3 Lecture 3: From PDEs to ODEs

In this lecture we will apply the theoretical framework outlined in the first two lectures to pattern formation problems on the plane. Many continuum physical systems can be described in terms of PDEs, often either reaction–diffusion systems or Navier–Stokes equations. Very often these physical systems show instabilities of an initially spatially uniform and time-independent state that result in a spatially periodic regular pattern. The symmetric bifurcation theory we have discussed enables us to draw up a classification *a priori* of possible spatially periodic patterns. The amplitude equations describing the branching behaviour and stability are determined by equivariance, and the specific physics of a system determines the values of the coefficients of nonlinear terms in the amplitude equations. The values of the coefficients determine which possible pattern is selected by the system.

The first step is to see how we can reduce a set of PDEs to a set of symmetric ODEs. Then we analyse the properties of the ODEs. We will focus only on PDEs that are Euclidean-symmetric and defined on a physical domain $\mathbb{R}^2 \times \Omega$ where Ω is bounded (for example the vertical direction for a fluid layer). We will see that there are various difficulties associated with the unbounded domain $\mathbb{R}^2$ and we in fact will circumvent this in a rather drastic way: we will consider instead a bounded region $D \subset \mathbb{R}^2$ with periodic boundary conditions on the edges.

3.1 Euclidean symmetry

The general framework which we consider applies to any physical problem governed by a set of m PDEs for a function $u : \mathbb{R}^2 \times \Omega \to \mathbb{R}^m$ with a bifurcation parameter we denote by μ. We write the PDEs schematically as

$$\frac{\partial u(x, \omega, t)}{\partial t} = \mathcal{F}(u, \mu), \tag{10}$$

for $x \in \mathbb{R}^2$, $\omega \in \Omega$, and assume there exists a trivial solution $u(x, \omega, t) = 0$ for all values of the real parameter μ. The state $u = 0$ is assumed to be invariant under the group $E(2)$ of Euclidean transformations in the plane. Exploring this in a bit more detail, we start from the natural action of $E(2)$ on $\mathbb{R}^2$. $E(2)$ is generated by rotations about the origin ρ^θ, a reflection m_x and translations $\mathcal{T}_{(\xi,\eta)}$ acting on (x, y) by:

$$\begin{aligned} R_\theta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ m_x(x_1, x_2) &= (x_1, -x_2), & (11) \\ T_{(\xi,\eta)}(x_1, x_2) &= (x_1 + \xi, x_2 + \eta). & (12) \end{aligned}$$

The elements R_θ and m_x generate the group $O(2)$ and hence $E(2) = O(2) \ltimes \mathbb{R}^2$, a semi-direct product of $O(2)$ and the translation group $\mathbb{R}^2$. This action of $E(2)$ on $\mathbb{R}^2$ leads to an action on a space $\mathcal{F}$ of (vector-valued) functions on $\mathbb{R}^2 \times \Omega$:

$$\begin{aligned} gu(x, \omega, t) &= \tilde{\rho}(g)u(g^{-1}x, \omega, t) & (13) \\ T_{(\xi,\eta)}u(x_1, x_2, \omega, t) &= u(x_1 - \xi, x_2 - \eta, \omega, t) & (14) \end{aligned}$$

where $g \in O(2)$ is composed of rotations R_θ and the reflection m_x. $\tilde{\rho}(g)$ is a representation of $O(2)$ on $\mathbb{R}^m$ which for this section we will take to be trivial (the <u>scalar</u> representation). Different choices for the representation ρ lead to <u>pseudoscalar</u> representations and rather different pattern forming behaviour. We will discuss this in section 3.5. In the action (13)-(14) we have assumed that g acts trivially on the bounded variables $\omega \in \Omega$. We assume that the $E(2)$-invariant state $u = 0$ loses stability to perturbations $\tilde{e}^{i\mathbf{k}\cdot\mathbf{x}}$ at a finite wavenumber $|\mathbf{k}| = k_c$ via a steady-state bifurcation at $\mu = 0$. This is the typical situation, but presents two serious problems from a bifurcation point of view:

Lattice	$H_{\mathcal{L}}$	ℓ_1	ℓ_2	$\mathbf{k}_1$	$\mathbf{k}_2$
Square	D_4	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$
Hexagonal	D_6	$(1,1/\sqrt{3})$	$(0,2/\sqrt{3})$	$(1,0)$	$(-1,\sqrt{3})/2$
Rhombic	D_2	$(1,-\cot\phi)$	$(0,\mathrm{cosec}\phi)$	$(1,0)$	$(\cos\phi,\sin\phi)$

Table 1: Holohedries and generators for planar lattices generated by two equal length vectors. For rhombic lattices $0<\phi<\pi/2$, $\phi\neq\pi/3$. Taken from [27], p130, table 5.1.

- wavenumbers near $k=k_c$ have growth rates that are arbitrarily close to zero,
- all Fourier modes $e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x})}$ with $|\mathbf{k}|=k_c$ are equivalent, and all are simultaneously marginal at the bifurcation point.

The first of these problems means that it is impossible to apply the centre manifold theorem directly to the PDE: we cannot separate the modes of instability into 'active' modes with neutral or positive growth rates and 'passive' modes with a growth rate uniformly less than $-\delta$ for some $\delta>0$ that could be eliminated by centre manifold reduction.

The second problem means that at the bifurcation point there is an uncountable number of modes corresponding to every horizontal direction: our 'centre manifold', if it existed, would have to be infinite dimensional.

Happily there is a way around both these problems that has been used repeatedly in the literature. It has the advantages of simplicity and relevance to numerical work, and the disadvantage in that with it we are able only to analyse 'parts' of the problem. This approach is to restrict our attention to perturbation modes which are periodic with respect to a doubly-periodic lattice in the plane. This corresponds numerically to 'approximating' unbounded domains with 'large' domains and applying periodic boundary conditions.

3.2 Spatially periodic lattices

A planar lattice $\mathcal{L}$ is a set of integer linear combinations of two independent vectors ℓ_1 and ℓ_2:

$$\mathcal{L} \quad = \quad \{n\ell_1+m\ell_2 : \; n,m\in\mathbb{Z}\}$$

There are three lattices generated by two wavevectors of the same length: square (vectors at 90°), hexagonal (vectors at 120°) and rhombic (vectors aligned at some other angle). The function space $\mathcal{F}$ restricts to a space of doubly-periodic functions:

$$\mathcal{F}_{\mathcal{L}} \quad = \quad \{u\in\mathcal{F} : u(x+\ell,\omega,t)=u(x,\omega,t)\ \forall\ \ell\in\mathcal{L}\}. \qquad (15)$$

Because $\mathcal{F}_{\mathcal{L}}$ is the fixed point subspace of the action of $\mathcal{L}$ on $\mathcal{F}$ it is flow-invariant and so solutions that are doubly-periodic are also solutions to the original PDE. Translational symmetries also decompose $\mathcal{F}$ into isotypic components made up of different Fourier components: this is a different way to say that eigenfunctions of the linearised operator for the PDE are Fourier plane waves.

