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Unearthing mathematical models for cell-surface dynamics

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Chapter 1

Introduction

We are going to consider partial differential equations posed on evolving surfaces. We will use the conservation of a scalar with a diffusive flux on a evolving hypersurface $\Gamma(t)$ to lead to the diffusion equation

$$\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v} = \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u) \tag{1.1}$$

on $\Gamma(t)$. When $\dot{u}$ is used it refers to the surface material derivative given by:

$$\dot{u} = \frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u. \tag{1.2}$$

1.1 Basic Notation, Surface derivatives, and Formulas

1.1.1 Tangential gradient

Suppose $\mathcal{N}(t) \subset \mathbb{R}^{n+1}$ is an open set which contains $\Gamma(t)$. For any function η defined on $\mathcal{N}(t)$ the tangential gradient on Γ is given by

$$\nabla_\Gamma \eta = \nabla \eta - \nabla \eta \cdot \boldsymbol{n}\boldsymbol{n} = (\underline{D}_1 \eta, ..., \underline{D}_{n+1} \eta)$$

$$\underline{D}_i \eta = \frac{d}{dx_i}\eta - \left[\sum_{i=1}^{n+1} \frac{d}{dx_i}(\eta) n_i \right] n_i$$

Where n is the normal to Γ and n_i is the ith component of n. [1]

1.1.2 Laplace-Beltrami

The tangential divergence of the tangential gradient gives the Laplace-Beltrami operator on $\Gamma(t)$:

$$\Delta_\Gamma \eta = \nabla_\Gamma \cdot \nabla_\Gamma \eta = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i \eta$$

. [1]

1.1.3 Notation

Consider the n dimensional surface $\Gamma(t) \subset \mathbb{R}^{n+1}$. Take $\Gamma(t)$ to be smooth, compact and oriented.
Take an arbitrary portion $\mathcal{M}(t) \subset \Gamma(t)$, where $\mu(x,t)$ is conormal to $\mathcal{M}(t)$; let $\mathcal{M}(t)$ have the boundary $\partial\mathcal{M}(t)$ with unit outer normal tangent to $\mathcal{M}(t)$ which is the conormal μ. Suppose P is a particle located at $\boldsymbol{X_p(t)}$ moving with the vector velocity field

$$\dot{\boldsymbol{X}}_{\boldsymbol{p}}\boldsymbol{(t)} = \boldsymbol{v}(\boldsymbol{X_p(t)}, t).$$

let $u = u(\mathbf{x}, t)$ be the density of a scalar in $\Gamma(t)$ and let the surface flux of u on $\Gamma(t)$ be $\mathbf{q}(\mathbf{x}, \mathbf{t})$.

1.1.4 Integration by Parts on a surface

The formula for integration by parts on $\Gamma(t)$ is

$$\int_{\mathcal{M}} \nabla_\Gamma \eta = -\int_{\mathcal{M}} \boldsymbol{\eta} \cdot H\boldsymbol{n} + \int_{\partial\mathcal{M}} \boldsymbol{\eta} \cdot \boldsymbol{\mu}, \tag{1.3}$$

where H denotes the mean curvature of Γ with respect to $\boldsymbol{n}$, which is given by

$$H = -\nabla_\Gamma \cdot \boldsymbol{n}, \tag{1.4}$$

where $\boldsymbol{n}$ is normal to the surface and μ is co-normal to $\mathcal{M}(t) \in \Gamma(t)$. [2]

1.1.5 Green's formula on a surface

On a surface $\Gamma(t)$ we use the following form of Green's formula, [2]

$$\int_\Gamma \nabla_\Gamma \xi \cdot \nabla_\Gamma \eta = \int_{\partial\Gamma} \xi \nabla_\Gamma \eta \cdot \boldsymbol{\mu} - \int_\Gamma \xi \Delta_\Gamma \eta. \tag{1.5}$$

1.1.6 Leibniz Formula

Lemma 1.1. *Let f be a function defined on $\mathcal{N}(t)$ such that the material derivative is defined. Then*

$$\frac{d}{dt}\int_\Gamma f = \int_\Gamma (\dot{f} + f\nabla_\Gamma \cdot \boldsymbol{v}). \tag{1.6}$$

With the deformation tensor $D(v)_{ij} = \frac{1}{2}(\underline{D}_i v_j + \underline{D}_j v_i)(i, j = 1, ..., n)$,, where v_i is the $i'th$ component of the velocity of the surface v, this becomes,

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma f|^2 = \int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma \dot{f} + \frac{1}{2}\int_\Gamma |\nabla_\Gamma f|^2 \nabla_\Gamma \cdot \boldsymbol{v} - \int_\Gamma D((\boldsymbol{v})\nabla_\Gamma f \cdot \nabla_\Gamma f. \tag{1.7}$$

[2]

