Preliminary Course 2006

Series and Limits

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'There is an unfortunate, almost snobbish attitude on the part of some writers of textbooks, who present the reader with [the definition of convergence of a sequence] without a thorough preparation, as though an explanation were beneath the dignity of a mathematician.'

R. Courant and H. Robbins *What is Mathematics?* OUP, 1941.

In these two lectures we will look at a rigorous idea of 'a limit', and some of the consequences. This careful reasoning gives us a secure foundation for talking about more complicated ideas such as what it means for a function to be 'continuous' and, later, 'differentiable'. There are important reasons for doing this, both in applied mathematical terms (we need to be able to tell when a mathematical argument is giving answers that are realistic, or not) and in pure mathematical terms where we are called to investigate the consequences of 'natural' mathematical assumptions.

The development of a rigorous idea of 'a limit' hindered mathematical progress in Europe from the time of Zeno of Elea (c. 490BC - c. 430BC) until the mid-seventeeth century with the rapid development of the differential calculus by Leibnitz, Newton and many others. Having struggled with these quasi-philosophical ideas for so long, the foundations were laid for the subsequent developments in the eighteenth and nineteenth centuries (by Gauss, Riemann, Poincaré and many many others) on which modern mathematics is built.

So we do not have to fight and win these battles again, but, as the quote above illustrates, we should take the foundations carefully, and they are not always straightforward, or, indeed, simple to explain.

These notes are in three sections: sequences and convergence, series, and continuity. Everything will be couched in terms of a single real variable $x \in \mathbb{R}$. Extensions to $\mathbf{x} \in \mathbb{R}^n$, are mostly pretty straightforward. Slightly more demanding extensions to $\mathbb{C}$ have some nice consequences which will be fully realised in the Part IB course Complex Analysis - quite far in the future at the moment.

In this summary, definitions of terms are indicated by <u>underlined</u> words. $\Box$ indicates the end of a proof. The 'for all', 'implies' and 'tends to' symbols $\forall$, $\Rightarrow$ and $\rightarrow$ are used frequently. log denotes the natural logarithm, i.e. to base e.

1 Sequences and convergence

Before we discuss series, we will start with sequences which are, of course, very closely related. A sequence $a_1, a_2, a_3, \ldots$, which will also be written $\{a_n\}_{n\geq 1}$ is, for this course, an ordered collection of real numbers. A few examples that you may have thought about before are:

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots \tag{1}$$

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots \tag{2}$$

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots \tag{3}$$

Our first idea of a sequence usually involves 'timelike' motion through a list of values, and we often think something like:

'As I go further along the sequence in (1) the difference between a_n and 0 becomes as small as I want.'

Let's extend this idea to an imaginary dialogue:

'If I went far enough along the sequence, the difference between a_n and 0 would become as small as *you* could demand from me.'

This is the essence of the mathematical definition of a limit; it is subtly different from, and much better than, the 'timelike' notion in which we think about n first and watch how a_n behaves as n increases. What is different? We have **reversed the order**. We first demand that the a_n behave in a certain way, and then, afterwards, work out how large n has to be to guarantee this behaviour.

Convergence should be thought of as a claim made by one participant (let us say, Alice) and challenged by another (let us say, Bob) [1]. Alice claims a_n tends to a limit a as n gets larger and larger. Bob is not so sure, and thinks it might not. He replies that if it does 'converge to a limit', he would expect that, far along the sequence, you get close to a and remain close to a. Bob then states a value ε and says 'I would like you to find a positive integer N such that $|a_n - a| < \varepsilon$ *for all* $n \geq N$. Alice agrees to calculate such an N, and shows Bob that a suitable N can be found, for the value ε that he came up with. Bob tries again, with a smaller ε this time. Alice comes up with a (larger) N for which the condition is still satisfied.

If Alice can calculate N *for any* ε that Bob demands, then Bob admits defeat and the conclusion is that the sequence converges, to the limit a. But if Alice cannot construct an N is every case, i.e. for every $\varepsilon > 0$ then the sequence does not 'converge to the limit a'. Note that the sequence might still possibly converge, it is just that the limit is not a.

