Series and Limits - Answer sheet

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Starred questions are interesting but less important than the others.

1. Show, using the definitions carefully, that if $\{a_n\}_{n\geq 1} \to a$ and $\{b_n\}_{n\geq 1} \to b$ as $n \to \infty$ then

$$\text{(i)}\ \ a_n + b_n \to a + b \ \text{ as } \ n \to \infty, \qquad \text{(ii)}\ \ a_n b_n \to ab \ \text{ as } \ n \to \infty.$$

Answer: (i) Given $\varepsilon > 0$ there exist N_1 and N_2 such that $|a_n - a| < \varepsilon/2$ for all $n > N_1$, and $|b_n - b| < \varepsilon/2$ for all $n > N_2$. So, for all $n > \max\{N_1, N_2\}$ we have

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so $a_n + b_n \to a + b$ as $n \to \infty$.

(ii) The same argument works here. A slight embellishment is to say, given $\varepsilon > 0$ there exists N_1 such that $|a_n - a| < \min\{\varepsilon, 1\}$. Define N_2 as before. Then, if $n > \max\{N_1, N_2\}$ we have

$$\begin{aligned} |a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| &\leq |a_n||b_n - b| + |b||a_n - a| \\ &< (|a| + 1)\varepsilon + |b|\varepsilon = \text{constant} \times \varepsilon \end{aligned}$$

so now, replacing ε with $\varepsilon/(1 + |a| + |b|)$ if you wish, we have convergence of $a_n b_n$.

2. Find the partial sum S_N of the first N terms of these series (note that they start at different values of n!), and hence determine whether they converge:

$$\text{(i)}\ \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right), \qquad \text{(ii)}\ \sum_{n=0}^{\infty} (-2)^n.$$

Answers: (i) $S_N = \log(N + 1)$. (ii) $S_N = \frac{1}{3}(1 - (-2)^N)$. Neither of them converges.

3. *Achilles and the Tortoise.* In this well-known paradox due to Zeno, we imagine that the Greek hero Achilles is racing against a tortoise. Sensing that the tortoise is at more than a slight disadvantage, Achilles gives him a headstart of d metres. Both begin to run, at constant but very unequal speeds, at time $t = 0$. The paradox is that it will take Achilles a certain length of time to get to where the tortoise started from, but in that interval the tortoise will have crawled further. Achilles will have to now cover that new distance, but in that time the tortoise will again have crawled forward. So Achilles can, in fact, never overtake the tortoise.

   Refute the paradox, supposing that the tortoise runs at v ms^{-1} and Achilles runs at rv ms^{-1} where $r > 1$, by showing that Achilles and the tortoise are level after a finite time $\frac{d}{v(r-1)}$ seconds.

Answer: There is a geometric series to sum here. It takes Achilles a time $t_1 = d/(rv)$ to reach the tortoise's starting point. In this time the tortoise has moved an additional distance $v \times d/(rv) = d/r$ metres. It takes Achilles an additional time $t_2 = \frac{d/r}{rv}$ to cover this distance, and so on. They are level after a time

$$T = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{d}{r^n v} = \frac{d}{v}\frac{1/r}{1 - 1/r} = \frac{d}{v(r-1)}$$

Check that the behaviour of T in the cases $r \gg 1$, and r close to 1, agree with your intuition. A sketch of the motion in the (x, t) plane might be useful.

Hint: $a_m = S_m - S_{m-1}$; *use the definition of convergence and the Triangle Inequality.* Note that the reverse implication is not true.

Answer: By assumption, $S_m \to L$ *as* $m \to \infty$, *i.e. given* $\varepsilon > 0$ *there exists* $M : |S_m - L| < \varepsilon/2 \;\forall\; m \geq M$. *So*

$$|a_m| = |S_m - S_{m-1}| \leq |S_m - L| + |L - S_{m-1}| \leq \varepsilon \;\forall m \geq M+1$$

by the Triangle Inequality. So all $|a_m$, *for* m *sufficiently large, are less than any given* ε, *i.e. the sequence* $\{a_m\} \to 0$ *as* $m \to \infty$. □

5. By rearranging the terms of the absolutely convergent series $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, show that

$$\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Answer: split into a sum over n *odd plus a sum over* n *even:*

$$\sum_{n\geq1}^{\infty} 1/n^2 = \sum_{n \text{ odd}}^{\infty} 1/n^2 + \sum_{m\geq1}^{\infty} 1/(2m)^2 = \sum_{n \text{ odd}}^{\infty} 1/n^2 + \frac{1}{4}\sum_{n\geq1}^{\infty} 1/n^2$$

and recombine the last term with the left-hand side to get the result.

6. (i) Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which sums to $\log 2 \approx 0.693$, but which is not absolutely convergent, can be rearranged in the order

$$S = \underbrace{\frac{1}{1} + \frac{1}{3} - \frac{1}{2}}_{m=1} + \underbrace{\frac{1}{5} + \frac{1}{7} - \frac{1}{4}}_{m=2} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}}_{m=3} + \cdots.$$

(ii) By considering the terms of the rearranged series grouped in threes as indicated, show that the series can be written as

$$S = \sum_{m=1}^{\infty} \frac{8m-3}{2m(4m-3)(4m-1)}$$

(iii) Show that the above series for S is convergent by comparison with $\sum 1/n^2$, and that it contains only positive terms. Evaluate the first term and deduce that the sum S is not equal to $\log 2$.

