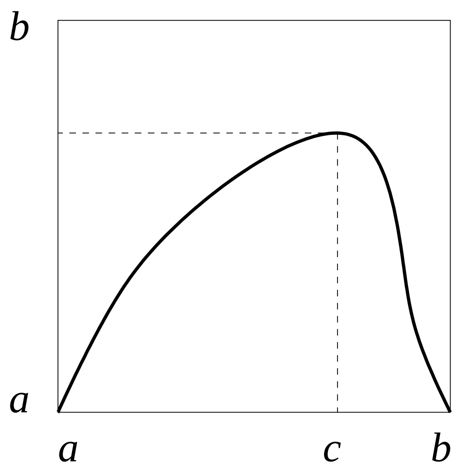
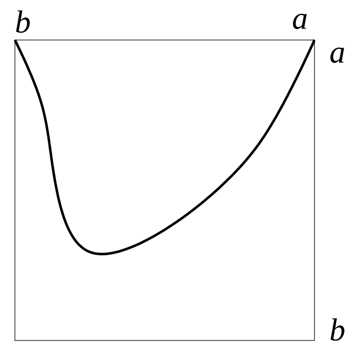
7.7 Unimodal Maps

Definition A unimodal map on the interval $[a, b]$ is a continuous map $F : [a, b]$ into $[a, b]$ such that

- $F(a) = F(b) = a$ and
- $\exists\, c \in (a, b)$ such that F is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$:



Note: a map of the form

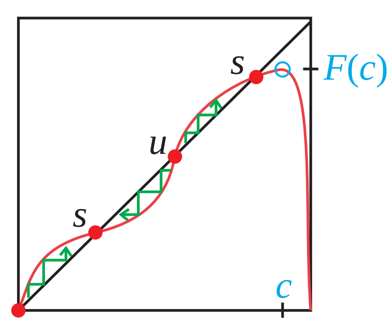


is effectively unimodal under $x \mapsto -x$ and $F \mapsto -F$.

Definition An orientation reversing fixed point (ORFP) of a unimodal map F is a fixed point in the interval (c, b) where F is decreasing.

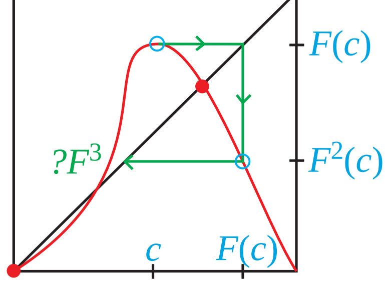
Lemma:
(1) If $F(c) \leq c$ then all solutions tend to fixed points, which lie in $[a, F(c)]$.
(2) If $F(c) > c$ then there is a unique ORFP $x_0 \in (c, F(c))$.
(3) If $F(c) > c$ then orbits either tend to fixed points in $[a, F^2(c)]$ or are attracted into $[F^2(c), F(c)]$.

Proof:
(1) $F([a, c]) = F([c, b]) = [a, F(c)] \subseteq [a, c]$. So after one iteration $x \in [a, c]$, and F is strictly increasing on this interval, hence $x < y \iff F(x) < F(y)$:



If $x_1 < F(x_1)$ then x_i increases monotonically to the nearest fixed point.
If $x_1 > F(x_1)$ then x_i decreases monotonically to the nearest fixed point.

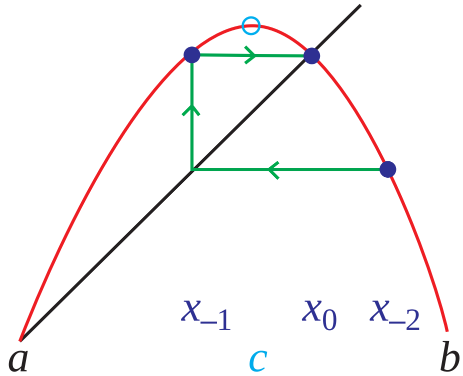
(2) Apply the Intermediate Value Theorem (IVT) to $g(x) \equiv F(x) - x$ on $[c, F(c)]$ noting that $F(c) > c \Rightarrow F^2(c) < F(c)$.



