**Definition** A unimodal map on the interval \([a, b]\) is a continuous map \(F : [a, b] \to [a, b]\) such that

- \(F(a) = F(b) = a\) and
- \(\exists c \in (a, b)\) such that \(F\) is strictly increasing on \([a, c]\) and strictly decreasing on \([c, b]\):

Note: a map of the form

\[ a \quad b \quad c \]

is effectively unimodal under \(x \mapsto -x\) and \(F \mapsto -F\).

**Definition** An orientation reversing fixed point (ORFP) of a unimodal map \(F\) is a fixed point in the interval \((c, b)\) where \(F\) is decreasing.

**Lemma:**

1. If \(F(c) \leq c\) then all solutions tend to fixed points, which lie in \([a, F(c)]\).
2. If \(F(c) > c\) then there is a unique ORFP \(x_0 \in (c, F(c))\).
3. If \(F(c) > c\) then orbits either tend to fixed points in \([a, F^2(c)]\) or are attracted into \([F^2(c), F(c)]\).

**Proof:**

1. \(F([a, c]) = F([c, b]) = [a, F(c)] \subseteq [a, c]\). So after one iteration \(x \in [a, c]\), and \(F\) is strictly increasing on this interval, hence \(x < y \iff F(x) < F(y)\):

If \(x_1 < F(x_1)\) then \(x_i\) increases monotonically to the nearest fixed point.
If \(x_1 > F(x_1)\) then \(x_i\) decreases monotonically to the nearest fixed point.

2. Apply the Intermediate Value Theorem (IVT) to \(g(x) \equiv F(x) - x\) on \([c, F(c)]\) noting that \(F(c) > c \Rightarrow F^2(c) < F(c)\).
(3) Exercise. (Cases split on whether $F^3(c) < F^2(c)$ or vice versa.) □

Lemma
(4) If $F$ has an ORFP $x_0$ then $\exists x_{-1} \in (a,c)$ and $x_{-2} \in (x_0,b)$ such that $F(x_{-2}) = x_{-1}$ and $F(x_{-1}) = x_0$.

Proof:

Apply the IVT first to the open interval $(c,b)$, and second, to the open interval $(a,c)$:

Firstly, $x_0 \in (c,b) \Rightarrow F(c) > F(x_0) = x_0$ and also $x_0 = F(x_0) > F(b) = F(a)$ so $g(x) \equiv F(x) - x_0$ has $g(c) > 0$ and $g(a) < 0$ so, by the IVT $\exists x_{-1} \in (a,c)$ such that $F(x_{-1}) = x_0$.

Secondly, since $x_{-1} \in (a,c)$ we have that $F(b) = a < x_{-1} < x_0 = F(x_0)$ so $g(x) \equiv F(x) - x_{-1}$ satisfies $g(b) < 0$ and $g(x_0) > 0$ which implies that $\exists x_{-2} \in (x_0,b)$ such that $F(x_{-2}) = x_{-1}$. □

Note: $F^2(x_{-2}) = F^2(x_{-1}) = F^2(x_0) = x_0$, and also $x \in [x_{-1}, x_0] \Rightarrow F^2(x) \in [F^2(c), x_0]$ and $x \in [x_0, x_{-2}] \Rightarrow F^2(x) \in [x_0, F(c)]$. Therefore $F^2$ has the graph

Theorem 1
If $F$ has an ORFP $x_0$ with preimages $x_{-1}$ and $x_{-2}$ as above then
either (i) $F^2$ has a horseshoe on $J_L \equiv [x_{-1}, x_0]$ and $J_R \equiv [x_0, x_{-2}]$

- or (ii) all solutions tend to fixed points of $F^2$

- or (iii) $F^2$ is a unimodal map with an ORFP on both $J_L$ and $J_R$.

**Proof:**

Which of the three cases we are in is decided by the value of $F^2(c)$:

(i) If $F^2(c) < x_{-1}$ (which is equivalent to $F(c) > x_{-2}$) then it is clear from the sketch $F^2$ has horseshoes:

(ii) If $F^2(c) > c$ then all solutions on $J_L \cup J_R$ tend to fixed points of $F^2$ (note that the graph within $J_R$ could cross and recross the diagonal, resembling the figure in the proof of statement (1) above). Hence all solutions on $[a, x_{-1}] \cup [x_{-2}, b]$ either tend to fixed points of $F$ or are attracted into $[F^2(c), F(c)] \subset J_L \cup J_R$.

