0	truncated PDEs

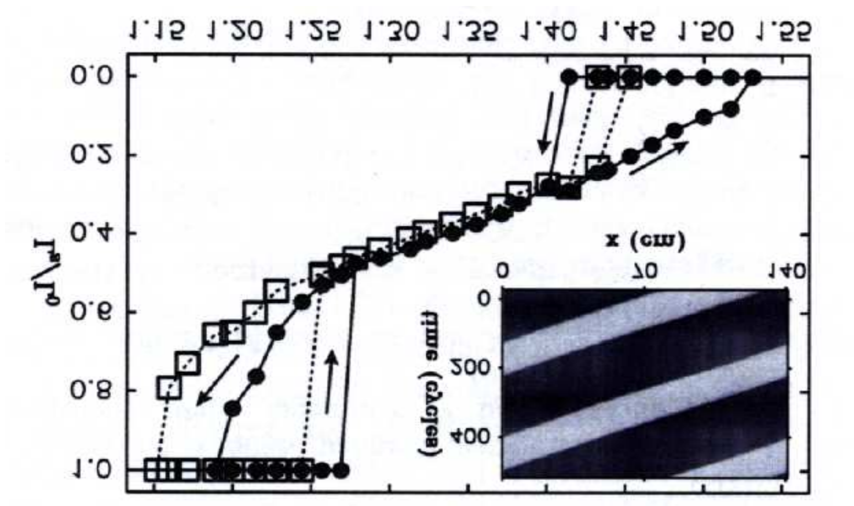
- Power law: $\frac{R}{\zeta} = 38.3\,Q^{0.862}$. (!!)

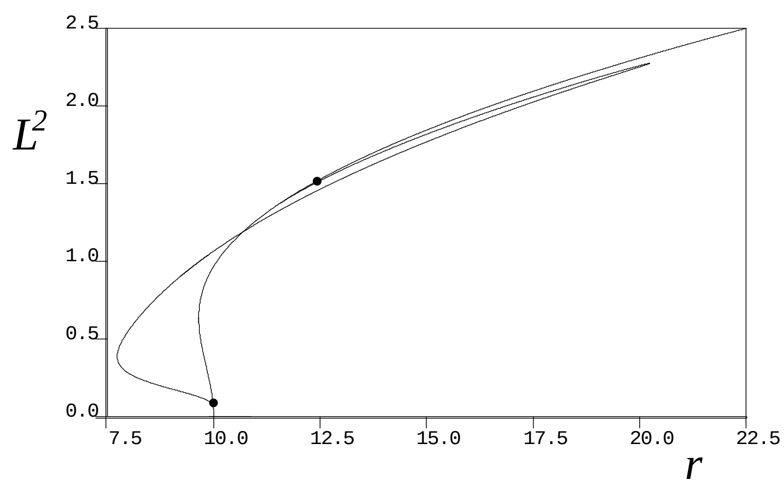
J.H.P. Dawes, Localised convection cells in the presence of a vertical magnetic field. *J. Fluid Mech.* **570**, 385–406 (2007)

uture work: granular Faraday experimen



Left: Experiment, Right: molecular dynamics simulation





A. Götzendorfer, J. Kreft, C.A. Kruelle and I. Rehberg, Sublimation of a vibrated granular monolayer: coexistence of gas and solid *Phys. Rev. Lett.* **95**, 135704 (2005)

Summary

- Supercritical pattern-forming instabilities generate regular structures
- Subcritical instabilities generate localised patterns
- Large-scale modes enhance localisation
- New asymptotic derivation, leading to a nonlocal Ginzburg–Landau equation

Further work:

- Oscillatory localised states in granular Faraday experiment (Tsimring & Aranson 1997, Yochelis, Burke & Knobloch 2006, Winterbottom, Matthews & Cox 2007)
- Non-variational dynamics - travelling pulses, e.g. in nonlinear optics models (McKenna & Champneys 1996)
- Localised turbulence (Prigent, Dauchot et al, Barkley & Tuckerman)