(3) Exercise. (Cases split on whether $F^3(c) < F^2(c)$ or vice versa.) □

Lemma
(4) If F has an ORFP x_0 then $\exists\ x_{-1} \in (a, c)$ and $x_{-2} \in (x_0, b)$ such that $F(x_{-2}) = x_{-1}$ and $F(x_{-1}) = x_0$.

Proof:

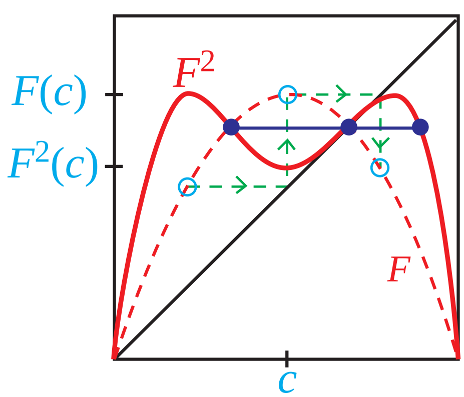


Apply the IVT first to the open interval (c, b), and second, to the open interval (a, c):

Firstly, $x_0 \in (c, b) \Rightarrow F(c) > F(x_0) = x_0$ and also $x_0 = F(x_0) > F(b) = F(a)$ so $g(x) \equiv F(x) - x_0$ has $g(c) > 0$ and $g(a) < 0$ so, by the IVT $\exists\ x_{-1} \in (a, c)$ such that $F(x_{-1}) = x_0$.

Secondly, since $x_{-1} \in (a, c)$ we have that $F(b) = a < x_{-1} < x_0 = F(x_0)$ so $g(x) \equiv F(x) - x_{-1}$ satisfies $g(b) < 0$ and $g(x_0) > 0$ which implies that $\exists\ x_{-2} \in (x_0, b)$ such that $F(x_{-2}) = x_{-1}$. □

Note: $F^2(x_{-2}) = F^2(x_{-1}) = F^2(x_0) = x_0$, and also $x \in [x_{-1}, x_0] \Rightarrow F^2(x) \in [F^2(c), x_0]$ and $x \in [x_0, x_{-2}] \Rightarrow F^2(x) \in [x_0, F(c)]$. Therefore F^2 has the graph



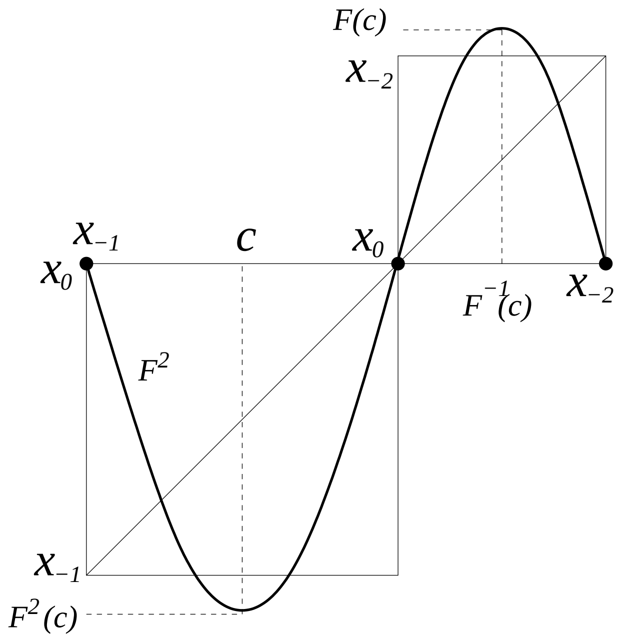
Theorem 1
If F has an ORFP x_0 with preimages x_{-1} and x_{-2} as above then

- either (i) F^2 has a horseshoe on $J_L \equiv [x_{-1}, x_0]$ and $J_R \equiv [x_0, x_{-2}]$
- or (ii) all solutions tend to fixed points of F^2
- or (iii) F^2 is a unimodal map with an ORFP on both J_L and J_R.