(iii) If $x_{-1} < F^2(c) < c$ then $F^2$ is a unimodal map on $J_L$ and $J_R$ with ORFPs that correspond to a 2-cycle for $F$. The attracting set is split between two disjoint subintervals $[F^2(c), F^4(c)]$ and $[F^3(c), F(c)]$, as indicated on the figure below:
Now, applying Theorem 1 successively to \( F^2, F^4, F^8, \ldots \) we can deduce

**Theorem 2**

If \( F \) has an ORFP then

- either (i) \( \exists N \) such that \( F^{2N} \) has a horseshoe and \( F \) is chaotic
- or (ii) \( \exists N \) such that all solutions tend to fixed points of \( F^{2N} \) and \( F \) has \( 2^m \)-cycles for \( 0 \leq m \leq N - 1 \)
- or (iii) there are \( 2^m \)-cycles \( \forall m \), and the attracting set is a Cantor set formed by the infinite intersection of the attracting subintervals of \( F^{2m} \).

**Proof**

By induction. See also Glendinning, pages 313–317.
Numerical investigation indicates that, for the logistic map and other similar unimodal maps with a quadratic maximum (e.g. $x_{n+1} = \mu \sin x_n$), the distances between parameter values $\mu_k$ at which successive period-doubling bifurcations occur approach an asymptotic geometrical relationship:

$$\lim_{k \to \infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta = 4.6692016 \ldots$$

Moreover, the successive forms of the logistic map restricted to the interval $[x, x_0]$, then flipped and rescaled, appear to converge to a limiting functional form.

These properties can in fact be proved, and yield insight into the ‘universal’ nature of the period-doubling transition to chaos.

To get some idea into what is going on, we will work with a slightly modified class of unimodal maps $G_r(x)$ such that $G_r(0) = 1$ always, and $G'_r(0) = 0$, i.e. the maps are centred on $x = 0$ and take their maximum value of unity there.

The simplest example would be the one-parameter family $G_r(x) = 1 - rx^2$, where $r$ is the bifurcation parameter, which is topologically conjugate to the standard logistic family $\mu x(1-x)$ so the well-known period-doubling cascade to chaos is preserved.

A typical member of the family $G_r$ is shown in the figure below:

We introduce notation for particular sets of unimodal maps which are of interest:

**Definition:** Let $S_k$ be the set of all unimodal maps $G$ defined on $[-1,1]$, having a quadratic maximum at $(x = 0, G = 1)$, and having a $2^k$-cycle which is at the point of undergoing a period-doubling bifurcation.

From the figure above we can see that for a map $G_r \in S_k$ we can restrict the map to a smaller interval $[\lambda, -\lambda] \subset [-1,1]$ and examine the dynamics of $G^2_r$ on this subinterval. We do this
explicitly by introducing the rescaled co-ordinate \( y = \lambda - x \) so that \( y \in [-1, 1] \). Note that \( \lambda < 0 \).

This motivates the following definition of an operator \( T \) acting on (families of) unimodal maps \( G_r \):

\[
TG_r(y) := \frac{1}{\lambda} G^2_r(\lambda y),
\]

where \( \lambda = G^2_r(0) \) is the rescaling factor that makes \( TG_r \) into a unimodal map on \([-1, 1]\) again.

Since \( T \) involves taking the composition of \( G_r \) with itself we can also see that \( TS_k = S_{k-1} \) since if \( G_r \) has a \( 2^k \)-cycle at a period-doubling bifurcation point, then \( TG_r \) has a \( 2^{k-1} \)-cycle which is at a period-doubling bifurcation point.

Therefore \( T \) acts on the space of unimodal maps in a manner which can be sketched cartoon-style as follows:

The function space of unimodal maps \( G(x) \). Sets \( S_k \) are indicated by vertical lines and the action of \( T \) is indicated by the arrows. The map \( G^* \) is a fixed point of \( T \).

The operator \( T \) performs a renormalisation of the family of maps \( \{G_r\} \) and hence, itself, forms a dynamical system on the space of unimodal maps. It turns out that repeatedly applying \( T \) to the family \( \{G_r\} \) yields convergence to a unique family of unimodal maps, and moreover, \( T \) has a unique fixed point \( G^* \) corresponding to a map that can be renormalised infinitely often. Such a map must have a \( 2^k \)-cycle for all \( k \) and so \( G^* \) must correspond to a map at the accumulation point of the period-doubling cascade. This is indicated in the cartoon above (where the thick lines cross).

Considering now maps of the form \( G(x) = 1 + ax^2 + bx^4 + O(x^6) \) we can find an approximate solution to the functional equation \( TG = G \) as follows.

