Series and Limits - Problem sheet

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Starred questions are interesting but less important than the others.

1. Show, using the definitions carefully, that if $\{a_n\}_{n\geq 1} \to a$ and $\{b_n\}_{n\geq 1} \to b$ as $n \to \infty$ then

$$\text{(i)} \quad a_n + b_n \to a + b \quad \text{as} \quad n \to \infty, \qquad \text{(ii)} \quad a_n b_n \to ab \quad \text{as} \quad n \to \infty.$$

2. Find the partial sum S_N of the first N terms of these series (note that they start at different values of n!), and hence determine whether they converge:

$$\text{(i)} \ \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right), \qquad \text{(ii)} \ \sum_{n=0}^{\infty} (-2)^n.$$

3. *Achilles and the Tortoise.* In this well-known paradox due to Zeno, we imagine that the Greek hero Achilles is racing against a tortoise. Sensing that the tortoise is at more than a slight disadvantage, Achilles gives him a headstart of d metres. Both begin to run, at constant but very unequal speeds, at time $t = 0$. The paradox is that it will take Achilles a certain length of time to get to where the tortoise started from, but in that interval the tortoise will have crawled further. Achilles will have to now cover that new distance, but in that time the tortoise will again have crawled forward. So Achilles can, in fact, never overtake the tortoise.

   Refute the paradox, supposing that the tortoise runs at v ms^{-1} and Achilles runs at rv ms^{-1} where $r > 1$, by showing that Achilles and the tortoise are level after a finite time $\frac{d}{v(r-1)}$ seconds.

4. Show that if $S_N \equiv \sum_{n=1}^{N} a_n$ converges then $a_n \to 0$ as $n \to \infty$.

   Hint: $a_m = S_m - S_{m-1}$; use the definition of convergence and the Triangle Inequality. Note that the reverse implication is not true.

5. By rearranging the terms of the absolutely convergent series $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, show that

$$\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

6. (i) Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$, which sums to $\log 2 \approx 0.693$, but which is not absolutely convergent, can be rearranged in the order

$$S = \underbrace{\frac{1}{1} + \frac{1}{3} - \frac{1}{2}}_{m=1} + \underbrace{\frac{1}{5} + \frac{1}{7} - \frac{1}{4}}_{m=2} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}}_{m=3} + \cdots.$$

(ii) By considering the terms of the rearranged series grouped in threes as indicated, show that the series can be written as

$$S = \sum_{m=1}^{\infty} \frac{8m-3}{2m(4m-3)(4m-1)}$$

(iii) Show that the above series for S is convergent by comparison with $\sum 1/n^2$, and that it contains only positive terms. Evaluate the first term and deduce that the sum S is not equal to $\log 2$.

8. By considering the sequence of continuous functions $\{f_n(x)\}$ defined for $0 \le x \le 1$ by

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le 1/n \\ 0 & \text{if} 1/n \le x \le 1 \end{cases},$$

show that the limit function of a sequence of continuous functions is not necessarily continuous.

* 9. Explain why it is plausible that

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

By comparing the coefficient of x^3 in this expression with that in the standard power series for $\sin x$, show that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Do you think this is a valid argument?

10. Let $f(x)$ be a continuous function which has a 'period three orbit', i.e. there exist points $x_0 < x_1 < x_2$ such that $f(x_i) = x_{i+1}$, taking i mod 3. Sketch a possible graph of $y = f(x)$, adding the diagonal line $y = x$.

(i) By considering $g(x) = f(x) - x$ show that there exists a point c such that $f(c) = c$. (This is called a 'fixed point' for $f(x)$.)

* (ii) By considering $h(x) = f(f(x)) - x$, show that there exist points c_1, c_2 (not equal to each other) such that $f(c_1) = c_2$ and $f(c_2) = c_1$. (This is, naturally, called a 'period two orbit'.)

Comments and queries are welcome, and should be sent to `J.H.P.Dawes@damtp.cam.ac.uk`.

October 3, 2006