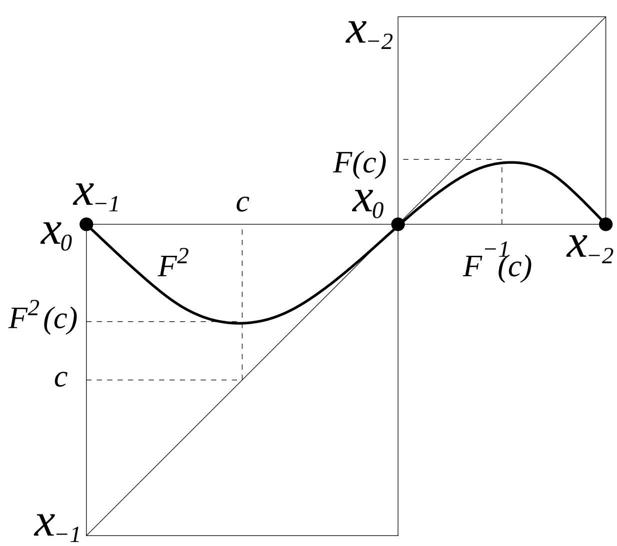
Proof:

Which of the three cases we are in is decided by the value of $F^2(c)$:

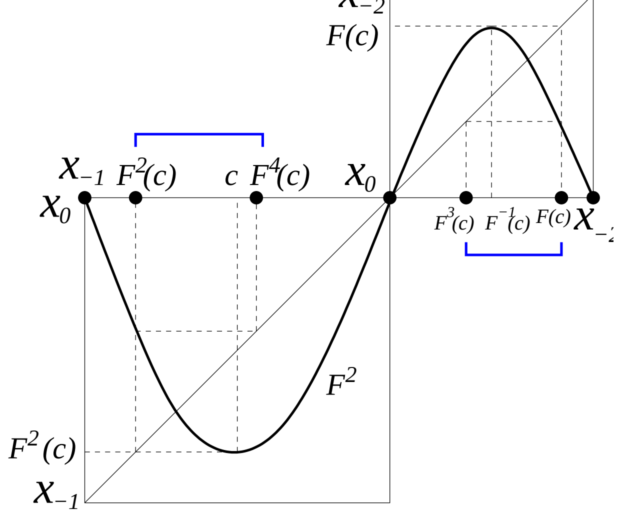
(i) If $F^2(c) < x_{-1}$ (which is equivalent to $F(c) > x_{-2}$) then it is clear from the sketch F^2 has horseshoes:



(ii) If $F^2(c) > c$ then all solutions on $J_L \cup J_R$ tend to fixed points of F^2 (note that the graph within J_R could cross and recross the diagonal, resembling the figure in the proof of statement (1) above). Hence all solutions on $[a, x_{-1}] \cup [x_{-2}, b]$ either tend to fixed points of F or are attracted into $[F^2(c), F(c)] \subset J_L \cup J_R$.



(iii) If $x_{-1} < F^2(c) < c$ then F^2 is a unimodal map on J_L and J_R with ORFPs that correspond to a 2-cycle for F. The attracting set is split between two disjoint subintervals $[F^2(c), F^4(c)]$ and $[F^3(c), F(c)]$, as indicated on the figure below:



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Now, applying Theorem 1 successively to $F^2, F^4, F^8, \ldots$ we can deduce

Theorem 2

If F has an ORFP then

- either (i) $\exists\ N$ such that F^{2^N} has a horseshoe and F is chaotic
- or (ii) $\exists\ N$ such that all solutions tend to fixed points of F^{2^N} and F has 2^m-cycles for $0 \leq m \leq N-1$
- or (iii) there are 2^m-cycles $\forall m$, and the attracting set is a Cantor set formed by the infinite intersection of the attracting subintervals of F^{2^m}.