1.1.7 Divergence Theorem

If $\boldsymbol{a} = \boldsymbol{a(x,t)}$ is a function on the surface $\Gamma(t)$ and $\boldsymbol{\mu}(\boldsymbol{x}, t)$ is conormal to $\mathcal{M} \subset \Gamma(t)$

$$\int_{\mathcal{M}} \nabla_\Gamma \cdot \boldsymbol{a} = \int_{\partial\mathcal{M}} \boldsymbol{a} \cdot \boldsymbol{\mu} \tag{1.8}$$

[2]

1.1.8 Gronwall's Lemma

Lemma 1.2. *Gronwall's lemma in integration form says that if*

$$f(t) \leq \alpha(t) + \int_a^t \beta(s)f(s).ds$$

then the following is true

$$f(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_a^t \beta(r).dr}.ds.$$

[2]

Chapter 2

Derivation of the Diffusion equation on $\Gamma(t)$

2.0.1 Conservation Law

The conservation law is that for every portion $\mathcal{M}(t) \subset \Gamma(t)$, the following holds, [1]

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\partial\mathcal{M}(t)} \mathbf{q}\cdot\boldsymbol{\mu}. \tag{2.1}$$

Where u is the density of a scalar quantity, q is the surface flux [3] and μ is the unit normal to $\partial\mathcal{M}(t)$. Then using integration by parts this becomes

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} \nabla_\Gamma\cdot\boldsymbol{q} + \int_{\mathcal{M}(t)} \boldsymbol{q}\cdot\boldsymbol{n}H. \tag{2.2}$$

Combining this with the use of Leibniz's formula we get,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma\cdot\boldsymbol{v}) = \int_{\mathcal{M}(t)} \nabla_\Gamma\cdot\boldsymbol{q} + \int_{\mathcal{M}(t)} \boldsymbol{q}\cdot\boldsymbol{n}H \tag{2.3}$$

Since the components of $\boldsymbol{q}$ normal to $\mathcal{M}(t)$ do not contribute to the flux we may assume that $\boldsymbol{q}$ is a tangent vector, and therefore $\boldsymbol{q}\cdot\boldsymbol{n} = 0$. Hence we have,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma\cdot\boldsymbol{v}) = \int_{\mathcal{M}(t)} \nabla_\Gamma\cdot\boldsymbol{q} \tag{2.4}$$

$$\int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma\cdot\boldsymbol{v} - \nabla_\Gamma\cdot\boldsymbol{q}) = 0. \tag{2.5}$$

Since $\mathcal{M}(t)$ is an arbitrary subset $\Gamma(t)$, we have the following:

$$\dot{u} + u\nabla_\Gamma\cdot\boldsymbol{\mu} - \nabla_\Gamma\cdot\boldsymbol{q} = 0. \tag{2.6}$$

for $x \in \Gamma(t),\ t \in (0, T]$ By taking $\boldsymbol{q}$ to be a diffusive flux i.e. ,

$$\boldsymbol{q} = -\mathcal{D}_0\nabla_\Gamma u, \tag{2.7}$$

where $\mathcal{D}_0$ is a symmetric diffusion tensor. [4] (2.6) now becomes,

$$\dot{u} + u\nabla_\Gamma\cdot\boldsymbol{v} + \nabla_\Gamma\cdot(\mathcal{D}_0\nabla_\Gamma u) = 0. \tag{2.8}$$

A solution to this equation requires $u \subset C^2(\Gamma(t))$.

2.0.2 Boundary Conditions

Either $\partial\Gamma = \emptyset$ or Neumann Homogeneous Boundary Conditions [5] $\mathcal{D}_0 \nabla_\Gamma u \cdot \boldsymbol{\mu} = 0$ when $u = 0$ on $\partial\Gamma$ or Dirichlet Homogeneous Conditions $\varphi = 0$ [6].
Multiply (2.8) by an adequate test function and integrate over Γ.

$$\int_\Gamma (\dot{u}\varphi + u\varphi \nabla_\Gamma \cdot \boldsymbol{v} - \varphi \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u)) = 0,$$

$$\int_\Gamma (\dot{u}\varphi + u\varphi \nabla_\Gamma \cdot \boldsymbol{v}) - \int_\Gamma \varphi \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u) = 0. \tag{2.9}$$

The product rule tell us that $\nabla_\Gamma \cdot (\varphi \mathcal{D}_0 \nabla_\Gamma u) - \nabla_\Gamma \varphi \cdot \mathcal{D}_0 \nabla_\Gamma u = \varphi \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u)$ which we can use to get

$$\int_\Gamma (\dot{u}\varphi + u\varphi \nabla_\Gamma \cdot \boldsymbol{v}) - \int_\Gamma \nabla_\Gamma \cdot (\varphi \mathcal{D}_0 \nabla_\Gamma u) + \int_\Gamma \nabla_\Gamma \varphi \cdot \mathcal{D}_0 \nabla_\Gamma u = 0,$$