Suppose that the \( k^{th} \) approximation to \( G^* \) is a map of the form \( G_k(x) = 1 + a_k x^2 + b_k x^4 + \ldots \).

Let \( \lambda \) be the value of \( G^2_k(0) = G_k(1) \). Renormalise \( G^2_k \) so that \( G_{k+1}(0) = 1 \) by defining

\[
G_{k+1}(y) = TG_k \equiv \frac{G^2_k(\lambda y)}{\lambda} \quad \text{say, where } \lambda = G^2_k(0)
\]

We are interested in a function \( G^* \) that is fixed under the functional map \( T \).

**First approximation.** As our first try we take \( G_k = 1 + a_k x^2 + O(x^4) \) so that we have \( G_k(1) = 1 + a_k = \lambda_k \)

\[
\Rightarrow \quad G_{k+1} = TG_k = \frac{1 + a_k \{1 + a_k[(1 + a_k)x^2] \}^2}{1 + a_k} = 1 + 2a_k^2 (1 + a_k)x^2 + O(x^4)
\]
i.e. we have reduced the problem to solving the 1D map
\[ a_{k+1} = 2a_k^2(1 + a_k) \]
which has an unstable fixed point \( a = -\frac{1}{2}(1 + \sqrt{3}) = -1.37 \Rightarrow \lambda = -0.37 \). At the fixed point the Jacobian is \( 4 + \sqrt{3} = 5.73 \): this value is an estimate of the unstable eigenvalue of \( TG^* \) and hence is an estimate of the convergence rate \( \delta \).

**Second approximation.** For a more accurate attempt we include the \( O(x^4) \) terms: take \( G_k = 1 + a_k x^2 + b_k x^4 + O(x^6) \) so that \( G_k(1) = 1 + a_k + b_k = \lambda_k \). Comparing the coefficients gives the 2D map
\[
\begin{align*}
    a_{k+1} & = 2a_k(a_k + 2b_k)\lambda_k \\
    b_{k+1} & = (2a_kb_k + a_k^2 + 4b_k^2 + 6a_k^2 b_k)\lambda_k^3 
\end{align*}
\]
which has a fixed point at \( a = -1.5222, b = 0.1276, \lambda = -0.3946 \). Similar computation of the eigenvalues of the Jacobian matrix (now a \( 2 \times 2 \) matrix) gives eigenvalues \(-0.49 \) and \( 4.844 \). This second value is a better approximation to \( \delta \).

In fact, numerical solution shows that the functional equation \( TG = G \) has a fixed point
\[
G^*(x) = 1 - 1.52736x^2 + 0.10482x^4 - 0.02671x^6 - 0.00352x^8 + \ldots
\]
\Rightarrow \lambda = G^*(1) = -0.3995

Including many more higher-order terms and computing the Jacobian matrix numerically yields a single eigenvalue \( \delta = 4.6692016 \ldots \) (sometimes called ‘Feigenbaum’s constant’) outside the unit circle (i.e. a single unstable direction), and an infinite spectrum of eigenvalues inside the unit circle (stable directions). So in the function space, \( G^* \) is a hyperbolic fixed point of \( T \). All this has been made precise by O.E. Lanford (1982) and other authors.

Since the stable manifold \( S_\infty \) of \( G^* \) occupies ‘all but one dimension’ of the possible space of functions, typical one-parameter families will cross \( S_\infty \) transversely as we vary one parameter, which to some degree explains why such a transition to chaos via a period-doubling cascade appears so frequently in nonlinear systems.

The map \( G = \mu_\infty x(1 - x), \mu_\infty = 3.5700 \ldots \), is on the stable manifold of \( G^* \), and varying \( \mu \) around \( \mu_\infty \) gives situation (i) if \( \mu > \mu_\infty \) (\( G^{2^N} \) has a horseshoe for some \( N \)) or situation (iii) if \( \mu < \mu_\infty \) (\( G^{2^N} \) has no ORFP for some \( N \) and cycle lengths divide \( 2^N \)).

If \( \mu_\infty - \mu = O(\delta^{-N}) \) then, roughly speaking, it takes \( O(N) \) renormalisations for the perturbation to grow to \( O(1) \) and eliminate the ORFP, thus explaining why \( \mu_\infty - \mu_k \sim A\delta^{-k} \) as \( k \to \infty \).

Renormalisation is a powerful idea that has been applied to many other dynamical systems problems, and reveals similar universal features (for example in maps of the circle).

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