Proof

By induction. See also Glendinning, pages 313–317.

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Universality and 'Feigenbaum's Constant'

Numerical investigation indicates that, for the logistic map and other similar unimodal maps with a quadratic maximum (e.g. $x_{n+1} = \mu \sin x_n$), the distances between parameter values μ_k at which successive period-doubling bifurcations occur approach an asymptotic geometrical relationship:

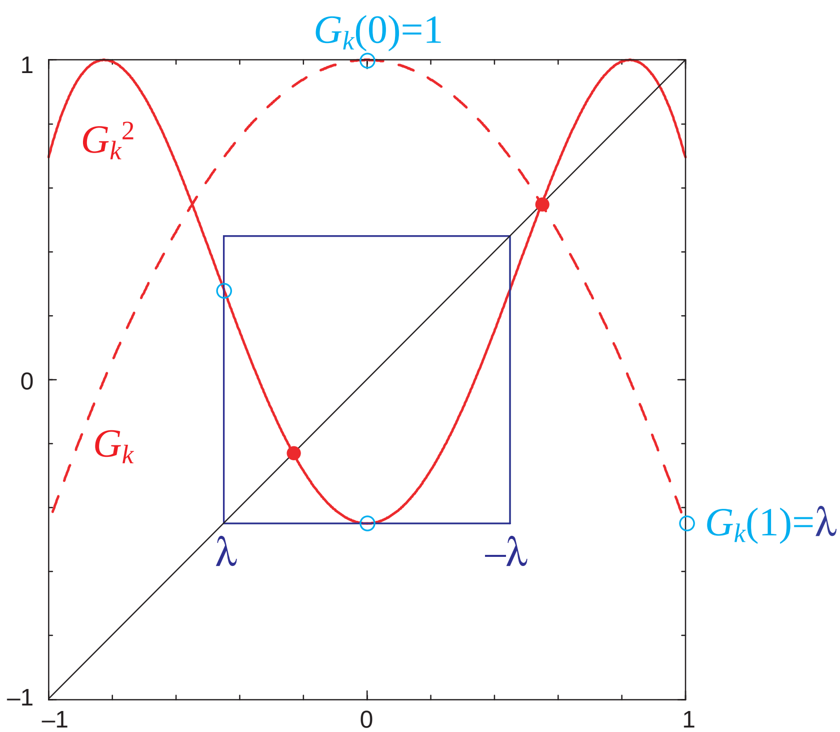
$$\lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta = 4.6692016\ldots.$$

Moreover, the successive forms of the logistic map restricted to the interval $[x_{-1}, x_0]$, then flipped and rescaled, appear to converge to a limiting functional form.

These properties can in fact be proved, and yield insight into the 'universal' nature of the period-doubling transition to chaos.

To get some idea into what is going on, we will work with a slightly modified class of unimodal maps $G(x)$ such that $G_r(0) = 1$ always, and $G_r'(0) = 0$, i.e. the maps are centred on $x = 0$ and take their maximum value of unity there.

The simplest example would be the one-parameter family $G_r(x) = 1 - rx^2$, where r is the bifurcation parameter, which is topologically conjugate to the standard logistic family $\mu x(1-x)$ so the well-known period-doubling cascade to chaos is preserved.
A typical member of the family G_r is shown in the figure below:



We introduce notation for particular sets of unimodal maps which are of interest:

Definition: Let S_k be the set of all unimodal maps G defined on $[-1, 1]$, having a quadratic maximum at $(x = 0, G = 1)$, and having a 2^k-cycle which is at the point of undergoing a period-doubling bifurcation.