With the use of the Divergence theorem on the second term this becomes

$$\int_\Gamma \dot{u}\varphi + u\varphi \nabla_\Gamma \boldsymbol{v} - \int_{\partial\Gamma} (\varphi \mathcal{D}_0 \nabla_\Gamma u) \cdot \boldsymbol{\mu} + \int_\Gamma \mathcal{D}_0 \nabla_\Gamma \varphi \cdot \nabla_\Gamma u = 0$$

And the boundary conditions then give

$$\int_\Gamma (\dot{u}\varphi + u\varphi \nabla_\Gamma \cdot \boldsymbol{v}) + \int_\Gamma \mathcal{D}_0 \nabla_\Gamma \varphi \cdot \nabla_\Gamma u = 0. \tag{2.10}$$

Using $(\dot{u\varphi}) = \dot{u}\varphi + u\dot{\varphi}$, we get

$$\int_\Gamma ((\dot{u\varphi}) - u\dot{\varphi} + u\varphi \nabla_\Gamma \cdot \boldsymbol{v}) + \int_\Gamma \mathcal{D}_0 \nabla_\Gamma \varphi \cdot \nabla_\Gamma u = 0.$$

Leibniz's formula then tells us

$$\frac{d}{dt} \int_\Gamma u\varphi - \int_\Gamma u\dot{\varphi} + \int_\Gamma \mathcal{D}_0 \nabla_\Gamma \varphi \cdot \nabla_\Gamma u = 0. \tag{2.11}$$

Chapter 3

Weak solution, existence and uniqueness

3.0.1 Weak Solution

Definition 3.1 (Weak solution)**.** Let $\mathcal{G}_{T_0} = \bigcup_{t\in[0,T_0]} \Gamma(t)x\{t\}$ and $\mathcal{D}_0 \in \mathbb{R}^+$. A function $u \in H^1(\mathcal{G}_{T_0})$ is a weak solution to (1.1) if for almost every $t \in (0, T_0)$

$$\int_\Gamma \dot{u}\varphi + \int_\Gamma u\varphi\nabla_\Gamma \boldsymbol{v} + \int_\Gamma \mathcal{D}_0\nabla_\Gamma u \cdot \nabla_\Gamma\varphi = 0, \quad \forall\varphi(\cdot, t) \in H^1(\Gamma(t)) \tag{3.1}$$

.

Lemma 3.1. *let u be a weak solution to (1.1) then*

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma |\nabla_\Gamma u|^2 + \frac{1}{2}\int_\Gamma u^2\nabla_\Gamma \cdot \boldsymbol{v} = 0 \tag{3.2}$$

Proof. Since (3.1) holds for all φ choose $\varphi = u$, and take $\mathcal{D}_0$=1 (3.1) then we have,

$$\int_\Gamma \dot{u}u + \int_\Gamma u^2\nabla_\Gamma \boldsymbol{v} + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = 0,$$

using Leibniz formula on $\int_\Gamma u^2\nabla_\Gamma \cdot \mathbf{v}$

$$\int_\Gamma \dot{u}u + \frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = \int_\Gamma \dot{(u)}^2,$$

$$\frac{1}{2}\int_\Gamma \dot{(u)}^2 + \frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = \int_\Gamma \dot{u^2},$$

$$\frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = \frac{1}{2}\int_\Gamma \dot{u^2},$$

$$\frac{1}{2}\left[\frac{d}{dt}\int_\Gamma u^2\right] + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = \frac{1}{2}\left[\int_\Gamma \dot{u^2} - \frac{d}{dt}\int_\Gamma u^2\right]$$

Using Leibniz formula

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = -\frac{1}{2}\int_\Gamma (u^2)\nabla_\Gamma \cdot \boldsymbol{v}$$

which gives

$$\frac{d}{dt}\left[\int_\Gamma u^2\right] + \frac{1}{2}\int_\Gamma (u^2)\nabla_\Gamma \cdot \boldsymbol{v} + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u = 0 \tag{3.3}$$

$$\frac{d}{dt}\left[\int_\Gamma u^2\right] + \int_\Gamma |\nabla_\Gamma u|^2 + \frac{1}{2}\int_\Gamma (u^2)\nabla_\Gamma \cdot \boldsymbol{v} = 0 \tag{3.4}$$

□

Lemma 3.2. *let u be a weak solution to* (1.1) *then*

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Gamma |\nabla_\Gamma u|^2\right] + \int_\Gamma (\dot{u})^2 = \frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 \nabla_\Gamma \cdot \boldsymbol{v} - \int_\Gamma D(v)\nabla_\Gamma u \cdot \nabla_\Gamma u - \int_\Gamma \dot{u} u \nabla_\Gamma \cdot \boldsymbol{v} \tag{3.5}$$