On the subspace $\mathcal{F}_{\mathcal{L}}$ we no longer have the full symmetry group $E(2)\equiv O(2)\ltimes\mathbb{R}^2$. The Euclidean symmetries lead to a smaller symmetry group $\Gamma_{\mathcal{L}}=H_{\mathcal{L}}\ltimes T^2_{\mathcal{L}}$ which is a semi-direct product of two groups. $T^2_{\mathcal{L}}\cong\mathbb{R}^2/\mathcal{L}$ is the group of translational symmetries of the system modulo the lattice translations (which preserve the lattice itself). It is a normal subgroup of $\Gamma_{\mathcal{L}}$. The group $H_{\mathcal{L}}$ is the subgroup of $O(2)$ (rotations and reflections about the origin) which preserves the lattice. It is often referred to as the holohedry of the lattice. Generators for the square, hexagonal and rhombic lattices are given in table 1

To each lattice we associate a dual lattice $\mathcal{L}^*$ generated by two linearly independent wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ that satisfy $\mathbf{k}_i\cdot\ell_j=\delta_{ij}$ (the Kronecker δ_{ij}):

$$\mathcal{L}^* \quad = \quad \{n\mathbf{k}_1+m\mathbf{k}_2 : \; n,m\in\mathbb{Z}\}.$$

The dual lattices for the square and rhombic cases are illustrated in figure 3.2. The dual lattice in the hexagonal case is illustrated in figure 3.3.

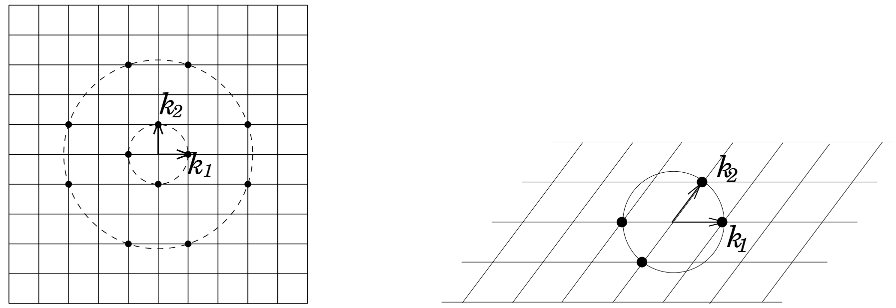

Figure 6: (a) Dual lattice $\mathcal{L}^*$ for the square case. Two critical circles are shown, intersecting the lattice in 4 and 8 points respectively. (b) Dual lattice $\mathcal{L}^*$ for the rhombic case.

3.3 Steady bifurcation on a hexagonal lattice

Many spatially extended physical systems are observed to form regular patterns with hexagonal symmetry. In this section we will show how symmetric bifurcation behaviour enables us to derive amplitude equations that model the appearance of hexagonal patterns, and gives stability conditions that, in turn, control pattern selection. We will consider only the scalar case. The results in this section are discussed by Hoyle [32], section 5.4; by Golubitsky & Stewart [27], and by Golubitsky et al [25] in 'case study' 4. The original references are papers by Buzano & Golubitsky [7] and Golubitsky, Swift and Knobloch [24].

For an $E(2)$-equivariant PDE all horizontal directions are equivalent. Restricting attention to the lattice of allowed wavevectors $\mathcal{L}^*$ we find, in the hexagonal case, that typically the lattice intersects the circle of critical wavevectors in 6 or 12 points. By taking finer and finer lattices $\mathcal{L}^*$ we can in fact ensure that there are as many intersection points as we wish between the circle and $\mathcal{L}^*$. This corresponds to considering a larger and larger physical domain with periodic boundary conditions. Since the observed experimental patterns have a wavelength which is far smaller than the typical box size, we restrict our attention to the cases with small numbers of intersection points, and hence coarser lattices. Finer lattices lead to the presence of 'hidden symmetries' which affect the calculations in intuitively natural, but slightly complicated, ways [18; 15]. The 6 point intersection is the simplest, and called the 'fundamental' representation.

The 'fundamental' case

Suppose that a solution to the linear problem is $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}f(\omega)$ where $\mathbf{k}$ lies on the critical circle $|\mathbf{k}| = k_c$ and $f(\omega)$ is a vector in $\mathbb{R}^m$ that describes the solution for the bounded variables. Then we look for a solution in the form

$$u(x,\omega,t) \;=\; \sum_{j=1}^{3} z_j(t)\mathrm{e}^{\mathrm{i}\mathbf{k}_j\cdot\mathbf{x}}f(\omega) + \text{c.c.} \tag{16}$$

where $z_1,\ldots,z_3$ are amplitudes of the three plane waves, and $+c.c.$ denotes the complex conjugate, in order that $u(x,\omega,t)$ is real.

The $E(2)$ symmetry is reduced to $D_6 \ltimes T^2$ which acts on the centre manifold $\mathbb{C}^3$ with coordinates (z_1, z_2, z_3) by

$$\rho(z_1,z_2,z_3) \;=\; (\bar{z}_2,\bar{z}_3,\bar{z}_1) \tag{17}$$

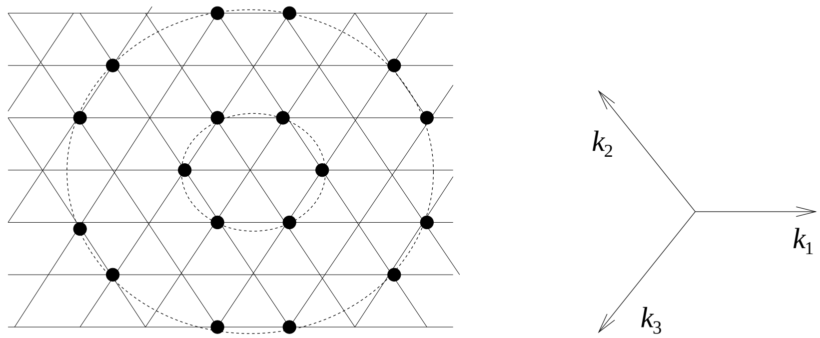

Figure 7: (a) Dual lattice $\mathcal{L}^*$ for the hexagonal case. Two critical circles are shown, intersecting the lattice in 6 and 12 points respectively. (b) Wavevectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ for the fundamental case. Note that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ so $\mathbf{k}_1$ and $\mathbf{k}_2$ are sufficient to generate the lattice.