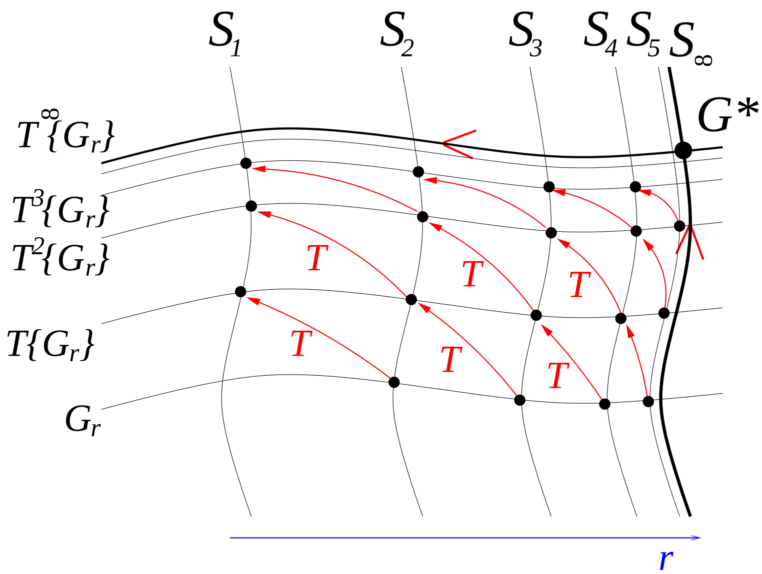
From the figure above we can see that for a map $G_r \in S_k$ we can restrict the map to a smaller interval $[\lambda, -\lambda] \subset [-1, 1]$ and examine the dynamics of G_r^2 on this subinterval. We do this explicitly by introducing the rescaled co-ordinate $y = \lambda^{-1}x$ so that $y \in [-1, 1]$. Note that $\lambda < 0$. This motivates the following definition of an operator $\mathcal{T}$ acting on (families of) unimodal maps G_r:

$$\mathcal{T}G_r(y) \quad := \quad \frac{1}{\lambda}G_r^2(\lambda y),$$

where $\lambda = G_r^2(0)$ is the rescaling factor that makes $\mathcal{T}G_r$ into a unimodal map on $[-1, 1]$ again.

Since $\mathcal{T}$ involves taking the composition of G_r with itself we can also see that $\mathcal{T}S_k = S_{k-1}$ since if G_r has a 2^k-cycle at a period-doubling bifurcation point, then $\mathcal{T}G_r$ has a 2^{k-1}-cycle which is at a period-doubling bifurcation point.

Therefore $\mathcal{T}$ acts on the space of unimodal maps in a manner which can be sketched cartoon-style as follows:



The function space of unimodal maps $G(x)$. Sets S_k are indicated by vertical lines and the action of T is indicated by the arrows. The map G^* is a fixed point of T.

The operator $\mathcal{T}$ performs a renormalisation of the family of maps $\{G_r\}$ and hence, itself, forms a dynamical system on the space of unimodal maps. It turns out that repeatedly applying $\mathcal{T}$ to the family $\{G_r\}$ yields convergence to a unique family of unimodal maps, and moreover, $\mathcal{T}$ has a unique fixed point G^* corresponding to a map that can be renormalised infinitely often. Such a map must have a 2^k-cycle for all k and so G^* must correspond to a map at the accumulation point of the period-doubling cascade. This is indicated in the cartoon above (where the thick lines cross).

Considering now maps of the form $G(x) = 1 + ax^2 + bx^4 + O(x^6)$ we can find an approximate solution to the functional equation $\mathcal{T}G = G$ as follows.

Suppose that the k^{th} approximation to G^* is a map of the form $G_k(x) = 1 + a_k x^2 + b_k x^4 + \ldots$. Let λ be the value of $G_k^2(0) = G_k(1)$. Renormalise G_k^2 so that $G_{k+1}(0) = 1$ by defining

$$G_{k+1}(y) = \mathcal{T}G_k \equiv \frac{G_k^2(\lambda y)}{\lambda} \quad \text{say, where } \lambda = G_k^2(0)$$

We are interested in a function G^* that is fixed under the functional map $\mathcal{T}$.