Proof. Choose $\varphi = \dot{u}$ and $\mathcal{D}_0 = 1$ in (3.1) then we have,

$$\int_\Gamma (\dot{u})^2 + \int_\Gamma u\dot{u}\nabla_\Gamma \boldsymbol{v} + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \dot{u} = 0,$$

Using Leibniz formula with a diffusion (1.7) this becomes

$$\int_\Gamma (\dot{u})^2 + \int_\Gamma u\dot{u}\nabla_\Gamma \boldsymbol{v} + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u|^2 - \frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 \nabla_\Gamma \cdot \boldsymbol{v} + \int_\Gamma D(v)\nabla_\Gamma u \cdot \nabla_\Gamma u = 0,$$

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u|^2 + \int_\Gamma (\dot{u})^2 = \frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 \nabla_\Gamma \cdot \boldsymbol{v} - \int_\Gamma D(v)\nabla_\Gamma u \cdot \nabla_\Gamma u - \int_\Gamma u\dot{u}\nabla_\Gamma \boldsymbol{v} = 0, \tag{3.6}$$

□

If we integrate (3.2) over t we get,

$$\frac{1}{2}\int_0^{T_0}\frac{d}{dt}\int_{\Gamma(t)} u^2 + \int_0^{T_0}\int_{\Gamma(t)} |\nabla_\Gamma u|^2 + \frac{1}{2}\int_0^{T_0}\int_{\Gamma(t)} u^2 \nabla_\Gamma \cdot \boldsymbol{v} = 0$$

$$\frac{1}{2}\left(\int_{\Gamma(T_0)} u^2 - \int_{\Gamma(0)} u^2\right) + \int_0^{T_0}\int_{\Gamma(t)} |\nabla_\Gamma u|^2 + \frac{1}{2}\int_0^{T_0}\int_{\Gamma t} u^2 \nabla_\Gamma \cdot \boldsymbol{v} = 0$$

By setting $f(t) = \frac{1}{2}\int_{\Gamma(t)} u^2$ we get that,

$$f(T_0) + \int_0^{T_0}\int_{\Gamma(t)} |\nabla_\Gamma u|^2 = f(0) - \int_0^{T_0} f(t)\int_{\Gamma t} \nabla_\Gamma \cdot \boldsymbol{v},$$

$$f(T_0) + \int_0^{T_0}\int_{\Gamma(t)} |\nabla_\Gamma u|^2 \le f(0) - \int_0^{T_0} f(t)\int_{\Gamma t} \nabla_\Gamma \cdot \boldsymbol{v},$$

let $\beta(t) = -\int_{\Gamma(t)} \nabla_\Gamma \cdot \boldsymbol{v}$, Set $g(t) = f(t) + \int_0^t |\nabla_\Gamma u|^2$ then we have:

$$g(T_0) \le f(0) + \int_0^{T_0} f(t)\beta(t),$$

and we use Gronwall's Lemma in integral form (1.2) to get,

$$g(T_0) \le f(0) + \int_0^{T_0} f(0)\beta(t)e^{\int_0^{T_0}\beta(r).dr}$$

$$\int_0^{T_0}\int_{\Gamma(t)}|\nabla_\Gamma u|^2+\frac{1}{2}\int_{\Gamma(T_0)}u^2\leq ||u||_0{}^2-\int_0^{T_0}||u||_0{}^2\int_{\Gamma(t)}(\nabla_\Gamma\cdot\boldsymbol{v})e^{\int_0^{T_0}\int_{\Gamma(r)}\nabla_\Gamma\cdot\boldsymbol{v}.dr}$$

since $e^{\int_0^{T_0}\int_{\Gamma(r)}\nabla_\Gamma\cdot\boldsymbol{v}.dr}=c_0$, and then $\int_{\Gamma(t)}c_0\nabla_\Gamma\cdot\boldsymbol{v}=c_1$ it is clear that

$$\int_0^{T_0}\int_{\Gamma(t)}|\nabla_\Gamma u|^2+\int_{\Gamma(T_0)}u^2\leq c||u_0||^2{}_{L^2(\Gamma(t))}$$

$$\exists T^*\in(0,T_0)\ s.t\ \forall t\in(0,T_0)\int_{\Gamma(t)}u^2\leq\int_{\Gamma(T^*)}u^2$$

and hence

$$\sup_{t\in(0,T_0)}||u||^2_{L^2(\Gamma(t)}=\int_{\Gamma(T^*)}u^2$$

$$\sup_{t\in(0,T_0)}||u||^2_{L^2(\Gamma(t)}+\int_0^{T_0}||\nabla_\Gamma u||^2_{L^2(\Gamma(t))}\leq c||u_0||^2_{L^2(\Gamma(0))}\tag{3.7}$$