$$m_x(z_1, z_2, z_3) = (z_1, z_3, z_2) \tag{18}$$
$$\tau_{\mathbf{p}}(z_1, z_2, z_3) = (z_1 \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{p}}, z_2 \mathrm{e}^{-\mathrm{i}\mathbf{k}_2\cdot\mathbf{p}}, z_3 \mathrm{e}^{-\mathrm{i}\mathbf{k}_3\cdot\mathbf{p}}). \tag{19}$$

where $\mathbf{p} = (p_1, p_2)$ is a translation vector in the two-torus $T^2 = \mathbb{R}^2/\mathcal{L}$. This group action follows directly from the definition of the action on functions: $gu(x) \equiv u(g^{-1}x)$. For example, in detail for the $\pi/3$ rotation ρ:

$$\begin{aligned}
\rho u(x) &\equiv u(\rho^{-1}x) = \rho(z_1 \mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + z_2 \mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + z_3 \mathrm{e}^{\mathrm{i}\mathbf{k}_3\cdot\mathbf{x}} + \text{c.c.}) \\
&= z_1 \mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\rho^{-1}\mathbf{x}} + z_2 \mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\rho^{-1}\mathbf{x}} + z_3 \mathrm{e}^{\mathrm{i}\mathbf{k}_3\cdot\rho^{-1}\mathbf{x}} + \text{c.c.} \\
&= z_1 \mathrm{e}^{\mathrm{i}\rho\mathbf{k}_1\cdot\mathbf{x}} + z_2 \mathrm{e}^{\mathrm{i}\rho\mathbf{k}_2\cdot\mathbf{x}} + z_3 \mathrm{e}^{\mathrm{i}\rho\mathbf{k}_3\cdot\mathbf{x}} + \text{c.c.} \\
&= z_1 \mathrm{e}^{-\mathrm{i}\mathbf{k}_3\cdot\mathbf{x}} + z_2 \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + z_3 \mathrm{e}^{-\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.}
\end{aligned}$$

i.e. the action on (z_1, z_2, z_3) is as in (17).

Axial branches and isotropy subgroups

We can easily check that the representation of $D_6 \ltimes T^2$ given by (17) - (19) is absolutely irreducible. Then the equivariant branching lemma can be applied to determine the generic branching behaviour in the bifurcation. From table 2 we see that there are two axial branches: *rolls* and *hexagons*. There are two types of hexagons because points (x, x, x) with $x > 0$ and $x < 0$ are not on the same group orbit. Physically there are differences in the hexagonal planforms they represent; see figure 8. Up-hexagons have isolated regions of up-flow ('plumes' in the thermal convection setting) surrounded by connected regions of downflow. For down-hexagons the flow directions are reversed in the two regions. There is no 'up/down' symmetry connecting the two flow states: in the case that there is an up/down symmetry we have a different bifurcation problem: this case is discussed in section 3.4.

We will discuss the computation of amplitude equations informally, arguing only for the form of the equations up to cubic order, rather than presenting a general formulation. Considering the first amplitude equation $\dot{z}_1 = f_1(z_1, z_2, z_3)$ we see that translation equivariance (19) means that the only terms that can appear at quadratic and cubic orders are

$$\bar{z}_2\bar{z}_3, \qquad z_1|z_1|^2, \qquad z_1|z_2|^2, \qquad z_1|z_3|^2.$$

The equivariance of the first of these follows from

$$\tau_{\mathbf{p}}(\bar{z}_2\bar{z}_3) = \bar{z}_2 \mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{p}} \bar{z}_3 \mathrm{e}^{\mathrm{i}\mathbf{k}_3\cdot\mathbf{p}} = \bar{z}_2\bar{z}_3 \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{p}}$$

	Point (z_1, z_2, z_3)	Isotropy subgroup Σ (generators)	dim Fix(Σ)
Trivial	$(0,0,0)$	$D_6 \ltimes T^2$	0
Rolls	$(x,0,0)$	$O(2) \times \mathbb{Z}_2$	1
		m_x, $\tau_{(0,p_2)}$, ρ^3	
+ Hexagons	(x,x,x), $x > 0$	D_6	1
		ρ, m_x	
− Hexagons	(x,x,x), $x < 0$	D_6	1
		ρ, m_x	
Rectangles	(x_1, x_2, x_2)	$D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	2
		ρ^3, m_x	
Triangles	(z,z,z)	D_3	2
		m_x, ρ	

Table 2: Scalar bifurcation on the hexagonal lattice, 'fundamental'case: fixed point subspaces and isotropy subgroups. $x_j \in \mathbb{R}$, $z \in \mathbb{C}$. The +/− Hexagon branches have the same symmetry but are not on the same group orbit.

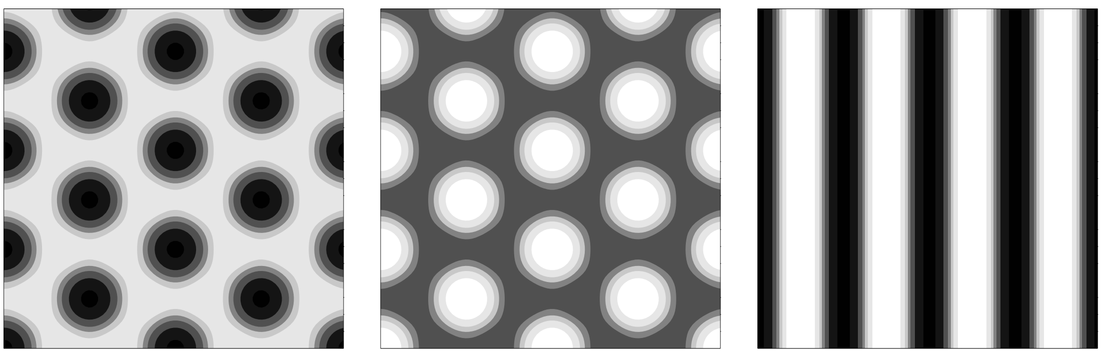

Figure 8: Planforms for axial branches for bifurcation on a hexagonal planar lattice: (a) up-hexagons; (b) down-hexagons; (c) rolls.

