ELEMENTS OF PRIME ORDER IN THE UPPER CENTRAL SERIES OF A GROUP OF PRIME-POWER ORDER

A. CARANTI, C. M. SCOPPOLA, AND G. TRAUSTASON

ABSTRACT. We investigate the occurrence of elements of order p in the upper central series of a finite p-group.

1. Introduction

In his Mathematics Stack Exchange post [Sik], Igor Sikora asked the following question, which originates with Cihan Okay.

Question 1.1. Let $p > 2$ be a prime. Is there a finite p-group G with the following properties?

(1) G is generated by elements of order p,
(2) G is non-abelian, and
(3) for every pair of non-commuting elements $x, y \in G$ of order p, their product xy has order greater than p.

A dihedral group of order $2^n \geq 8$ provides an example of a group which satisfies (1)–(3) for $p = 2$, hence the requirement for the prime p to be odd.

In an answer [Car] to the above post, the first author showed that there is no group satisfying the properties of Question 1.1, building on the following

Lemma 1.2. *Let p be an odd prime. Let G be a finite, non-abelian p-group, which is generated by elements of order p.*

Then there is an element of order p in $Z_2(G) \setminus Z(G)$.

A negative answer to Question 1.1 is then obtained as follows. Let $t \in Z_2(G) \setminus Z(G)$ have order p. Since $t \notin Z(G)$, and G is generated by elements of order p, there is an element $x \in G$ of order p which does not commute with t. Since $t \in Z_2(G)$, we have that $1 \neq [t, x] \in Z(G)$,

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so that the group $\langle x, t \rangle$ has class two, and thus

$$(xt)^p = x^p t^p [t, x]^{\binom{p}{2}} = [t^{\binom{p}{2}}, x] = 1,$$

as $p > 2$. Igor Sikora kindly asked to include the answer in the paper [COS24]; it appears there as Lemma 4.8 and its proof.

This note originated from our desire to reconcile Lemma 1.2 with some well-known examples. On the one hand there are finite p-groups of arbitrary nilpotence class where all elements of order p lie in the centre. And then certain finite p-groups of maximal class provide examples of groups G of arbitrary nilpotence class $c \geq p-1$ where there are elements of order p in the subsets $Z_i(G) \setminus Z_{i-1}(G)$, for $1 \leq i \leq p-1$, and then only in $Z_c(G) \setminus Z_{c-1}(G)$. (These examples will be described in detail in Subsection 2.2.)

Theorem 1.4 below extends Lemma 1.2, and shows that the examples of maximal class are in some sense typical.

Definition 1.3.

(1) Let G be a finite p-group of nilpotence class c.
   (a) For $1 \leq i \leq c$, the *i-th layer* of the upper central series is the set
   $$Z_i(G) \setminus Z_{i-1}(G).$$
   (b) The *p-spectrum* of G is the set
   $$\{1 \leq i \leq c : \text{there is an element of order } p \text{ in } Z_i(G) \setminus Z_{i-1}(G)\}$$

(2) Let G be a group, and $\sigma \in G$, A *left-normed commutator of length $l \geq 1$ starting with σ* is defined recursively as σ, for $l = 1$, and for $l > 1$ as $[g, y]$, for some y in G, where g is a left-normed commutator of length $l - 1$ starting with σ. We will be writing such commutators as
$$\begin{aligned}&[[\sigma, y_1], y_2] = [\sigma, y_1, y_2],\\ &[[[\sigma, y_1], y_2], y_3] = [\sigma, y_1, y_2, y_3],\\ &\text{etc.}\end{aligned}$$

Theorem 1.4. *Let p be a prime.*

(1) Let G be a finite p-group.
Assume there is an element σ of order p in the k-th layer
$$Z_k(G) \setminus Z_{k-1}(G)$$
of the upper central series, for some $k \geq 2$.
   (a) Then the p-spectrum of G contains the set
   $$\{1, 2, \dots, \min\{k, p-1\}\}.$$
   (b) Among the elements of order p in the layers
   $$1, 2, \dots, \min\{k, p-1\}$$
   there are left-normed commutators starting with σ.