Formally:

The sequence $\{a_n\}_{n\geq 1}$ converges to a limit a if, given any $\varepsilon > 0$ there exists an integer $N > 0$ such that $|a_n - a| \leq \varepsilon \quad \forall\, n \geq N$.

[1] Alice and Bob have cameo roles in many courses, including Special Relativity (where they might well be twins), and Quantum Information Theory (where they are forever trying to send each other messages, usually with little success).

We write

$$a_n \to a \text{ as } n \to \infty$$

or, equivalently,

$$\lim_{n\to\infty} a_n = a.$$

Let's go back to our initial examples and apply this definition.

(1) $a_n = 1/n$. The limit is 0 since given an $\varepsilon > 0$, $|a_n - 0| < \varepsilon$ for all $n > 1/\varepsilon$. So we can take N to be the next integer larger than $1/\varepsilon$. This prescription works for all ε, however small.

(2) $a_n = n/(n+1)$. The limit is 1 since given an $\varepsilon > 0$ we find that

$$|a_n - 1| = \left|\frac{n+1-n}{n+1}\right| = \left|\frac{1}{n+1}\right| < \varepsilon \tag{4}$$

So again we can take $n > 1/\varepsilon$ and satisfy the definition of convergence.

(3) $a_n = (-1)^n$. The limit is not 1 since if we're given $\varepsilon = 1/2$ then I can't find an N such that $|(-1)^n - 1| < \varepsilon$ for all $n \geq N$. Similarly, the limit is not -1. In fact, no fixed real number a works, so the sequence does not converge.

It is useful at this point to state a defining property of the real numbers that we will need, and a very useful inequality.

Axiom: Any increasing sequence in $\mathbb{R}$, which is bounded above, converges.

This immediately implies, incidentally, that (2) converges because $a_n < a_{n+1} < 1$ for all n. Note that these strict inequalities are relaxed to $\leq$ for the limit: $\lim_{n\to\infty} a_n \leq 1$ - compare this with $\lim_{n\to\infty} 1/n = 0$.

Theorem 1 (The Triangle Inequality) *Let a, b and c be real numbers. Then*

$$|a - b| \leq |a - c| + |c - b|$$

Note: this is true in $\mathbb{R}^n$.

Proof: (which you should write out the details of). By multiplying out, show that, for $x, y \in \mathbb{R}$: $(|x| + |y|)^2 \geq (x + y)^2$. Then set $x = a - c$, $y = c - b$. □

Now that we have a definition of a limit, we should be able to prove some 'obvious' properties of limits. Here is one such 'obvious' property proved in complete detail. As exercises, there are other properties for which the proof is very similar: you are strongly encouraged to write out proofs of these, along similar lines.

Theorem 2 (Limits are unique) *Suppose $a_n \to a$ as $n \to \infty$ and $a_n \to b$ as $n \to \infty$. Then $a = b$.*

Proof: Suppose $a \neq b$. We will deduce a contradiction. Set, rather stringently, $\varepsilon = \frac{1}{4}|a - b| > 0$. Then, from the definitions of convergence we know

$$\begin{array}{c}\text{there exists } N_1\text{: } \forall\ n > N_1,\ \ |a_n - a| < \varepsilon \\ \text{there exists } N_2\text{: } \forall\ n > N_2,\ \ |a_n - b| < \varepsilon\end{array}$$

Consider $n > \max\{N_1, N_2\}$. Then

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon = \frac{1}{2}|a - b|$$

which is a contradiction. So in fact $|a - b| = 0$, i.e. $a = b$. □

2 Series

Let a_n be a sequence in $\mathbb{R}$. The corresponding series is given by summing the terms: $\sum_{n=1}^{\infty} a_n$. Being careful, we really only want to sum the first N terms (this is called the N^{th} *partial sum* S_N) and then look at the limit as we allow $N \to \infty$:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N \quad = \quad \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

if this exists (i.e. if this is a convergent limit). So in fact, to investigate the convergence of this sum, we are back to a question about sequences; the convergence of the sequence of partial sums S_N. So we can apply the definition for the convergence of sequences that we already have.

Some well-known series (defined by their partial sums) are

- $S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$
  This is called the Harmonic Series. It does not converge as $N \to \infty$.