so $\bar{z}_2\bar{z}_3$ transforms as z_1 under translational symmetries. Equivariance with respect to the π rotation ρ^3 implies

$$\begin{aligned} \rho^3 \dot{z}_1 &= \rho^3 f_1(z_1, z_2, z_3) = f_1(\rho^3(z_1, z_2, z_3)) = f_1(\bar{z}_1, \bar{z}_2, \bar{z}_3) \\ \Rightarrow \dot{\bar{z}}_1 &= f_1(\bar{z}_1, \bar{z}_2, \bar{z}_3) = \overline{f_1(z_1, z_2, z_3)} \end{aligned}$$

so all coefficients in $f_1(z_1, z_2, z_3)$ are forced to be real. Also, the reflection symmetry m_x forces the amplitude equation to be symmetric in z_2 and z_3. Hence the first amplitude equation takes the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, z_3) \equiv \mu z_1 + \varepsilon \bar{z}_2 \bar{z}_3 + a_1 z_1 |z_1|^2 + a_2 z_1 (|z_2|^2 + |z_3|^2) \\ &\quad + \bar{z}_2 \bar{z}_3 (b_1 |z_1|^2 + b_2 (|z_2|^2 + |z_3|^2) + c_1 z_1^2 z_2 z_3 + O(5). \end{aligned} \tag{20}$$

Applying the rotations ρ^2 and ρ^4 yields the amplitude equations for z_2 and z_3:

$$\dot{z}_2 = f_2(z_1, z_2, z_3) \equiv \mu z_2 + \varepsilon \bar{z}_1 \bar{z}_3 + a_1 z_2 |z_2|^2 + a_2 z_2 (|z_1|^2 + |z_3|^2)$$

$$+\bar{z}_1\bar{z}_3(b_1|z_2|^2 + b_2(|z_1|^2 + |z_3|^2) + c_1 z_2^2 z_1 z_3 + O(5), \tag{21}$$

$$\begin{aligned} \dot{z}_3 &= f_3(z_1, z_2, z_3) \equiv \mu z_3 + \varepsilon \bar{z}_1 \bar{z}_2 + a_1 z_3|z_3|^2 + a_2 z_3(|z_1|^2 + |z_2|^2) \\ &\quad + \bar{z}_1\bar{z}_2(b_1|z_3|^2 + b_2(|z_1|^2 + |z_2|^2) + c_1 z_3^2 z_1 z_2 + O(5). \end{aligned} \tag{22}$$

From the amplitude equations we can write down branching equations for the various solution types, and hence calculate stability in terms of the various normal form coefficients.

Stability of rolls and hexagons

Stability calculations can, of course, be carried out by brute force, writing down the 6×6 Jacobian matrix from (20) - (22) and evaluating it on each solution branch. Such calculations reveal, for example, that although only the axial branches of rolls and hexagons exist near the bifurcation point, a branch of rectangle equilibria exists in $\mu > 0$ for an open set of coefficients and gives rise to secondary bifurcations, as shown in figure 9.

But there is a faster way, exploiting the symmetric structure of the Jacobian, and the isotypic decomposition discussed in section 1.2. By theorem 2 the isotypic decomposition gives a set of coordinates in which the Jacobian $L \equiv Df(0,0)$ is block-diagonal. Finding these coordinates vastly simplifies the eigenvalue calculations. First we choose a solution branch and its isotropy subgroup Σ. Then, to find the right coordinates we write $\mathbb{R}^n$ as a direct product of subspaces on which Σ acts irreducibly. If Σ acts by a different irreducible representation on each subspace then life is very straightforward: the Jacobian is diagonal, and we can read off the eigenvalues as these diagonal entries. We now summarise the details for rolls and hexagons: the results are presented in table 3.

Rolls. The isotropy subgroup $\Sigma_R \cong O(2) \times \mathbb{Z}_2$ acts as the identity I on the one dimensional subspace $\mathrm{Fix}(\Sigma_R) = (x, 0, 0)$. This is the first isotypic component.

On the subspace $(\mathrm{i}x, 0, 0)$ we see that m_x and $\tau_{(0,\eta)}$ act as I and ρ^3 acts as $-I$. So $O(2) \times \mathbb{Z}_2$ acts as $\mathbb{Z}_2$ on this subspace, another irreducible representation. With respect to the usual coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ where $z_j = x_j + \mathrm{i}y_j$, and writing $f_j(z_1, z_2, z_3) = f_j^r + \mathrm{i}f_j^i$ in real and imaginary parts, we can compute that these two one-dimensional isotypic components give

$$L \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial f_1^r}{\partial x_1} & \frac{\partial f_1^r}{\partial y_1} & \cdots & \\ \frac{\partial f_1^i}{\partial x_1} & \frac{\partial f_1^i}{\partial y_1} & \cdots & \\ \vdots & & & \\ & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \lambda = \frac{\partial f_1^r}{\partial x_1}$$

and

$$L \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial f_1^r}{\partial x_1} & \frac{\partial f_1^r}{\partial y_1} & \cdots & \\ \frac{\partial f_1^i}{\partial x_1} & \frac{\partial f_1^i}{\partial y_1} & \cdots & \\ \vdots & & & \\ & & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \lambda = \frac{\partial f_1^i}{\partial y_1}$$

We can now evaluate the derivatives $\frac{\partial f_1^r}{\partial x_1}$ and $\frac{\partial f_1^i}{\partial y_1}$ on the solution branch. The first one of these has sign $\mathrm{sgn}(a_1)$ which gives the direction of branching. The second gives zero corresponding to neutral stability to perturbations in the direction of the wavevector $\mathbf{k}_1$: there is a continuous group orbit of roll solutions given by translations in the $(1,0)$ direction.

For the remaining eigenvalues, consider the (four real dimensional) subspace $(0, z_2, z_3)$ on which $m_x(z_2, z_3)$ acts as $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$, where I_2 denotes the 2×2 identity matrix, $\rho^3(z_2, z_3) = (\bar{z}_2, \bar{z}_3)$ and $\tau_{(0,\eta)}(z_2, z_3) = (z_2 \mathrm{e}^{-\mathrm{i}\sqrt{3}\eta/2}, z_3 \mathrm{e}^{\mathrm{i}\sqrt{3}\eta/2})$. Working in real coordinates (x_2, y_2, x_3, y_3) we find that any commuting matrix is forced to take the form

$$\begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{pmatrix}$$

where $a = \partial f_2^r / \partial x_2$ and $c = \partial f_2^r / \partial x_3$. Hence the eigenvalues are $a \pm c$, each of multiplicity two.

