Lecture 10 – Splines

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10.1 Introduction to Splines
First applied by Poirier & Garber (1974) in a study of profit rates in three distinct periods. They treated time by means of a linear spline:

Period 1 - \( y_t = \alpha_1 + \beta_1 t + u_t \) \( t \leq a \)
Period 2 - \( y_t = \alpha_2 + \beta_2 t + u_t \) \( a < t \leq b \)
Period 3 - \( y_t = \alpha_3 + \beta_3 t + u_t \) \( b < t \)

Figure 1.1

We might perform three separate regressions on this data and this result would look like Figure 1.2. There is nothing in the unrestricted estimation process to ensure that the functions meet at the join points (or knots) \( t = a \) and \( t = b \). Figure 1.3 illustrates a linear spline, or piecewise linear function, which eliminates instantaneous jumps or discontinuities in the function at the ‘join’, or ‘knot’.

Figure 1.2

Here the x axis is time. It need not be and one can fit a spline to, e.g., wages regressed on years experience [The Mincer curve]. The linear spline may be fitted in two ways, and these are outlined in the following two sections.
10.2 Estimating a Spline – Method One

The function is given as:

\[ w_{1t} = t \]
\[ w_{2t} = \begin{cases} 0 & \text{if } t \leq a \\ t - a & \text{if } a < t \end{cases} \]
\[ w_{3t} = \begin{cases} 0 & \text{if } t \leq b \\ t - b & \text{if } b < t \end{cases} \]

Figure 2.1

The function is then re-parametised:

\[ y_t = \alpha + \delta_1 w_{1t} + \delta_2 w_{2t} + \delta_3 w_{3t} + u_t \]

Figure 2.2

Comparing Figure 1.1 and 2.2 we get:

\[ \beta_1 = \delta_1 \]
\[ \beta_2 = \delta_1 + \delta_2 \]
\[ \alpha_2 = \alpha_1 - \delta_2 a \]
\[ \beta_3 = \delta_1 + \delta_2 + \delta_3 \]
\[ \alpha_3 = \alpha_2 - \delta_3 b \]

Figure 2.3

Fitting Figure 2.3 by OLS is OK. Testing the significance of \( \delta_1 (= \beta_1) \) is asking whether there is a trend in the first period, \( \delta_2 \) whether the slope in the second period is significantly different from in the first, and \( \delta_3 \) whether the slope in the third period is significantly different from in the second.
10.3 Estimating a Spline – Method Two

An alternative method, which is the primary focus of the lecture, is to use restricted least squares. Looking at Figure 1.1 the restrictions implied by the join points are:

\[
\begin{align*}
\alpha_1 + \beta_1 a &= \alpha_2 + \beta_2 a \\
\alpha_2 + \beta_2 b &= \alpha_3 + \beta_3 b
\end{align*}
\]

Figure 3.1

These may be set up as:

\[
\begin{bmatrix}
1 & a & -1 & -a & 0 & 0 \\
0 & 0 & 1 & b & -1 & -b \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\alpha_3 \\
\beta_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

Figure 3.2

Figure 3.2 in matrix form is:

\[
R\beta = r
\]

Figure 3.3

Now let us define the restricted least squares estimator as \( b_* \), This must satisfy our restrictions:

\[
Rb_* = r
\]

Figure 3.4

The assumed model is \( y = X\beta + u \). The objective is to choose \( b_* \) so as to minimize the sum of squared residuals subject to satisfying this constraint. Therefore we set up a Lagrangian, minimising the sum of squares subject to Figure 3.4.
\[
\phi = (y - Xb_s)'(y - Xb_s) - 2\lambda'(Rb_s - r)
\]
\[
\frac{\partial \phi}{\partial b_s} = -2X'\dot{y} + 2X'Xb_s - 2R'\lambda = 0
\]
\[
\frac{\partial \phi}{\partial \lambda} = -2(Rb_s - r) = 0
\]