Similar to (3.2) if we integrate (3.5) over t we get,

$$\int_0^{T_0}\int_\Gamma D(\boldsymbol{v})\nabla_\Gamma u\cdot\nabla_\Gamma u+\int_0^{T_0}\int_{\Gamma(t)}\dot{u}^2+\int_0^{T_0}\int_{\Gamma(t)}u\dot{u}\nabla_\Gamma\cdot v+\frac{1}{2}\int_0^{T_0}\frac{d}{dt}\int_{\Gamma(t)}|\nabla_\Gamma u|^2-\frac{1}{2}\int_0^{T_0}\int_{\Gamma(t)}|\nabla_\Gamma u|^2\nabla_\Gamma\cdot v=0$$

$$\int_0^{T_0}\int_\Gamma D(\boldsymbol{v})\nabla_\Gamma u\cdot\nabla_\Gamma u\frac{1}{2}\left(\int_{\Gamma(T_0)}|\nabla_\Gamma u|^2-\int_{\Gamma(0)}|\nabla_\Gamma u|^2\right)=$$

$$-\int_0^{T_0}\int_{\Gamma(t)}u\dot{u}\nabla_\Gamma\cdot v+\frac{1}{2}\int_0^{T_0}\int_{\Gamma(t)}|\nabla_\Gamma u|^2\nabla\cdot v-\int_0^{T^0}\int_{\Gamma(t)}\dot{u}^2$$

If we set $g(T_0)=\frac{1}{2}\int_{\Gamma(T_0)}|\nabla_\Gamma u|^2$, $\alpha_1(T_0)=\int_0^{T^0}\int_{\Gamma(t)}u\dot{u}\nabla_\Gamma\cdot v$, $\alpha_2(T_0)=\frac{1}{2}||u_0||^2_{H(\Gamma(0))}$, $\alpha_3(T_0)=\int_\Gamma D(v)\nabla_\Gamma u\cdot\nabla_\Gamma u$, $\alpha_4(T_0)=\int_{\Gamma(t)}\dot{u}^2$ and $\beta_2(t)=\int_{\Gamma(t)}\nabla_\Gamma\cdot v$, then,

$$g(T_0)+\alpha_3(T_0)+\alpha_4(T_0)\leq\alpha_1(T_0)+\alpha_2(T_0)+\int_0^{T_0}g(t)\beta_2(t)$$

and by using Gronwall's integration lemma again

$$g(T_0)+\alpha_3(T_0)+\alpha_4(T_0)\leq\alpha_1(T_0)+\alpha_2(T_0)+\int_0^{T_0}(\alpha_1(t)+\alpha_2(t))e^{\int_0^{T_0}\beta_2(r).dr}$$

$$\frac{1}{2}\int_{\Gamma(T_0)}|\nabla_\Gamma u|^2+\int_\Gamma D(v)\nabla_\Gamma u\cdot\nabla_\Gamma u+\int_{\Gamma(t)}\dot{u}^2\leq$$

$$\int_0^{T^0}\int_{\Gamma(t)}u\dot{u}\nabla_\Gamma\cdot v+\frac{1}{2}||u_0||^2_{H(\Gamma(0))}+\int_0^{T_0}(\alpha_1(t)+\alpha_2(t))e^{\int_0^{T_0}\int_{\Gamma(r)}\nabla_\Gamma\cdot v.dr}$$

$$\frac{1}{2}\int_{\Gamma(T_0)}|\nabla_\Gamma u|^2\leq c\frac{1}{2}||u_0||^2_{H(\Gamma(0))}$$

$\exists T^*\in(0,T_0)\ s.t\ \forall t\in(0,T_0)$
$\int_{\Gamma(t)}|\nabla_\Gamma u|^2\leq\int_{\Gamma(T^*)}|\nabla_\Gamma u|^2$ and hence $\sup_{t\in(0,T_0)}||u||^2_{H(\Gamma(0))}=\int_{\Gamma(T^*)}|\nabla_\Gamma u|^2$, and therefore,

$$\sup_{t\in(0,T_0)}||u||^2_{H(\Gamma(0))}+\int_0^{T_0}||u||^2_{H(\Gamma(t))}\leq c\frac{1}{2}||u_0||^2_{H(\Gamma(0))}\tag{3.8}$$

Theorem 3.0.1. *let $u \in H^1(\Gamma(0))$. Then there exists a unique weak solution to* (1.1) *and* (3.2) *and* (3.5) *hold.*

let $\varphi_j^0, j \in \mathbb{N}$, denote the eigen functions of the Laplace-Beltrami operator on the initial surface, Δ_{Γ_0}.
Let $\phi = \phi(y,t), y \in \Gamma_0, 0 \leq t \leq T_0$ and denote the diffeomorphism between $\Gamma_0$and $\Gamma(t)$.
Set $\varphi_j(\phi(\cdot,t),t) = \varphi_j^0$.
This gives a countable dense subset $\{\varphi_j(\cdot,t)|j \in \mathbb{N}\}$ of $H^1(\Gamma(t))$.
For $j = 1, 2, ...N$ the transport property $\varphi_j = 0$ on Γ holds.