Hexagons. Hexagons have an isotropy subgroup $\Sigma_H \cong D_6$ which acts trivially on the subspace (x, x, x): this yields an eigenvalue

$$\frac{\partial f_1^r}{\partial x_1} + \frac{\partial f_1^r}{\partial x_2} + \frac{\partial f_1^r}{\partial x_3} \equiv \frac{\partial f_1^r}{\partial x_1} + 2\frac{\partial f_1^r}{\partial x_2}, \tag{23}$$

using symmetry. It can similarly be checked that on the subspace $(\mathrm{i}x, \mathrm{i}x, \mathrm{i}x)$ ρ acts as -1 and m_x acts as $+1$: this is another isotypic component, and gives the eigenvalue

$$\frac{\partial f_1^i}{\partial y_1} + \frac{\partial f_1^i}{\partial y_2} + \frac{\partial f_1^i}{\partial y_3}. \tag{24}$$

The other four eigenvalues come from two different two-dimensional isotypic components:

$$\mathbb{R}\{(1, -1, 0), (0, 1, -1)\} \qquad \text{and} \qquad \mathbb{R}\{(\mathrm{i}, -\mathrm{i}, 0), (0, -\mathrm{i}, \mathrm{i})\}.$$

On the first of these, ρ acts as $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and m_x acts as $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$; these generate an irreducible representation isomorphic to the 2D irrep of D_3. The matrices generating the irrep are not orthogonal by our (bad but obvious) choice of coordinates. After checking absolute irreducibility we can conclude that the eigenvalue

$$\frac{\partial f_1^r}{\partial x_1} - \frac{\partial f_1^r}{\partial x_2} \tag{25}$$

occurs, with multiplicity two.

On the final subspace, $\mathbb{R}\{(\mathrm{i}, -\mathrm{i}, 0), (0, -\mathrm{i}, \mathrm{i})\}$, there is a faithful action of D_6 generated by ρ acting as $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ (note that $\rho^3 = -I$, so ρ has order six), and by m_x acting as $\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$. This is absolutely irreducible and so the eigenvalue is

$$\frac{\partial f_1^i}{\partial y_1} - \frac{\partial f_1^i}{\partial y_2} \tag{26}$$

again, with multiplicity two. Direct calculation shows that this eigenvalue is zero. The translational symmetries of the problem imply that hexagons have two zero eigenvalues because there is a two-dimensional (in fact a two-torus) group orbit of hexagon solutions given by translations in the two independent directions in the plane. From (26) being zero we can now simplify (24):

$$\frac{\partial f_1^i}{\partial y_1} + \frac{\partial f_1^i}{\partial y_2} + \frac{\partial f_1^i}{\partial y_3} \equiv 3\frac{\partial f_1^i}{\partial y_1}.$$

Table 3 lists the branching equations and eigenvalues for the axial branches rolls and hexagons, and for rectangles. The rolls branch has a single zero eigenvalue corresponding to neutral stability

Name	Branching equation	Eigenvalues	Stability conditions	Multiplicity
Rolls	$0 = \mu + a_1 x^2 + \cdots$	$\frac{\partial f_1^r}{\partial x_1}$	$\mathrm{sgn}(a_1)$	
		$\frac{\partial f_1^i}{\partial y_1}$	0	
		$\frac{\partial f_2^r}{\partial x_2} \pm \frac{\partial f_2^r}{\partial x_3}$	$\pm\varepsilon x + (a_2 - a_1)x^2$	2 each
$\pm$ Hexagons	$0 = \mu + \varepsilon x$	$\frac{\partial f_1^r}{\partial x_1} + 2\frac{\partial f_1^r}{\partial x_2}$	$\varepsilon x + 2(a_1 + 2a_2)x^2$	
	$+(a_1 + 2a_2)x^2 + \cdots$	$3\frac{\partial f_1^i}{\partial y_1}$	$\mp 3\varepsilon x + 3(c_1 - b_1 - 2b_2)x^3$	
		$\frac{\partial f_1^r}{\partial x_1} - \frac{\partial f_1^r}{\partial x_2}$	$\mp 2\varepsilon x + 2(a_1 - a_2)x^2$	2
		$\frac{\partial f_1^i}{\partial y_1} - \frac{\partial f_1^i}{\partial y_2}$	0	2

Table 3: Scalar bifurcation on the hexagonal lattice, 'fundamental'case: branching equations, eigenvalue expressions and those expressions evaluated near $x = 0$, for the axial branches rolls and hexagons. A solution branch is stable when all the stability conditions are negative.

to perturbations in the direction of the wavevector $\mathbf{k}_1$: there is a continuous group orbit of roll solutions given by translations in the $(1, 0)$ direction. Notice that translations in the $(0, 1)$ direction have no effect. By contrast, the hexagon branches have two zero eigenvalues.

Figure 9 sketches a typical bifurcation diagram in the case $a_2 - a_1 < 0$, $a_1 < 0$, for which rolls bifurcate supercritically and are stable for large positive μ. Notice that hexagons bifurcate transcritically. Both up-hexagons and the trivial (no pattern) solution are stable for $\mu_1 = \varepsilon^2/[4(a_1 + 2a_2)] < \mu < 0$ and both rolls and up-hexagons are stable in the range $\mu_2 = \varepsilon^2 a_1/(a_1 - a_2)^2 < \mu < -\varepsilon^2(2a_1 + a_2)/(a_1 - a_2)^2 = \mu_3$. The bifurcation at $\mu = \mu_2$ is a pitchfork, and the bifurcation at $\mu = \mu_3$ is transcritical (in fact it is the bifurcation with D_3 symmetry on $\mathbb{R}^2$ that we met in the last lecture).

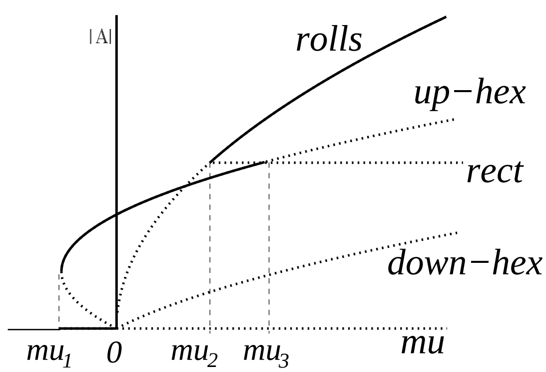

Figure 9: Typical bifurcation diagram for bifurcation on a hexagonal planar lattice, in the case $a_2 - a_1 < 0$ and $a_1 < 0$.