(2) (a) *Given any $1 \le k \le p-1$ and $c \ge k$, there is a finite p-group G of class c whose p-spectrum is*

$$\{1, \dots, k\} .$$

(b) *Given any $n \ge 1$, and any sequence*

$$p \le c_1 < c_2 < \dots < c_n \le c$$

of integers, there is a finite p-group G of class c whose p-spectrum is

$$\{1, \dots, p-1, c_1, c_2, \dots, c_n\} .$$

Remark 1.5.

(1) Part (2) of Theorem 1.4 shows that part (1) provides the only restriction on the occurrence of elements of order p in the layers of the upper central series of a finite p-group.
(2) Not every central series will do in part (1) of Theorem 1.4; this is discussed in Subsection 2.5.

We are grateful to Cihan Okay and Igor Sikora for sharing on Mathematics Stack Exchange the nice Question 1.1, which led to this note.

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2. Proof of Theorem 1.4

2.1. **Proof of part** (1)**.** Let $\sigma \in Z_k(G) \setminus Z_{k-1}(G)$ have order p. For $i \le \min(k, p-1)$ the set

$$\{j : j \ge i, \text{ and in } Z_j(G) \setminus Z_{j-1}(G) \text{ there is a left-normed commutator which starts with } \sigma \text{ and has order } p\ \}$$

is non-empty, as by assumption it contains k. Let m be its minimum.

Suppose by way of contradiction that $m > i$. Let $x \in Z_m(G) \setminus Z_{m-1}(G)$ be a left-normed commutator starting with σ having order p. Since $x \in Z_m(G)$, we have that for all $y \in G$ the element $[x, y]$ lies in $Z_{m-1}(G)$. Since $x \notin Z_{m-1}(G)$, there are elements $y \in G$ such that $[x, y] \in Z_{m-1}(G) \setminus Z_{m-2}(G)$ (see also Subsection 2.5). Since $Z_{m-2}(G) \ge Z_{i-1}(G)$, for such a y we have also $[x, y] \notin Z_{i-1}(G)$; in particular $[x, y] \ne 1$.

Consider an arbitrary central series of G written in descending order as $G = G_1 \ge G_2 \ge \dots$ Let s be the greatest integer for which there is an element y in G_s such that $[x, y] \in Z_{m-1}(G) \setminus Z_{i-1}(G)$. Since for such a y the element $[x, y]$ lies in G_{s+1}, we have that $[x, [x, y]]$ lies in $Z_{i-1}(G)$, and thus in $Z_{p-2}(G)$, as $i \le p-1$. Thus, setting $H = \langle\, x, [x, y]\,\rangle$, we obtain that

$$\frac{\langle\, x, [x, y]\,\rangle\, Z_{p-2}(G)}{Z_{p-2}(G)}$$

is abelian. We have also

$$\begin{aligned}&Z_{p-2}(HZ_{p-2}(G)) =\\&= \{g \in HZ_{p-2}(G) : [g, x_1, \dots, x_{p-2}] = 1, \text{ for all } x_i \in HZ_{p-2}(G)\} \geq\\&\quad \geq \{g \in HZ_{p-2}(G) : [g, x_1, \dots, x_{p-2}] = 1, \text{ for all } x_i \in G\} \geq\\&\qquad \geq Z_{p-2}(G),\end{aligned}$$

so that the group $HZ_{p-2}(G)$, and its subgroup H, have nilpotence class at most $p-1$. Therefore H is regular in the sense of Philip Hall ([Hup67, III.10.1]). Clearly $x^y = y^{-1}xy = x[x, y]$ is an element of order p in H, so that by regularity ([Hup67, III.10.5])

$$[x, y]^p = (x^{-1}x^y)^p = 1.$$

Thus $[x, y] \in Z_{m-1}(G) \setminus Z_{i-1}(G)$ is a left-normed commutator starting with σ having order p in $Z_j(G) \setminus Z_{j-1}(G)$, for some $m > j \geq i$, contradicting the definition of m.

