4.3 The Poincaré–Bendixson Theorem

[See also Glendinning, pp132 - 136.]

Theorem (Poincaré–Bendixson): If the forward orbit $\mathcal{O}^+(\mathbf{x})$ of a point $\mathbf{x}$ remains in a closed, bounded set $K \subset \mathbb{R}^2$ that contains no fixed points then $\omega(\mathbf{x})$ is a periodic orbit.

[**

Remark: Through any non-fixed point one can construct a line segment, called a *unidirectional interval* (UI) or *local transversal*, such that all trajectories crossing the UI do so from the same side.

Lemma: If a trajectory crosses a UI several times then the intersections either move monotonically along the UI or the trajectory is closed and periodic.

Proof of Lemma: If the trajectory is not closed then one of the diagrams

occurs. In each case the trajectory leaves the hatched area and cannot return. □

(Note the implicit use of the Jordan curve lemma, which is why P–B is restricted to $\mathbb{R}^2$).

Proof of P–B Theorem:

1. If $\mathcal{O}^+(\mathbf{x}) \subset$ a closed, bounded set K then this implies $\omega(\mathbf{x})$ is non-empty and $\omega(\mathbf{x}) \subset K$.
2. Consider any $\mathbf{y} \in \omega(\mathbf{x})$. The aim is to show that $\mathcal{O}^+(\mathbf{y})$ is periodic, so we now investigate properties of $\omega(\mathbf{y})$.
3. $\mathcal{O}^+(\mathbf{y}) \subseteq \omega(\mathbf{x}) \subset K$ (by invariance of $\omega(\mathbf{x})$), so $\omega(\mathbf{y})$ is also non-empty, and $\omega(\mathbf{y}) \subset \omega(\mathbf{x})$. Pick a point $\mathbf{z} \in \omega(\mathbf{y})$.
4. Then (by the definition of ω-limit set and continuity) $\mathcal{O}^+(\mathbf{y})$ must have an infinite sequence of intersections with the UI through $\mathbf{z}$. Choose any two such intersection points, say $\mathbf{y}_1$ and $\mathbf{y}_2$.
5. Both $\mathbf{y}_1$ and $\mathbf{y}_2 \in \mathcal{O}^+(\mathbf{y}) \subseteq \omega(\mathbf{x})$, so (by the definition of $\omega(\mathbf{x})$) there is a subsequence of intersections of $\mathcal{O}^+(\mathbf{x})$ with the UI that tends to $\mathbf{y}_1$ and another subsequence that tends to $\mathbf{y}_2$.
6. But the intersections of $\mathcal{O}^+(\mathbf{x})$ and the UI move monotonically along the UI (by the lemma) and cannot have subsequences tending to different limits. Hence we must in fact have that $\mathbf{y}_1 = \mathbf{y}_2$, and that these are equal to $\mathbf{z}$, so that the intersection of $\omega(\mathbf{y})$ and the UI through $\mathbf{z}$ is exactly one point, i.e. $\mathbf{z}$. Hence $\mathcal{O}^+(\mathbf{y})$ is periodic. □

**]