	Point (z_1, z_2, z_3)	Isotropy subgroup Σ (generators)	dim Fix(Σ)
Trivial	$(0,0,0)$	$D_6 \ltimes T^2 \times \mathbb{Z}_2$	0
Rolls	$(x,0,0)$	$O(2) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $m_x,\ \tau_{(0,p_2)},\ \rho^3,\ m_h \circ [\pi, 0]$	1
Hexagons	(x,x,x)	D_6 $\rho,\ m_x$	1
Triangles	$(\mathrm{i}x, \mathrm{i}x, \mathrm{i}x)$	$\tilde{D}_6$ $m_x,\ \rho \circ m_h$	1
Patchwork Quilt	$(0,x,x)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $m_x,\ \rho^3,\ m_h \circ [0, \frac{2\pi}{\sqrt{3}}]$	1

Table 4: Scalar bifurcation on the hexagonal lattice with midplane reflection symmetry, in the 'fundamental'case: axial branches, fixed point subspaces and isotropy subgroups. $x_j \in \mathbb{R}$. The patchwork quilt solution is often just called 'rhombs'.

3.4 Hexagonal patterns with a midplane reflection symmetry

In many thermal convection experiments the fluid is close to the idealised case in which fluid properties such as viscosity and thermal conductivity do not vary significantly with temperature, and the fluid is almost incompressible. In this case the fluid is said to obey the Boussinesq approximation, and the governing Navier–Stokes equations have an extra symmetry of reflection in a horizontal plane at the mid-level of the fluid layer. In other words, hot rising fluid behaves symmetrically to cold falling fluid. In this situation, and when the boundary conditions at the upper and lower sides of the fluid layer are the same, there is an extra symmetry in the bifurcation problem because the part of the eigenfunction that deals with the vertical structure is even about the midplane. In terms of the amplitudes (z_1, z_2, z_3) the extra symmetry m_h acts as -1:

$$m_h(z_1, z_2, z_3) \quad = \quad -(z_1, z_2, z_3).$$

The full symmetry group of the problem is now $D_6 \ltimes T^2 \times \mathbb{Z}_2$ and the analysis of the previous subsection needs to be reworked to take this symmetry into account.

In this case it transpires that there are four axial branches. These are listed in table 4. Compared to the original problem without midplane symmetry, the 'up-hexagons' and 'down-hexagons' now lie on the same group orbit, so their stability and branching directions must be the same. Rolls have gained an extra symmetry which forces the regions of upflow and downflow to be symmetrically related, which was not necessarily the case in the original problem. The amplitude equations are simplified because the action of m_h removes the quadratic terms in (20)-(22). The cubic truncation turns out not to be sufficient to distinguish the relative stability of hexagons and triangles. This can be seen easily from the form of the cubic terms: they do not contain the phase information that distinguishes (x,x,x) from $(\mathrm{i}x, \mathrm{i}x, \mathrm{i}x)$. Terms at fifth order are needed to determine stability completely. The form of the first amplitude equation, up to fifth order, is

$$\begin{aligned} \dot{z}_1 \quad &= \quad \mu z_1 + z_1(a_1|z_1|^2 + a_2(|z_2|^2 + |z_3|^2)) + z_1(f_{11}|z_1|^4 + f_{12}|z_1|^2|z_2|^2 + \cdots \\ &\quad \cdots + f_{33}|z_3|^4) + \bar{z}_2\bar{z}_3(d_1 z_1 z_2 z_3 + d_2 \bar{z}_1\bar{z}_2\bar{z}_3) + O(7), \end{aligned} \tag{27}$$

with the equations for $\dot{z}_2$ and $\dot{z}_3$ following by rotation symmetries as before. The calculations of the eigenvalues follows from isotypic decomposition as before, and is summarised in table 5. The calculations for rolls and hexagons are very similar to those in the previous case. The calculation for triangles is similar to that for hexagons. The calculation for the patchwork quilt solution is set out below.

Patchwork Quilt. The symmetries m_x, ρ^3 and $[0, 2\pi/\sqrt{3}]\circ m_h$ that generate the isotropy subgroup of patchwork quilt all act as $+1$ on the subspace $(0, x, x)$; the first isotypic component. On the subspace $(0, x, -x)$ m_x acts as -1 but the other two generators act as $+1$ still. This defines a second isotypic component, with eigenvalue

$$\frac{\partial f_2^r}{\partial x_2} - \frac{\partial f_3^r}{\partial x_3}.$$

Similarly, on the subspace $(0, \mathrm{i}x, \mathrm{i}x)$, ρ^3 acts as -1 but m_x and $[0, 2\pi/\sqrt{3}] \circ m_h$ act as $+1$. Hence

$$\frac{\partial f_2^i}{\partial y_2} + \frac{\partial f_3^i}{\partial y_3} \tag{28}$$

must be an eigenvalue. The fourth isotypic component is $(0, \mathrm{i}x, -\mathrm{i}x)$ on which m_x and ρ^3 act as -1 and $[0, 2\pi/\sqrt{3}] \circ m_h$ acts as $+1$. The fourth eigenvalue is

$$\frac{\partial f_2^i}{\partial y_2} - \frac{\partial f_3^i}{\partial y_3}. \tag{29}$$

Both (29) and (28) turn out to be zero: as expected because there is again a group orbit of patchwork quilt equilibria forming a two-torus.

The last two eigenvalues must come from perturbations in the first coordinate: isotypic components are $(x, 0, 0)$ and $(\mathrm{i}x, 0, 0)$ on which m_x acts as $+1$ and $[0, 2\pi/\sqrt{3}]\circ m_h$ acts as -1. On the first of these ρ^3 acts as $+1$ and on the second, ρ^3 acts as -1. So the actions are irreducible and different to all those we determined for the other isotypic components. The resulting eigenvalues are

$$\frac{\partial f_1^r}{\partial x_1} \quad \text{and} \quad \frac{\partial f_1^i}{\partial y_1},$$

which are in fact equal when evaluated on the cubic truncation.

All four branches bifurcate as pitchforks due to the absence of quadratic terms: in each fixed point subspace of an isotropy subgroup there is an element of the normaliser that acts as -1 ensuring that each bifurcation is a symmetric pitchfork. From the stability conditions, some observations on the possible bifurcation behaviours near $\mu = 0$ is can be made:

- if all four axial branches bifurcate supercritically then exactly one is stable,
- the stability of rolls is determined by cubic order terms,
- if rolls are unstable then a fifth order term determines which of hexagons and triangles is stable,
- patchwork quilt is unstable except possibly in the degenerate case $a_1 = a_2$.

Planforms for the triangle and patchwork quilt solutions are shown in figure 10.

