Isaac Newton Institute, Cambridge

Training Course: Pattern Formation, 1 – 5 August 2005

Problem sheet 2: Symmetric bifurcations

1. Show that absolute irreducibility implies irreducibility.

2. The group orbit $\mathcal{G}x$ of a point $x \in \mathbb{R}^n$ is defined to be $\mathcal{G}x = \{gx : g \in \mathcal{G}\}$. Show that $\Sigma_{gx} = g\Sigma_x g^{-1}$, i.e. that points on the same group orbit have conjuagte isotropy subgroups. (see lecture notes, page 10).

3. Show that the natural 2D action of D_3, generated by

$$\rho(z) = \mathrm{e}^{2\pi \mathrm{i}/3} z, \qquad \text{and} \qquad m_x(z) = \bar{z}$$

where $z = x + \mathrm{i}y$ is a coordinate on $\mathbb{R}^2$, is absolutely irreducible. Either work explicitly in coordinates or argue geometrically.

4. Using the normal form from the lecture notes, check by hand, from the Jacobian matrix, that the nontrivial branch in a steady-state bifurcation with D_3 symmetry is unstable on both sides of the bifurcation point.

5. *Rotating hexagonal lattice.* Consider the bifurcation problem on a hexagonal lattice in a rotating system. The symmetry group is now $\mathbb{Z}_6 \ltimes T^2$ generated by ρ and $\tau_{\mathbf{p}}$ as in the notes. Show that the amplitude equations are now

$$\dot{z}_1 \quad = \quad \mu z_1 + \varepsilon \bar{z}_2 \bar{z}_3 - a z_1 |z_1|^2 - b z_1 |z_2|^2 - c z_1 |z_3|^2$$

plus symmetric versions for z_2 and z_3. Now consider the case where all the amplitudes are real, so that the equivariant ODEs (truncated at cubic order) take the form

$$\begin{aligned}
\dot{x}_1 &= x_1[\mu - a x_1^2 - b x_2^2 - c x_3^2] + x_2 x_3 \\
\dot{x}_2 &= x_2[\mu - a x_2^2 - b x_3^2 - c x_1^2] + x_1 x_3 \\
\dot{x}_3 &= x_3[\mu - a x_3^2 - b x_1^2 - c x_2^2] + x_1 x_2
\end{aligned}$$

Describe the bifurcation, at $\mu = 0$, of the branch of solutions with $x_1 = x_2 = x_3$. Show that there is a secondary Hopf bifurcation from this branch when $x = \mu(b+c-2a)/(4a+b+c)$, assuming that the frequency $2\sqrt{3}\mu(c-b)/(4a+b+c)$ is non-zero (and that the usual non-degeneracy conditions on the nonlinear terms hold).
[Hint: use the structure of the Jacobian matrix to find its eigenvalues.]

6. [I. Melbourne, *Dyn. Stab. Syst.* **1**, 293–321 (1986)]. Consider the group $\Gamma = \mathbb{O} \times \mathbb{Z}_2^c$ of rotations and reflections of a cube in $\mathbb{R}^3$, centred at the origin and aligned with the coordinate axes. A representation of Γ on $\mathbb{R}^3$ is defined by the following matrices representing elements that generate the group:

$$\kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad r_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

(a) Show that this representation of Γ is absolutely irreducible.

(b) By determining their isotropy subgroups show that three (group orbits of) distinct equilibria are guaranteed to bifurcate from the origin in a generic Γ-symmetric steady-state bifurcation problem.

(c) Determine the normal form amplitude equations for the bifurcation, up to cubic order.

Hint: to simplify notation define κ_y, κ_z and r_z by analogy with the matrices above.

7. *(Madruga, Riecke and Pesch, preprint, 2005).* Consider the following modification to the ODEs for steady-state bifurcation on a hexagonal lattice:

$$\begin{aligned}
\dot{x}_1 &= \mu x_1 - x_2 x_3(a + b\mu) - x_1[x_1^2 + c(x_2^2 + x_3^2)] \\
\dot{x}_2 &= \mu x_2 - x_1 x_3(a + b\mu) - x_2[x_2^2 + c(x_3^2 + x_1^2)] \\
\dot{x}_3 &= \mu x_3 - x_1 x_2(a + b\mu) - x_3[x_3^2 + c(x_1^2 + x_2^2)]
\end{aligned}$$

(again restricting attention to the subspace where all amplitudes are real). The modification is that the coefficient of the quadratic terms now depends on the bifurcation parameter μ. This models thermal convection in a fluid that departs rapidly from the Boussinesq approximation as the Rayleigh number increases. For convenience the coefficient of x_1^3 has been scaled to be -1. Assume $a, b > 0$.

Change variables to $A_1 = x_1/(a + b\mu)$, $T = (a + b\mu)^2 t$ and hence interpret the resulting bifurcation diagram in terms of figure 9 (lecture notes, p29) in the case $a/b > \mu_3$, where μ_3 is as defined in figure 9. Discuss the stability of the hexagon branch at large $\mu \gg 1$.

8. *Hopf bifurcation with $\mathbb{Z}_2$ symmetry [GS (2002), p93].* Let $\mathbb{Z}_2 = \{I, \kappa\}$ act on $\mathbb{R}$ nontrivially, i.e. $\kappa(x) = -x$. Show that a generic Hopf bifurcation with this symmetry is symmetric under $\mathbb{Z}_2 \times S^1$ acting on $\mathbb{R}^2$ by

$$\begin{aligned}
\kappa \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -x \\ -y \end{pmatrix} \\
\tau_\theta \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

Apply the Equivariant Hopf Theorem to conclude there exists a unique branch of periodic orbits. Give the spatiotemporal symmetry group of these orbits.

9. *Hopf bifurcation with D_3 symmetry [GS (2002), p94].* Consider three identical, and symmetrically bidirectionally coupled, cells governed by equations

$$\begin{aligned}
\dot{x}_0 &= f(x_0, x_1, x_2) \\
\dot{x}_1 &= f(x_1, x_2, x_0) \\
\dot{x}_2 &= f(x_2, x_0, x_1)
\end{aligned}$$

where $x_j \in \mathbb{R}^k$ describes the state of each cell, and $f(x_0, x_1, x_2) = f(x_0, x_2, x_1)$. Suppose that $x_0 = x_1 = x_2 = 0$ is a D_3-symmetric equilibrium point. Let the complete system of ODEs in $\mathbb{R}^{3k}$ be $\dot{x} = F(x)$. Show that the Jacobian matrix DF takes the form

$$DF = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix},$$

where $A = \partial f/\partial x_0$ and $B = \partial f/\partial x_1$ are $k \times k$ matrices. Hence show that the eigenvalues of DF are

- those of $A + 2B$ with the same multiplicity as they have in $A + 2B$, and
- those of $A - B$ with twice the multiplicity.

Discuss the two generic Hopf bifurcations that could take place (call these the trivial and non-trivial cases).

In the non-trivial case the relevance action of $D_3 \times S^1$ on the centre manifold is generated by

$$\begin{aligned}
\rho_\phi(z_1, z_2) &= (\mathrm{e}^{-\mathrm{i}\phi} z_1, \mathrm{e}^{\mathrm{i}\phi} z_2) \\
\kappa(z_1, z_2) &= (z_2, z_1) \\
\tau_\theta(z_1, z_2) &= (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2).
\end{aligned}$$

This action is D_3-simple (check if you wish). Find three non-conjugate $\mathbb{C}$-axial isotropy subgroups and their fixed point subspaces. Interpret your results in terms of the dynamics of the original three coupled cells.