Our ansatz for a Galerkin solution of (3.1) from $X_N = span(\varphi_1(\cdot,t), \varphi_2(\cdot,t), ..., \varphi_N(\cdot,t))$ is

$$u_N(\boldsymbol{x},t) = \sum_{j=1}^{N} u_j(t)\varphi_j(\boldsymbol{x},t),$$

$$u_j(0) = (u_0, \varphi_j^0)_{L^2(\Gamma_0)}.$$

Due to the transport property we have

$$\dot{u}_N = \sum_{j=1}^{N} \dot{u}_j \varphi_j \ in \ X_n.$$

By linear Ordinary Differential Equation Theory we have existence and uniqueness of u_N satisfying:

$$\frac{d}{dt}\int_{\Gamma(t)} u_N\varphi + \int_{\Gamma(t)} \nabla_\Gamma u_N \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u_N \dot{\varphi}.$$

$$\forall \varphi(\cdot,t) \in X_N.$$

From Lemma (3.2) we can see that

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma {u_N}^2 + \int_\Gamma |\nabla_\Gamma u_N|^2 + \frac{1}{2}\int_\Gamma {u_N}^2 \nabla_\Gamma \cdot \boldsymbol{v} = 0 \tag{3.9}$$

$$\int_0^{T_0} \frac{1}{2}\frac{d}{dt}\int_\Gamma {u_N}^2 + \int_0^{T_0}\int_\Gamma |\nabla_\Gamma u_N|^2 + \int_0^{T_0}\frac{1}{2}\int_\Gamma {u_N}^2 \nabla_\Gamma \cdot \boldsymbol{v} = 0$$

$$\frac{1}{2}\int_{\Gamma(T_0)} {u_N}^2 - ||u_0||^2_{L^2(\Gamma)} + \int_\Gamma |\nabla_\Gamma u_N|^2 + \frac{1}{2}\int_\Gamma {u_N}^2 \nabla_\Gamma \cdot \boldsymbol{v} = 0$$

By setting $h(T_0) = \frac{1}{2}\int_{\Gamma(T_0)} {u_N}^2$, $\alpha_3(t) = ||u_0||^2_{L^2(\Gamma)}$, and $\beta_3(t) = -\int_{\Gamma(t)} \nabla_\Gamma \cdot \boldsymbol{v}$, we have:

$$h(T_0) \leq \alpha_3(t) + \int_0 T_0 h(t)\beta_3(t)$$

so we can use Gronwall's Lemma (1.2), to get,

$$h(T_0) \leq \alpha_3(t) + \int_0^{T_0} \alpha_3(t)\beta_3(t)e^{\int_0^{T_0} \beta_3(r).dr}$$

$$\frac{1}{2}\int_{\Gamma(T_0)} {u_N}^2) \leq ||u_0||^2_{L^2(\Gamma)}) - \int_0^{T_0} ||u_0||^2_{L^2(\Gamma)} \int_{\Gamma(t)} \nabla_\Gamma \cdot \boldsymbol{v} e^{\int_0^{T_0} - \int_{\Gamma(r)} \nabla_\Gamma \cdot \boldsymbol{v}.dr}$$

$$\frac{1}{2}\int_{\Gamma(T_0)} u_N{}^2) \leq c||u_0||^2_{L^2(\Gamma)})$$

$$\sup_{t\in(0,T_0)}\int_{\Gamma(t)} u_N(\cdot,t)^2.dA + \int_0 T_0 \int_{\Gamma(t)} |\nabla_\Gamma u_N|^2 \leq C \tag{3.10}$$

$C = C(\Gamma(t), t, u_0)$ C does not depend on N. From Lemma (3.5) we get

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u_N|^2 + \int_\Gamma |\dot{u_N}|^2 = \frac{1}{2}\int_\Gamma |\nabla_\Gamma u_N|^2 \nabla_\Gamma \cdot \boldsymbol{v} - \int_\Gamma D(v)\nabla_\Gamma u_N \cdot \nabla_\Gamma u_N - \int_\Gamma \dot{u_N} u_N \nabla_\Gamma \cdot \boldsymbol{v} \tag{3.11}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u_N|^2 + \int_\Gamma |\dot{u_N}|^2 \leq c\frac{1}{2}\int_\Gamma |\nabla_\Gamma u_N|^2 \nabla_\Gamma \cdot \boldsymbol{v} + c\int_\Gamma \dot{u_N} u_N \nabla_\Gamma \cdot \boldsymbol{v}$$