Answer: (i) check that all the terms will appear at some stage.

(ii) Add the fractions $1/(4m-3) + 1/(4m-1) - 1/(2m)$ *in the* m^{th} *triple.*

(iii) Let S_N *be the sum to* N *terms. Then take the first term separately and write* $m = n+1$ *in the remainder to get*

$$S_N < \frac{5}{6} + \sum_{n=1}^{N} \frac{8(n+1)}{2(n+1)(4n+1)(4n+3)} < \frac{5}{6} + \sum_{n=1}^{N} \frac{8n+8}{2n \times 4n \times 4n} = \frac{5}{6} + \frac{1}{4}\sum_{n=1}^{N}\frac{1}{n^2} + \frac{1}{4}\sum_{n=1}^{N}\frac{1}{n^3} < \frac{5}{6} + \frac{2}{4}\sum_{n=1}^{N}\frac{1}{n^2}.$$

Clearly the sum contains only positive terms, so S_N is an increasing sequence. First term $= 5/6 \approx 0.83 >$ $\log 2$, so the sum is strictly larger than $\log 2$.

8. By considering the sequence of continuous functions $\{f_n(x)\}$ defined for $0 \leq x \leq 1$ by

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{if} 1/n \leq x \leq 1 \end{cases},$$

Answer: Sketch the first few $f_n(x)$ to get a feel for the behaviour. For any fixed $x_0 > 0$ we have $f_n(x_0) = 0$ for all sufficiently large n, i.e. when $n > 1/x_0$. So $\lim_{n\to\infty} f_n(x_0) = 0$ for all $x_0 > 0$. Clearly $f_n(0) = 1$ for all n, so $\lim_{n\to\infty} f_n(0) = 1$ and the limit function is not continuous at $x = 0$.

In Analysis II a stronger form of convergence for sequences of functions is introduced: uniform convergence. It is proved that a uniform *limit of continuous functions is continuous. But in general limits of continuous functions need not be continuous.*

* 9. Explain why it is plausible that

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

By comparing the coefficient of x^3 in this expression with that in the standard power series for $\sin x$, show that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Do you think this is a valid argument?

Answer: Yes, it is a valid argument, but we don't have the theoretical tools to prove it at this point! In simple terms, we know that the standard power series for $\sin x$ converges everywhere in $\mathbb{R}$ - in this sense $\sin x$ is a very 'nice' function. The infinite product formula can, in some sense, be 'multiplied out' to give a power series for $\sin x$; this power series would then have to be the same power series as the standard one.

More detail: The more technical term for functions such as $\sin z$ that are analytic (complex-differentiable) in the whole of $\mathbb{C}$ is 'entire'; see Complex Analysis or Complex Methods. The Weierstrass Factorization Theorem (not in the undergraduate Tripos I think) asserts that entire functions can be written as an infinite product of terms, each of which has only a single zero. Thus both the infinite product and the standard power series for $\sin z$ 'make sense', i.e. are convergent, in a non-empty open set in the complex plane (in fact, in this case they both converge in the whole of $\mathbb{C}$). Hence they must take the same values on this 'overlap' region and we should expect the coefficients to be equal.

10. Let $f(x)$ be a continuous function which has a 'period three orbit', i.e. there exist points $x_0 < x_1 < x_2$ such that $f(x_i) = x_{i+1}$, taking i mod 3. Sketch a possible graph of $y = f(x)$, adding the diagonal line $y = x$.

(i) By considering $g(x) = f(x) - x$ show that there exists a point c such that $f(c) = c$. (This is called a 'fixed point' for $f(x)$.)

* (ii) By considering $h(x) = f(f(x)) - x$, show that there exist points c_1, c_2 (not equal to each other) such that $f(c_1) = c_2$ and $f(c_2) = c_1$. (This is, naturally, called a 'period two orbit'.)

Answer: (i) Consider the interval $[x_1, x_2]$. Note that $g(x)$ is continuous because $f(x)$ is. Also $g(x_1) = x_2 - x_1 > 0$ and $g(x_2) = x_0 - x_2 < 0$; now apply the Intermediate Value Theorem. Note that we must have $x_1 < c < x_2$ since neither x_1 nor x_2 are themselves fixed points.

(ii) Consider the interval $[x_0, x_1]$. As in part (i), $h(x)$ is continuous, and we see that $h(x_0) = x_2 - x_0 < 0$ and $h(x_1) = x_0 - x_1 > 0$. So, applying the IVT we find that there exists a period two point c_1 satisfying $x_0 < c_1 < x_1$.

It is important to check that this point c_1 is NOT the same point as the c we found in part (i): notice that if $f(c) = c$ then $f(f(c)) = c$. If we had considered the interval $[x_1, x_2]$ in part (ii) then we wouldn't have been able to rule this possibility out. A straightforward extension of this argument enables us to find distinct period-n points for any $n > 0$. Hence the dynamics of the iterated map $x_{j+1} = f(x_j)$ are pretty complicated. This application of the IVT is the basis of the result due to Li & Yorke (1975) known by the title of their paper: 'Period three implies chaos'.

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