Isaac Newton Institute, Cambridge

Training Course: Pattern Formation, 1 – 5 August 2005

Answers for problem sheet 2: Symmetric bifurcations

1. We show the logically equivalent statement, that reducible reps cannot be absolutely irreducible. Suppose that the real orthogonal representation $\tilde{\rho}(G)$ is reducible. Then there exist invariant subspaces V, $V^\perp$ such that $\mathbb{R}^n = V \oplus V^\perp$ and $\tilde{\rho}(G)$ acts irreducibly on V. Let $\dim V = m$. Then the matrix

$$A = \begin{pmatrix} aI_m & 0 \\ 0 & 0 \end{pmatrix}$$

where $a \neq 0$ is real, and I_m is the $m \times m$ identity matrix, commutes with the matrices in the representation. But A is not a multiple of the $n \times n$ identity matrix and so the representation is not absolutely irreducible.

2. This is proved in the lecture notes, page 10.

3. Geometrically, if a 2×2 real matrix commutes with a rotation by an angle less than π, it must be a rotation itself. Then, if this rotation also commutes with a reflection, it must be the identity. In coordinates, check that for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, if we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $a = d$ and $b = c = 0$.

4. Using the third-order truncation we find

$$Df|_{(x,y)} = \begin{pmatrix} \mu - 2ax & 2ay \\ 2ay & \mu + 2ax \end{pmatrix}$$

and the nontrivial branch is located approximately at $0 = \mu - ax + O(x^2)$, $y = 0$. So at leading order the Jacobian matrix is

$$Df|_{(x,0)} = \begin{pmatrix} -\mu & 0 \\ 0 & 3\mu \end{pmatrix}$$

i.e. the nontrivial is a saddle on both sides of $\mu = 0$ for $|\mu|$ small enough.

5. The solution branch $x_1 = x_2 = x_3$ ('hexagons') is located at $0 = \mu + x - (a+b+c)x^2$. For stability, compute the Jacobian matrix which, by symmetry, has the form

$$Df|_{hex} = \begin{pmatrix} P & Q & R \\ R & P & Q \\ Q & R & P \end{pmatrix}$$

which is circulant. So the eigenvectors are $e_1 = (1,1,1)^T$, $e_2 = (1,\omega,\omega^2)^T$ and $e_3 = (1,\bar{\omega},\bar{\omega}^2)$ where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. Evaluating the corresponding eigenvalues we find $v_1 = x - 2(a+b+c)x^2 = -2\mu - x$ and $v_2 = -2x - 2x^2(a + b\omega + c\omega^2)$. $v_3 = \bar{v}_2$. Taking real and imaginary parts of v_2 we have $Re(v_2) = 0$ when

$$x = \frac{\mu(b+c-2a)}{4a+b+c}$$

and, at this point the frequency $Im(v_2)$ is as required.

6.(a) It is enough to check commutivity with the group generators. If a matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

commutes with κ_x (forces $b = c = d = g = 0$), with r_x (forces $f = -h$ and $e = k$), and with r_y (forces $a = e$ and $f = 0$), then $A = aI$.

(b) The three axial branches are:

Subspace	Isotropy subgroup	Generators
$(x, 0, 0)$	D_4	$\kappa_y,\ r_x$
$(x, x, 0)$	$D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\kappa_z,\ \kappa_x \circ r_z$
(x, x, x)	D_3	$\kappa_y \circ r_x,\ \kappa_x \circ r_z$

(there is a group orbit of branches in each case). Geometrically these correspond to distorting the cube by pulling or pushing in the middle of a face, the middle of an edge, or at a vertex. Apply the equivariant branching lemma since they all have 1D fixed point subspaces.