Integrating over with respect to t over $(0, T_0)$ gives,

$$\frac{1}{2}\int_{\Gamma(T_0)} |\nabla_\Gamma u_N|^2 - \frac{1}{2}\int_{\Gamma(0)} |\nabla_\Gamma u_N|^2 \leq c\int_0^{T_0} \frac{1}{2}\int_\Gamma |\nabla_\Gamma u_N|^2 \nabla_\Gamma \cdot \boldsymbol{v} + c\int_0^{T_0}\int_\Gamma \dot{u_N} u_N \nabla_\Gamma \cdot \boldsymbol{v}$$

Using (3.10) $c\int_0^{T_0}\int_\Gamma \dot{u_N} u_N \nabla_\Gamma \cdot \boldsymbol{v} \leq C_0$, let $k(T_0) = \frac{1}{2}\int_{\Gamma(T_0)} |\nabla_\Gamma u_N|^2$ then we have,

$$k(T_0) \leq k(0) + C_0 + \int_0^{T_0} k(t)\beta(t)$$

So we can use Gronwall's Lemma ((1.2)) to get

$$k(T_0) \leq k(0) + C + \int_0^{T_0} ((k(0) + C_0)\,\beta(t) e^{\int_0^{T_0}\beta(r)}$$

$$\frac{1}{2}\int_{\Gamma(T_0)} |\nabla_\Gamma u_N|^2 \leq \frac{1}{2}\int_{\Gamma(0)} |\nabla_\Gamma u_N|^2 + C + \int_0^{T_0}\left(\frac{1}{2}\int_{\Gamma(0)} |\nabla_\Gamma u_N|^2 + C_0\right) C_1$$

We arrive at the estimate

$$\int_0^{T_0}\int_{\Gamma(t)} \dot{u}_N(\cdot,t)^2 dAdt + \sup_{t\in(0,T_0)}\int_{\Gamma(t)} |\nabla_\Gamma u_N(\cdot,t)|^2 dA.dt \leq C \tag{3.12}$$

(3.10) and (3.12) obtain the boundedness of $(u_N)_{N\in\mathbb{N}}$ in $H^1(\mathcal{G}_{T_0})$ thus

$$\exists u = u(\boldsymbol{x}, t) \in H^1(\mathcal{G}_{T_0})\ s.t\ u_N \rightharpoonup u\ (N \to \infty)\ in\ H^1(\mathcal{G}_{T_0}).$$

Chapter 4

System of Partial Differential Equations

We have looked at the linear diffusion equation on an evolving surface $\Gamma(t)$. Now we are going to consider a semi-linear system of equations on Γ.

4.0.1 Problem Setting

Consider the two functions $u,\ v:\ [0,T]\times\Gamma\to\mathbb{R},\ \Gamma\subset\mathbb{R}^N$; Γ is a bounded, sufficiently regular domain and $T>0$. The partial differential equations we are going to consider describes the interactions between u and w when the velocity of Γ is 0. These are:

$$\frac{\partial u(t,\boldsymbol{x})}{\partial t}=\mathcal{D}_u\Delta u(t,\boldsymbol{x})+f(u(t,\boldsymbol{x},w(t,\boldsymbol{x})),\quad (t,\boldsymbol{x})\in(0,T)\times\Gamma \tag{4.1}$$

and

$$\frac{\partial w(t,\boldsymbol{x})}{\partial t}=\mathcal{D}_w\Delta w(t,\boldsymbol{x})+g(u(t,\boldsymbol{x}),w(t,\boldsymbol{x})),\quad (t,\boldsymbol{x})\in(0,T)\times\Gamma \tag{4.2}$$

The motivation is to extend this to when u and w are functions over an evolving surface $\Gamma(t)$, which $\Gamma(t)$ is smooth, compact and oriented. And the velocity vector field of Γ is $\boldsymbol{v}$ and the normal vector field is $\boldsymbol{n}$.
Consider an arbitrary portion $\mathcal{M}(t)\subset\Gamma(t)$, The conservation law gives,

$$\frac{d}{dt}\int_{\mathcal{M}(t)}u=-\int_{\partial\mathcal{M}(t)}\boldsymbol{q_u}\cdot\boldsymbol{\mu}+\hat{f}(u,w) \tag{4.3}$$

$$\frac{d}{dt}\int_{\mathcal{M}(t)}w=-\int_{\partial\mathcal{M}(t)}\boldsymbol{q_w}\cdot\boldsymbol{\mu}+\hat{g}(u,w) \tag{4.4}$$