- $S_N = 1 + r + r^2 + r^3 + r^4 + \cdots + r^N$
  This is a Geometric Series (the term-by-term sequence is called a Geometric Progression). It converges if $|r| < 1$ as we will prove, in a moment.

- $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$
  Does this converge? What might be the limit?

- $S_N = \sum_{n=1}^{N} \frac{1}{n!}$ Does this converge? What might be the limit?

The first example we did, and the first in the list above, shows us that $a_n \to 0$ is not a sufficient condition to ensure convergence. But it is a necessary condition for convergence; there is a helpful hint on the problem sheet to enable you to prove this result:

Theorem 3 *If* $S_N \equiv \sum_{n=1}^{N} a_n$ *converges then* $a_n \to 0$ *as* $n \to \infty$.

Proof: left as an exercise!

In examples we come across in real life, on problem sheets, and in exam questions, most of the time we prove convergence by making comparisons between the series we have been given and standard, simpler, ones. These comparison ideas will be developed further in Analysis I where various tests for convergence or the lack of it will be derived (the Comparison Test, the Integral Test etc). We will consider only one (but possibly the most important one) of these here; testing against a Geometric Series.

2.1 Geometric Series and the Ratio Test

The standard geometric progression has partial sums

$$S_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots + r^N.$$

As this has only a finite number of terms we can manipulate it with ease:

$$\begin{aligned} rS_N - S_N &= r^{N+1} - 1 \\ \Rightarrow S_N &= \frac{1 - r^{N+1}}{1 - r} \end{aligned}$$

so we would guess that if $|r| < 1$ we have $S_N \to 1/(1 - r)$ as $N \to \infty$.

Proof: Assume $|r| < 1$, then

$$\left| S_N - \frac{1}{1-r} \right| = \left| \frac{r^{N+1}}{1-r} \right| \leq \left| \frac{r}{1-r} \right| |r|^N = C|r|^N$$

and we can guarantee that $C|r|^N < \varepsilon$ by taking $N > \frac{\log(\varepsilon/C)}{\log|r|}$.
So we have shown that we can make the difference between S_N and the limit as small as we please by taking N large enough. Hence S_N converges to the limit we guessed. □

Making a comparison with a geometric series is the idea behind the Ratio Test, due to D'Alembert.

Theorem 4 (The Ratio Test) *Let* $\{a_n\}_{n\geq 1}$ *be a sequence with only positive terms,* $a_n > 0$ *for all* n*. Then, if* $a_n/a_{n-1} \to \ell < 1$ *as* $n \to \infty$ *then the partial sums* $S_N \equiv \sum_{n=1}^{N} a_n$ *converge as* $N \to \infty$.

Proof: Since $\ell < 1$ we can pick an $\varepsilon > 0$ such that $\ell + \varepsilon < 1$. Then, for this ε we know that, by convergence of the ratio a_n/a_{n-1}, there exists an N such that

$$\left| \frac{a_n}{a_{n-1}} - \ell \right| < \varepsilon \quad \forall \quad n \geq N.$$

So, since $a_n > 0$ we know that $a_n/a_{n-1} < \ell + \varepsilon$ for all $n > N$. Hence, forming a product of $n - N$ of these ratios:

$$\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{N+1}}{a_N} < (\ell + \varepsilon)^{n-N}$$

So we can cancel a large number of terms on the left-hand side, and tidy up the equation to read

$$a_n \quad < \quad \frac{a_N}{(\ell+\varepsilon)^N} (\ell + \varepsilon)^n \quad \forall \, n > N.$$

Now we consider N as fixed, and consider the sum of terms $\tilde{S}_M$ from $n = N$ up to some higher limit $M > N$:

$$\tilde{S}_M = \sum_{n=N}^{M} a_n < \frac{a_N}{(\ell+\varepsilon)^N} \sum_{n=N}^{M} (\ell + \varepsilon)^n, \tag{5}$$

but the right-hand side is a Geometric Series and hence converges to a limit L as $M \to \infty$, so $\tilde{S}_M$ is an increasing sequence (in M, considering N to be fixed) which is bounded above, by L and hence, by the axiom for the real numbers, S_M converges. Comparing $\tilde{S}_M$ and the partial sums of our original series S_N we see that they differ only by a finite number of terms: $a_1 + \cdots + a_{N-1}$. Hence if $\tilde{S}_M$ converges as $M \to \infty$, so does $\sum_{n=1}^{\infty} a_n$. □

2.2 Absolute convergence

Absolute convergence is a stronger property than convergence; it gives us a class of series that are really well-behaved. The definition is

> The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.
> As one might hope, absolute convergence implies convergence.
> Series can be divided into three classes:
>
> - those that diverge,
> - those that converge but are not 'absolutely convergent',
> - those that are absolutely convergent.