3.5 Pseudoscalar planforms

In the previous section we assumed that the action of $E(2)$ on the function u in our pattern forming PDE (10) was as equation (13) - (14) with ρ the trivial representation. In this section we discuss another choice for the action of $E(2)$ that applies directly to the incompressible 2D Navier–Stokes equations. It turns out that any planar system of 'pattern forming' PDEs lies in one of two classes, *scalar* and *pseudoscalar*, and can be reduced to a PDE for a function taking values in $\mathbb{R}$ rather than $\mathbb{R}^m$; see Melbourne [41].

The pseudoscalar action of $E(2)$ on a real-valued function u is given by

$$gu(x, \omega, t) \quad = \quad \det(g)\ u(g^{-1}x, \omega, t) \tag{30}$$

Name	Branching equation	Eigenvalues	Stability conditions	Mult
Rolls	$0 = \mu + a_1 x^2 + \cdots$	$\frac{\partial f_1^r}{\partial x_1}$	$\mathrm{sgn}(a_1)$	
$(x, 0, 0)$		$\frac{\partial f_1^i}{\partial y_1}$	0	
		$\frac{\partial f_2^r}{\partial x_2} \pm \frac{\partial f_2^r}{\partial x_3}$	$\mathrm{sgn}(a_2 - a_1)$	2 each
Hexagons	$0 = \mu + (a_1 + 2a_2)x^2$	$\frac{\partial f_1^r}{\partial x_1} + 2\frac{\partial f_1^r}{\partial x_2}$	$\mathrm{sgn}(a_1 + 2a_2)$	
(x, x, x)	$+x^4(f_{11} + \cdots + f_{33} + d_1 + d2) + \cdots$	$3\frac{\partial f_1^i}{\partial y_1}$	$-\mathrm{sgn}(d_2)$	
		$\frac{\partial f_1^r}{\partial x_1} - \frac{\partial f_1^r}{\partial x_2}$	$\mathrm{sgn}(a_1 - a_2)$	2
		$\frac{\partial f_1^i}{\partial y_1} - \frac{\partial f_1^i}{\partial y_2}$	0	2
Triangles	$0 = \mu + (a_1 + 2a_2)x^2$	$\frac{\partial f_1^i}{\partial y_1} + 2\frac{\partial f_1^i}{\partial y_2}$	$\mathrm{sgn}(a_1 + 2a_2)$	
$(\mathrm{i}x, \mathrm{i}x, \mathrm{i}x)$	$+x^4(f_{11} + \cdots + f_{33} + d_1 - d2) + \cdots$	$3\frac{\partial f_1^r}{\partial x_1}$	$\mathrm{sgn}(d_2)$	
		$\frac{\partial f_1^i}{\partial y_1} - \frac{\partial f_1^i}{\partial y_2}$	$\mathrm{sgn}(a_1 - a_2)$	2
		$\frac{\partial f_1^r}{\partial x_1} - \frac{\partial f_1^r}{\partial x_2}$	0	2
Patchwork	$0 = \mu + (a_1 + a_2)x^2 + \cdots$	$\frac{\partial f_2^r}{\partial x_2} + \frac{\partial f_3^r}{\partial x_3}$	$\mathrm{sgn}(a_1 + a_2)$	
quilt		$\frac{\partial f_2^r}{\partial x_2} - \frac{\partial f_3^r}{\partial x_3}$	$\mathrm{sgn}(a_1 - a_2)$	
$(0, x, x)$		$\frac{\partial f_2^i}{\partial y_2} + \frac{\partial f_3^i}{\partial y_3}$	0	
		$\frac{\partial f_2^i}{\partial y_2} - \frac{\partial f_3^i}{\partial y_3}$	0	
		$\frac{\partial f_1^r}{\partial x_1}$	$-\mathrm{sgn}(a_1 - a_2)$	
		$\frac{\partial f_1^i}{\partial y_1}$	$-\mathrm{sgn}(a_1 - a_2)$	

Table 5: Scalar bifurcation on the hexagonal lattice, with midplane reflection symmetry, in the 'fundamental'case: branching equations, eigenvalue expressions and those expressions evaluated near $x = 0$, for the axial branches rolls and hexagons. A solution branch is stable when all the stability conditions are negative.

where $\det(g) = +1$ if g is composed only of translations and rotations, and $\det(g) = -1$ if g contains a reflection. An important example of a pseudoscalar PDE is the 2D Navier–Stokes equation written in terms of the streamfunction $\psi(x, y, t)$, see Bosch Vivancos et al. [3]. Let (u, v) be the components of the 2D velocity field, then $u = \partial_y \psi$, $v = -\partial_x \psi$ and

$$\partial_t \nabla^2 \psi + J[\psi, \nabla^2 \psi] \quad = \quad \nu \nabla^4 \psi + f(x, y, t),$$

where $J[p, q] = \partial_x p \partial_y q - \partial_x q \partial_y p$ and we have taken $\hat{z} \cdot \nabla \times$ the usual Navier–Stokes equation. $f(x, y, t)$ denotes a general forcing term.

The reduction to a spatially periodic lattice (here we will take a hexagonal one) goes through as before, and we examine perturbations to a putative solution $\psi = 0$ that are linear combinations of three plane waves, as (16):

$$\psi(x, y, t) \quad = \quad \sum_{j=1}^{3} z_j(t) \mathrm{e}^{\mathrm{i}\mathbf{k}_j \cdot \mathbf{x}} + \mathrm{c.c.} \tag{31}$$

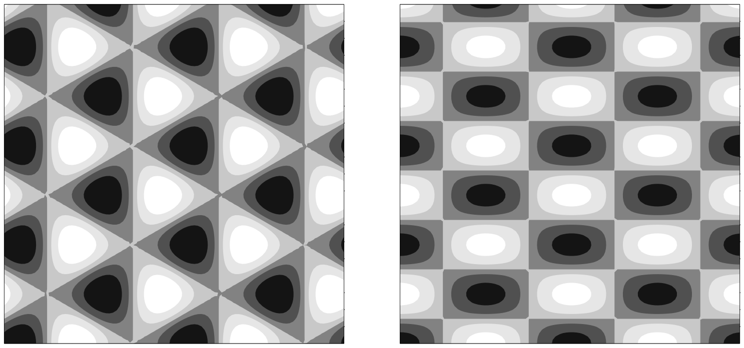

Figure 10: Planforms for axial branches for bifurcation on a hexagonal planar lattice with a midplane reflection symmetry: (a) triangles; (b) patchwork quilt (also called 'rhombs').