Where u and w are the densities of scalars on $\Gamma(t)$ and $\mathbf{q_u}$ and $\mathbf{q_w}$ denote the surface flux of u and w respectively such that $q_u=-\mathcal{D}_u\nabla_\Gamma u$, and $q_w=-\mathcal{D}_w\nabla_\Gamma w$. Using the integration by parts formula (1.3), we have

$$\int_{\partial\mathcal{M}}\boldsymbol{q_u}\cdot\boldsymbol{\mu}=\int_{\mathcal{M}}\nabla_\Gamma\cdot\boldsymbol{q_u}+\int_{\mathcal{M}}\boldsymbol{q_u}\cdot H\boldsymbol{n},$$

$$\int_{\partial\mathcal{M}} \boldsymbol{q_w} \cdot \boldsymbol{\mu} = \int_{\mathcal{M}} \nabla_\Gamma \cdot \boldsymbol{q_w} + \int_{\mathcal{M}} \boldsymbol{q_w} \cdot H\boldsymbol{n},$$

Where H is the mean curvature of Γ with respect to the normal $\boldsymbol{n}$, which is given by $H = -\nabla_\Gamma \cdot \boldsymbol{n}$. The components of $\boldsymbol{q_u}$, $\boldsymbol{q_w}$normal to $\mathcal{M}(t)$ do not contribute to the flux, so $\boldsymbol{q_u}$, and $\boldsymbol{q_w}$ are tangent vectors and hence $\boldsymbol{q_u} \cdot \boldsymbol{n} = 0$, $\boldsymbol{q_w} \cdot \boldsymbol{n} = 0$ and therefore,

$$\int_{\partial\mathcal{M}(\sqcup)} \boldsymbol{q_u} \cdot \boldsymbol{\mu} = \int_{\mathcal{M}} \nabla_\Gamma \cdot \boldsymbol{q_u},$$

$$\int_{\partial\mathcal{M}(t)} \boldsymbol{q_w} \cdot \boldsymbol{\mu} = \int_{\mathcal{M}} \nabla_\Gamma \cdot \boldsymbol{q_w}.$$

So we now have

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = -\int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_u} + \hat{f}(u,w), \tag{4.5}$$

$$\frac{d}{dt}\int_{\mathcal{M}(t)} w = -\int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_w} + \hat{g}(u,w). \tag{4.6}$$

We also have from Leibniz formula (1.6), that

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v}),$$

$$\frac{d}{dt}\int_{\mathcal{M}(t)} w = \int_{\mathcal{M}(t)} (\dot{w} + w\nabla_\Gamma \cdot \boldsymbol{v}),$$

so we get,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v}) = -\int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_u} + \hat{f}(u,w),$$

$$\frac{d}{dt}\int_{\mathcal{M}(t)} w = \int_{\mathcal{M}(t)} (\dot{w} + w\nabla_\Gamma \cdot \boldsymbol{v}) = -\int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_w} + \hat{g}(u,w),$$

rearranging gives,

$$\int_{\mathcal{M}(t)} (\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v}) + \int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_u} = \hat{f}(u,w)$$

$$\int_{\mathcal{M}(t)} (\dot{w} + w\nabla_\Gamma \cdot \boldsymbol{v}) + \int_{\mathcal{M}(t)} \nabla_\Gamma \cdot \boldsymbol{q_w} = \hat{g}(u,w),$$

Take $\hat{f}(u,w)$ and $\hat{g}(u,w)$ to be the anti-derivatives of $f(u,w)$ and $g(u,w)$ respectively. Since $\mathcal{M}(t)$ is an arbitrary portion of arbitrary size we then get

$$(\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v} + \nabla_\Gamma \cdot \boldsymbol{q_u}) = f(u,w)$$

$$(\dot{w} + w\nabla_\Gamma \cdot \boldsymbol{v} + \nabla_\Gamma \cdot \boldsymbol{q_w}) = g(u,w),$$

Using $q_u = -\mathcal{D}_u\nabla_\Gamma u$, and $q_w = -\mathcal{D}_w\nabla_\Gamma w$ this then becomes:

$$(\dot{u} + u\nabla_\Gamma \cdot \boldsymbol{v} - \nabla_\Gamma \cdot (\mathcal{D}_u\nabla_\Gamma u)) = f(u,w) \tag{4.7}$$

$$(\dot{w} + w\nabla_\Gamma \cdot \boldsymbol{v} - \nabla_\Gamma \cdot (\mathcal{D}_w\nabla_\Gamma w)) = g(u,w), \tag{4.8}$$

Assuming $\mathcal{D}_u$,and$\mathcal{D}_w$ do not depend $x(t)$,

$$\dot{u} = \mathcal{D}_u\Delta_\Gamma u + u\nabla_\Gamma \cdot \boldsymbol{v} + f(u,w)$$

$$\dot{w} = \mathcal{D}_w\Delta_\Gamma w - w\nabla_\Gamma \cdot \boldsymbol{v} + g(u,w),$$

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