First approximation. As our first try we take $G_k = 1 + a_k x^2 + O(x^4)$ so that we have $G_k(1) = 1 + a_k = \lambda_k$

$$\Rightarrow \quad G_{k+1} = \mathcal{T}G_k = \frac{1 + a_k\{1 + a_k[(1 + a_k)x]^2\}^2}{1 + a_k} = 1 + 2a_k^2(1 + a_k)x^2 + O(x^4)$$

i.e. we have reduced the problem to solving the 1D map

$$a_{k+1} = 2a_k^2(1 + a_k)$$

which has an unstable fixed point $a = -\frac{1}{2}(1 + \sqrt{3}) = -1.37 \Rightarrow \lambda = -0.37$. At the fixed point the Jacobian is $4 + \sqrt{3} = 5.73$: this value is an estimate of the unstable eigenvalue of TG^* and hence is an estimate of the convergence rate δ.

Second approximation. For a more accurate attempt we include the $O(x^4)$ terms: take $G_k = 1 + a_k x^2 + b_k x^4 + O(x^6)$ so that $G_k(1) = 1 + a_k + b_k = \lambda_k$. Comparing the coefficients gives the 2D map

$$\begin{aligned} a_{k+1} &= 2a_k(a_k + 2b_k)\lambda_k \\ b_{k+1} &= (2a_k b_k + a_k^3 + 4b_k^2 + 6a_k^2 b_k)\lambda_k^3 \end{aligned}$$

which has a fixed point at $a = -1.5222$, $b = 0.1276$, $\lambda = -0.3946$. Similar computation of the eigenvalues of the Jacobian matrix (now a 2×2 matrix) gives eigenvalues -0.49 and 4.844. This second value is a better approximation to δ.

In fact, numerical solution shows that the functional equation $\mathcal{T}G = G$ has a fixed point

$$\begin{aligned} G^*(x) &= 1 - 1.52736x^2 + 0.10482x^4 - 0.02671x^6 - 0.00352x^8 + \ldots \\ \Rightarrow \lambda &= G^*(1) = -0.3995 \end{aligned}$$

Including many more higher-order terms and computing the Jacobian matrix numerically yields a single eigenvalue $\delta = 4.6692016\ldots$ (sometimes called 'Feigenbaum's constant') outside the unit circle (i.e. a single unstable direction), and an infinite spectrum of eigenvalues inside the unit circle (stable directions). So in the function space, G^* is a hyperbolic fixed point of $\mathcal{T}$. All this has been made precise by O.E. Lanford (1982) and other authors.

Since the stable manifold S_∞ of G^* occupies 'all but one dimension' of the possible space of functions, typical one-parameter families will cross S_∞ transversely as we vary one parameter, which to some degree explains why such a transition to chaos via a period-doubling cascade appears so frequently in nonlinear systems.

The map $G = \mu_\infty x(1 - x)$, $\mu_\infty = 3.5700\ldots$, is on the stable manifold of G^*, and varying μ around μ_∞ gives situation (i) if $\mu > \mu_\infty$ (G^{2^N} has a horseshoe for some N) or situation (iii) if $\mu < \mu_\infty$ (G^{2^N} has no ORFP for some N and cycle lengths divide 2^N).

If $\mu_\infty - \mu = O(\delta^{-N})$ then, roughly speaking, it takes $O(N)$ renormalisations for the perturbation to grow to $O(1)$ and eliminate the ORFP, thus explaining why $\mu_\infty - \mu_k \sim A\delta^{-k}$ as $k \to \infty$.

Renormalisation is a powerful idea that has been applied to many other dynamical systems problems, and reveals similar universal features (for example in maps of the circle).

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