Name (z_1, z_2, z_3)	Streamfunction $\psi(x, y)$	Isotropy subgroup Σ (generators)	dim Fix(Σ)
Trivial $(0, 0, 0)$	0	$D_6 \ltimes T^2$	0
Pseudo-rolls $(x, 0, 0)$	$\cos x$	$O(2) \times \mathbb{Z}_2$ ρ^3, $\tau_{(0,p_2)}$, $m_x \circ [\pi, 0]$	1
Pseudo-hexagons (x, x, x)	$\cos \mathbf{k}_1 \cdot \mathbf{x} + \cos \mathbf{k}_2 \cdot \mathbf{x} + \cos \mathbf{k}_3 \cdot \mathbf{x}$	$\mathbb{Z}_6$ ρ	1
Pseudo-triangles $(\mathrm{i}x, \mathrm{i}x, \mathrm{i}x)$	$\sin \mathbf{k}_1 \cdot \mathbf{x} + \sin \mathbf{k}_2 \cdot \mathbf{x} + \sin \mathbf{k}_3 \cdot \mathbf{x}$	$\tilde{D}_3$ $m_x \circ \rho$, ρ^2	1
Pseudo-rectangles $(0, x, -x)$	$\cos \mathbf{k}_2 \cdot \mathbf{x} - \cos \mathbf{k}_3 \cdot \mathbf{x}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ m_x, ρ^3	1

Table 6: Pseudoscalar bifurcation on the hexagonal lattice, in the 'fundamental'case: axial branches, fixed point subspaces and isotropy subgroups. $x_j \in \mathbb{R}$.

where, as before, *c.c.* denotes complex conjugate, and we have ignored the bounded variables $\omega \in \Omega$. The action of the reduced symmetry group $D_6 \ltimes T^2$ on the centre manifold $\mathbb{C}^3$ spanned by the three mode amplitudes (z_1, z_2, z_3) is

$$\rho(z_1, z_2, z_3) = (\bar{z}_2, \bar{z}_3, \bar{z}_1) \quad (32)$$

$$m_x(z_1, z_2, z_3) = -(z_1, z_3, z_2) \quad (33)$$

$$\tau_{\mathbf{p}}(z_1, z_2, z_3) = (z_1 \mathrm{e}^{-\mathrm{i}\mathbf{k}_1 \cdot \mathbf{p}}, z_2 \mathrm{e}^{-\mathrm{i}\mathbf{k}_2 \cdot \mathbf{p}}, z_3 \mathrm{e}^{-\mathrm{i}\mathbf{k}_3 \cdot \mathbf{p}}). \quad (34)$$

where $\mathbf{p} = (p_1, p_2)$ is a translation vector in the two-torus $T^2 \cong \mathbb{R}^2/\mathcal{L}$. Note that the actions of ρ and $\tau_{\mathbf{p}}$ are unchanged, but that (33) differs from (18). This change has radical consequences: there are now four axial branches, rather than two, and the bifurcation problem with $D_6 \ltimes T^2$ symmetry now looks similar to the one with a midplane reflection studied in section 3.4. The axial branches are listed in table 6. The other difference is in the visualisation of these planforms: because orientation is important, and instead of plotting level sets of the streamfunction ψ, we need to plot planar velocity vectors. This produces planform pictures that have the required symmetries (and no more!), as shown in figure 11. Further details of stability, and pseudoscalar bifurcations on the

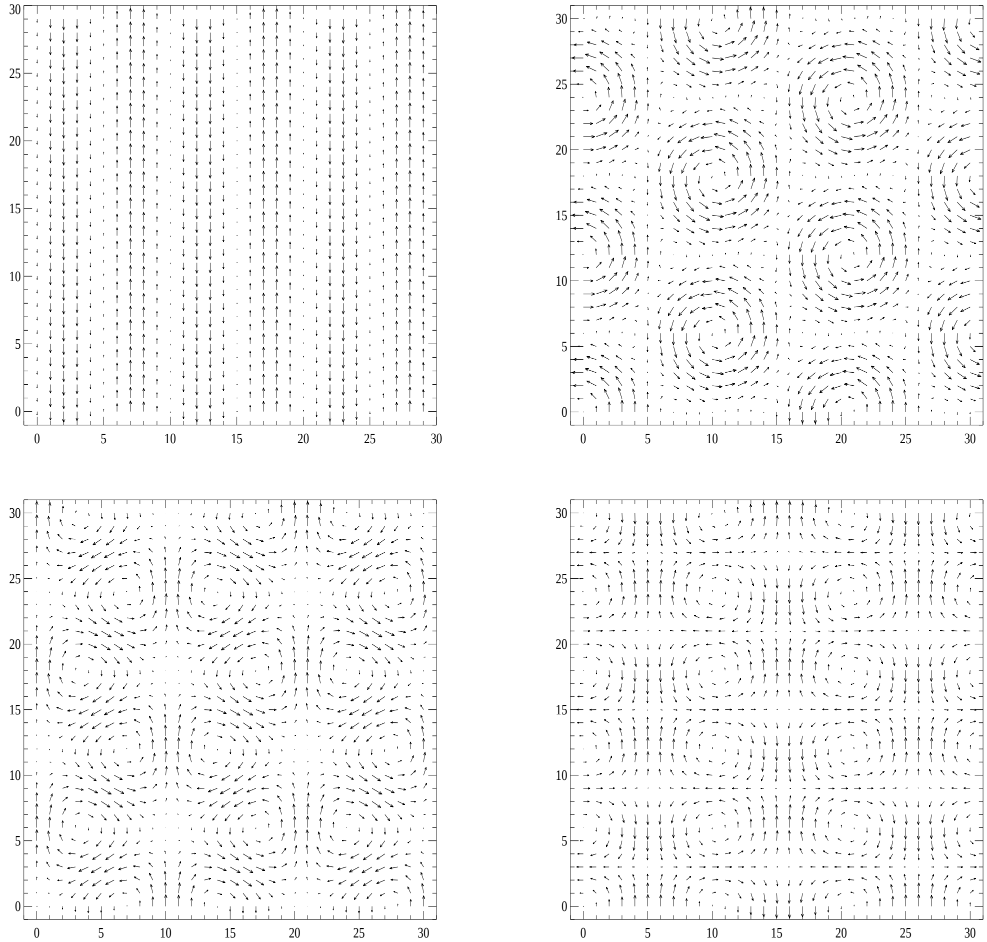

Figure 11: Pseudoscalar planforms for axial branches for bifurcation on a hexagonal planar lattice. Velocity vectors $(u, v) = (\psi_y, -\psi_x)$ are shown. (a) pseudo-rolls; (b) pseudo-hexagons; (c) pseudo-triangles (d) pseudo-rectangles.

planar rhombic and square lattices can be found in [5] and [27].