**Figure 3.5**

The 2 premultiplying \( \lambda' \) is for simplicity in the differentiation. \( \lambda' \) is a vector of Lagrangean multipliers one for each constraint. The first differentiation is discussed in greater detail in last terms lecture in deriving the OLS estimator. Rearranging the second equation in Figure 3.5 and premultiplying by \( R(X'X)^{-1} \) gives:

\[
Rb_s - R(X'X)^{-1}X'\dot{y} - R(X'X)^{-1}R'\lambda = 0
\]

**Figure 3.6**

Using the third equation in Figure 3.5, and the OLS formula \( b = (X'X)^{-1}X'Y \), we can get:

\[
r - Rb - R(X'X)^{-1}R'\lambda = 0
\]

\[
\lambda = [R(X'X)^{-1}R']^{-1}(r - Rb)
\]

**Figure 3.7**

Returning to the second equation in Figure 3.5 we can rearrange this to get:

\[
X'Xb_s = R'\lambda + X'\dot{y}
\]

\[
b_s = (X'X)^{-1}[R'\lambda + X'\dot{y}]
\]

**Figure 3.8**

Inserting Figure 3.7 into Figure 3.8 we get:

\[
b_s = (X'X)^{-1}X'\dot{y} + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - Rb)
\]

\[
= b + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - Rb)
\]

**Figure 3.9**

Here \( b \) is the unrestricted OLS estimator. Figure 3.9 defines the **restricted least squares estimator** for any set of restrictions (not just the ones we have set up relating to splines) embodied in \( Rb_s = r \).
This example used time, but works equally well with other continuous variables such as experience in the Mincer equation.

### 10.4 Cubic Splines

One disadvantage with the method presented so far is that the first derivative is not continuous. Thus at the join points there can be a sharp change in the slope, which is probably unrealistic. To overcome this we can use cubic or quadratic splines. We will illustrate with a cubic spline. Suppose we have a two-variable relationship with two known knots or splines at \(x_a\) and \(x_b\). Now instead of a linear relationship, we will use a third-degree polynomial in \(x\). i.e.:

\[
y_i = \alpha_{i1} + \beta_{i1} x + \beta_{i2} x^2 + \beta_{i3} x^3 + u\quad i = 1,2,3
\]

**Figure 4.1**

Where the subsets are defined by:

- \(i = 1\) \quad \(x \leq x_a\)
- \(i = 2\) \quad \(x_a < x \leq x_b\)
- \(i = 3\) \quad \(x_b < x\)

**Figure 4.2**

The restrictions implied by continuity at the knots are then:

\[
\begin{align*}
\alpha_{11} + \beta_{11} x_a + \beta_{12} x_a^2 + \beta_{13} x_a^3 &= \alpha_{21} + \beta_{21} x_a + \beta_{22} x_a^2 + \beta_{23} x_a^3 \\
\alpha_{21} + \beta_{21} x_b + \beta_{22} x_b^2 + \beta_{23} x_b^3 &= \alpha_{31} + \beta_{31} x_b + \beta_{32} x_b^2 + \beta_{33} x_b^3
\end{align*}
\]

**Figure 4.3**

We further impose continuity of the first derivatives with respect to \(x\) of the cubic spline function at the two join points, which implies:

\[
\begin{align*}
\beta_{11} + 2 \beta_{12} x_a + 3 \beta_{13} x_a^2 &= \beta_{21} + 2 \beta_{22} x_a + 3 \beta_{23} x_a^2 \\
\beta_{21} + 2 \beta_{22} x_b + 3 \beta_{23} x_b^2 &= \beta_{31} + 2 \beta_{32} x_b + 3 \beta_{33} x_b^2
\end{align*}
\]

**Figure 4.4**

Continuity of the second derivative implies:

\[
\begin{align*}
2 \beta_{12} + 6 \beta_{13} x_a &= 2 \beta_{22} + 6 \beta_{23} x_a \\
2 \beta_{22} + 6 \beta_{23} x_b &= 2 \beta_{32} + 6 \beta_{33} x_b
\end{align*}
\]

**Figure 4.5**
Thus the cubic spline merely allows discontinuities in the third derivatives at the join points. The cubic spline may be estimated by fitting Figure 4.1 and estimating the twelve parameters subject to the [b] restrictions set out as in Figure 3.9.

Think what the restriction matrix in 3.2 would look like with these extra restrictions added.

**Useful references:**