(c) Normal form to cubic order is

$$\begin{aligned} \dot{x} &= f_1(x, y, z) = \mu x + ax(y^2 + z^2) + bx^3 \\ \dot{y} &= f_2(x, y, z) = \mu y + ay(x^2 + z^2) + by^3 \\ \dot{z} &= f_3(x, y, z) = \mu z + az(x^2 + y^2) + bz^3 \end{aligned}$$

Justification: from κ_x, κ_y, κ_z we see that f_1 is odd in x and even in y and z. Absolute irreducibility implies that the linear terms are just μI. r_x equivariance implies that the coefficients of xy^2 and xz^2 are equal. Then the forms of f_2 and f_3 come from applying $\kappa_x \circ r_z$ and $\kappa_y \circ r_x$.

7. Applying the change of variables in the question we obtain

$$\frac{dA_1}{dT} = \frac{\mu}{(a + b\mu)^3} A_1 - A_2 A_3 - A_1^3 - cA_1(A_2^2 + A_3^2).$$

Define $\hat{\mu} = \mu/(a + b\mu)^3$. Then as μ increases from small negative values, $\hat{\mu}$ first increases approximately linearly, then reaches a maximum at $\mu = a/b$, $\hat{\mu} = 1/(4ab)$, then decreases to zero with further increases in μ. Now refer to figure 9 from the lecture notes and consider that, as μ increases we traverse the figure from left to right and then back to the left again. So hexagons undergo two D_3 transcritical bifurcations with the rectangle branch and are stable again for large positive μ.

8. Generic Hopf bifurcation has either (i) an irreducible but not absolutely irreducible group representation, or (ii) two copies of an absolutely irreducible rep. Since $\mathbb{Z}_2$ only has one non-trivial rep and it is absolutely irreducible we must be in case (ii) here. So the action of $\mathbb{Z}_2 \times S^1$ on $\mathbb{R}^2$ must be generated by κ and τ_θ as given (note: we could have $\tau_\theta(z) = \mathrm{e}^{\mathrm{i}m\theta} z$ for any integer m, but these cases are essentially the same, just with all solutions having an automatic additional $\mathbb{Z}_m$ symmetry). The spatio-temporal symmetry $\kappa \circ \tau_\pi$ acts trivially, so $\Sigma = \{I, \kappa \circ \tau_\pi\}$ is an isotropy subgroup with a two dimensional fixed point subspace and so is $\mathbb{C}$-axial, guaranteeing exactly one branch of periodic orbits in a generic Hopf bifurcation.

9. If a complex conjugate pair of eigenvalues of $A + 2B$ cross the imaginary axis we have no extra multiplicity due to symmetry. So this is standard Hopf bifurcation, and we expect a unique branch of periodic orbits that have full D_3 symmetry. A solution would take the form $(x(t), x(t), x(t))$ where $x(t)$ is time-periodic.

In the second case, where a complex conjugate pair of eigenvalues of $A - B$ cross the imaginary axis, the Jacobian matrix DF will have critical eigenvalues with twice the multiplicity, leading to the non-trivial case outlined in the question. Let T be the oscillation period. Then the three $\mathbb{C}$-axial branches are

Subspace	Isotropy subgroup	Generators	Oscillation pattern
(z, z)	$\mathbb{Z}_2$	κ	$(y(t), y(t), x(t))$
$(z, 0)$	$\mathbb{Z}_3$	$\rho_\phi \circ \tau_\phi$	$(x(t), x(t + T/3), x(t + 2T/3))$
$(z, -z)$	$\mathbb{Z}_2$	$\kappa \circ \tau_\pi$	$(x(t), x(t + T/2), \hat{y}(t))$

Notice that, in the third case, $\hat{y}(t)$ is forced by symmetry to have exactly half the period of $x(t)$: $\hat{y}(t) = \hat{y}(t + T/2)$. See GS (2002) p70 or GSS (1988) p389. In all cases (in normal form) there is the spatiotemporal symmetry $\rho_\pi \circ \tau_\pi$ - this fixes all points in $\mathbb{C}^2$, and has been omitted for clarity.