The simplest example of a series in the middle class (also known as a 'conditionally convergent' series) is $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$ - we saw above that by re-arranging the terms of this series we could make it either converge to a limit that was not $\log 2$, or indeed diverge. But for absolutely convergent series we can prove that we cannot change the convergence properties by re-ordering the terms.

This robustness to changing the summation order is an important feature if we wish to define functions $f(x)$ using series; we would be in serious difficulties if, for a fixed value of x, $f(x)$ had more than one possible value. We say that $f(x)$ is 'well-defined' if for a fixed x, $f(x)$ has only one possible value. Defining functions using (absolutely convergent) series turns out to be a very useful and, particularly in the complex plane, a natural, thing to do.

2.3 Power series

We define the series $\sum_{n=1}^{\infty} a_n x^n$ to be the power series with coefficients $\{a_n\}$. The description below will stick to $x \in \mathbb{R}$, but you should be aware that x can be replaced by $z \in \mathbb{C}$ with little change.

Most importantly, we need to know for what range of x is a power series convergent? The Ratio Test provides one method to establish this.

Examples:

(i) $f(x) = \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. This converges absolutely for all $x \in \mathbb{R}$ by the Ratio Test:

$$\left|\frac{x^{n+1}}{(n+1)!}\right| \Big/ \left|\frac{x^n}{n!}\right| \quad = \quad \frac{|x|}{n+1}$$

which tends to zero as $\to \infty$ for any fixed x, and so there definitely exists an N such that it is less than 1 for all $n > N$.

(ii) Similarly it can be shown that

$$\sin x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

converge (see problem sheet).

(iii) Consider $f(x) = \sum_{n=0}^{\infty} x^n$. By the Ratio Test we have

$$\frac{|x^{n+1}|}{|x^n|} = |x|$$

so we have convergence guaranteed if $|x| < 1$ (and in this case, no convergence if $|x| \geq 1$).

Examples such as the last of these lead to the idea that a power series may not make sense for all x; the idea of a 'radius of convergence':

The power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence R if $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < R$, and diverges for all x with $|x| > R$.

Notes: (i) we may or may not have convergence for $|x| = R$.

(ii) There are two exceptional cases: maybe $\sum_{n=1}^{\infty} a_n x^n$ never converges for non-zero x; in this case we set $R = 0$, or maybe $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$; in this case we might say '$R = \infty$'.

There is, therefore, a bit of care needed in stating something like

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{6}$$

because the left-hand side make sense for all $x \neq 1$ whereas the right-hand side only make sense if $|x| < 1$. So this equality only makes sense for $|x| < 1$ (otherwise $1+2+4+8+16+\cdots = -1$)!

It is relatively straightforward to show that if $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence R then the term-by-term derivative $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for all $|x < R$ and, in fact, has the same radius of convergence R as the original series. So within the radius of convergence we can differentiate as we would expect, as many times as we wish. So, staying within $|x| < 1$, we can use (6) to deduce power series for $1/(1-z)^2$, $1/(1-z)^3$ etc.

3 Continuity

Informally, a continuous function is one that can be drawn without taking your pen off the paper. A standard example of a function that is not continuous is $f(x) = 1/x$. Thinking more carefully, it appears that $1/x$ is actually fine as long as we keep away from $x = 0$. More complicated problems might be encountered in functions such as

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Is this continuous anywhere?

