3.2 Example: Basins of Attraction

Consider the two-dimensional system (written in the complex variable $z = x + iy$):

$$\dot{z} = e^{i\alpha} (z - z^3)$$

1. The fixed points are at $z = 0$ and $z = \pm 1$.

2. The Jacobian (again in complex form) is $e^{i\alpha}(1 - 3z^2)$. At $z = 0$ the eigenvalues are $\cos \alpha \pm i \sin \alpha$ and at $z = \pm 1$ they are $-2(\cos \alpha \pm i \sin \alpha)$. (Check this by taking real and imaginary parts!) Hence if $|\alpha| < \pi/2$ the origin is an unstable focus, whereas $\pm 1$ are stable foci and the only attracting sets.

3. The direction of trajectories is best considered in polars $z = re^{i\theta}$:

$$\Rightarrow \dot{r} + ir\dot{\theta} = re^{i\alpha} - r^3e^{i(\alpha + 2\theta)}$$

which leads to

$$\dot{r} = r \cos \alpha - r^3 \cos(\alpha + 2\theta),$$
$$r\dot{\theta} = r \sin \alpha - r^3 \sin(\alpha + 2\theta).$$

4. There is no point in calculating eigenvectors for foci. We can deduce the sense of rotation from the $\dot{\theta}$ equation; for example when $r$ is small and $\alpha > 0$ we see that $\dot{\theta} > 0$.

5. The behaviour of $\dot{r}$ as $r \to 0$ confirms the stability results for $z = 0$.

All trajectories emerging from $z = 0$ are attracted to $z = \pm 1$ except two which escape to infinity along the directions where $\dot{\theta} = 0$ and $\dot{r} > 0$ i.e. $\theta = (\pi - \alpha)/2$ and $\theta = (3\pi - \alpha)/2$. These trajectories form the boundaries of the basins of attraction of $z = \pm 1$.

Below is the phase portrait for $\alpha = 0.96\pi/2$. Refer to example sheet 2 for additional details.