Our initial idea of continuity is probably similar to the initial idea of convergence; as x gets closer and closer to some point a, we hope that $f(x)$ tends towards $f(a)$. As before, to make this a rigorous statement, we return to the idea of a challenge, or a game, between Alice and Bob. First, Alice fixes a point $x = a$ to check continuity at. Then Bob comes up with a value for ε measuring how close $f(x)$ should be to $f(a)$. Alice then has to try to calculate how close x has to be to a in order to guarantee Bob's condition is satisfied.

Formally,

$f(x)$ is <u>continuous at $x = a$</u> if, for any given $\varepsilon > 0$ we can find a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Notice that, as with the definition of the limit of a sequence, we have reversed the order in which we consider the independent (i.e. x) and dependent (i.e. f) variables; first assert how close you require $f(x)$ to be to $f(a)$, then go and compute how close x has to be to a to guarantee that this happens.

Finally, we say that $f(x)$ is <u>continuous</u> if $f(x)$ is continuous at all $x \in \mathbb{R}$.

3.1 The Intermediate Value Theorem

The rigorous definition of what it means to be a continuous function is necessary to prove, carefully, a number of 'obvious' results. The most important of these is the statement that a function which attains positive and negative values has at least one zero value. Continuity is crucial to this - consider for example the function defined by

$$f(x) = \begin{cases} +1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

which has no zeros, but which is also not continuous at any $x \in \mathbb{R}$.

Lemma 5 (Functions of sequences) *Suppose the sequence* $x_n \to a$ *as* $n \to \infty$*, and* $f(x)$ *is a function that is continuous at* $x = a$*. Then* $f(x_n) \to f(a)$ *as* $n \to \infty$*.*

Proof: Using the fact that f is continuous at $x = a$ we have that given any $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$|x - a| < \delta \qquad \Rightarrow \qquad |f(x) - f(a)| < \varepsilon.$$

Since the sequence $\{x_n\}$ converges, for the δ we need to use above, we know that there exists an N such that $|x_n - a| < \delta$ for all $n \geq N$. Putting these two statements together we have:

$$\forall\, n \geq N \qquad \text{we know that} \qquad |f(x_n) - f(a)| < \varepsilon, \tag{7}$$

and this is true for any given ε; that is, for any given ε we can find an N such that (7) holds. And this is exactly the definition of what it means for the sequence $\{f(x_n)\}$ to converge to the limit $f(a)$. □

Note: a convenient notation we shall use is to denote the closed interval $a \leq x \leq b$ by $[a, b]$. 'Closed' means that the endpoints $x = a$ and $x = b$ are included.

Theorem 6 (The Intermediate Value Theorem) *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, and* $f(a) < 0 < f(b)$*. Then there exists a value* c *such that* $a < c < b$ *and* $f(c) = 0$*.*

Proof: This argument is sometimes called 'repeated bisection': we will construct convergent sequences that locate a value c for which $f(c) = 0$.

First, set $a_0 = a$ and $b_0 = b$. We will construct a sequence of 'nested' subintervals $[a_n, b_n]$; 'nested' means $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$.

The inductive step in the construction is determined by checking the value of $f(x)$ at the midpoint of the interval $[a_n, b_n]$:

- If $f\left(\frac{a_n+b_n}{2}\right) > 0$ then let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$.
- If $f\left(\frac{a_n+b_n}{2}\right) < 0$ then let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$.

Of course, if $f\left(\frac{a_n+b_n}{2}\right) = 0$ then we have found a value for c and we stop here.

So, from the construction, we can see that the sequence of left-hand endpoints $\{a_n\}$ is increasing and bounded above by b, hence the sequence converges, say to a value ℓ. By our lemma above, since f is continuous, $f(a_n) \to f(\ell)$ as $n \to \infty$. Moreover, because $f(a_n) < 0$ for all n, we have $f(\ell) \leq 0$ (notice the loss of a strict inequality in passing to the limit).

Also, the sequence of right-hand endpoints $\{b_n\}$ is decreasing and bounded below, by a and converges. since $|b_n - a_n| = |b_0 - a_0|/2^n$ we in fact see that $\{b_n\}$ converges to the same limit ℓ. Since $f(b_n) > 0$ for all n, we know that $f(\ell) \geq 0$. The only way these conclusions can be reconciled is if $f(\ell) = 0$ and so we have found a root of $f(x)$. $\square$