2.2. **Proof of part** (2)**.** We begin by recalling the construction of the unique infinite pro-p group of maximal class, as lifted from [CDC22, Section 5]. For the theory of p-groups of maximal class, see [Bla58], [Hup67, III.14], and [LGM02].

Let p be a prime, and $\mathbb{Z}_p$ be the ring of p-adic integers. Let ω be a primitive p-th root of unity. ω has minimal polynomial

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}_p[x]$$

over $\mathbb{Z}_p$, so that the ring $\mathbb{Z}_p[\omega]$, when regarded as a $\mathbb{Z}_p$-module, is free of rank $p-1$.

The ring $\mathbb{Z}_p[\omega]$ is a discrete valuation ring, with maximal ideal $I = (\omega - 1)$. Consider the automorphism α of the group $E = (\mathbb{Z}_p[\omega], +)$ given by multiplication by ω. Clearly α has order p in $\mathrm{Aut}(E)$.

The infinite pro-p-group of maximal class is the semidirect product

$$M = \langle\, \alpha \,\rangle \ltimes E.$$

For $p = 2$ this is the infinite pro-2-dihedral group, in which all elements outside E have order 2. In general, the following lemma, which appears in [CDC22], is well known.

Lemma 2.1. *All elements of $M \setminus E$ have order p. In particular, M is generated as a pro-p-group by elements of order p.*

Proof. If $g \in E$ and $0 < i < p$, we have

$$\begin{aligned}(\alpha^i g)^p &= \alpha^{ip} g^{\alpha^{i(p-1)} + \alpha^{i(p-2)} + \cdots + \alpha^i + 1}\\&= g^{\alpha^{i(p-1)} + \alpha^{i(p-2)} + \cdots + \alpha^i + 1}\\&= (\omega^{i(p-1)} + \omega^{i(p-2)} + \cdots + \omega^i + 1)g\\&= 0,\end{aligned}$$

since ω^i is a conjugate of ω, for $0 < i < p$, so that it has the same minimal polynomial. □

Since E is an additive group, for $v \in E$ the commutator $[v, \alpha]$ represents the element $-v + v^\alpha = (-1+\omega)v$ of E. Therefore if we denote by $s_1 \in E$ the multiplicative unit of $\mathbb{Z}_p[\omega]$, and let $s_n = [s_{n-1}, \alpha]$, for $n > 1$, we will have $s_n = (\omega - 1)s_{n-1} = (\omega - 1)^{n-1} s_1$. Then we have that for $k \geq 2$ the k-th term $\gamma_k(M)$ of the lower central series of M is the closed subgroup spanned by $\{s_i : i \geq k\}$. Moreover, since $(\omega - 1)^{p-1} = p\zeta$, for some unit ζ in $\mathbb{Z}_p[\omega]$, we have, for $k \geq 1$,

$$s_{k+p-1} = [s_k, \underbrace{\alpha, \dots, \alpha}_{p-1}] = p\zeta s_k, \tag{2.1}$$

where the repeated commutator is left normed.

For $c \geq 2$ the group $M_c = M/\gamma_{c+1}(M)$ is thus a finite p-group of order p^{c+1} and maximal class c, with

$$Z_{c-1}(M_c) = \langle\, s_2, \dots, s_c \,\rangle\, \gamma_{c+1}(M)/\gamma_{c+1}(M).$$

Lemma 2.1 shows that $M_c \setminus Z_{c-1}(M_c)$ contains elements of order p, and it is immediately seen from (2.1) that the set of elements of order dividing p in $Z_{c-1}(M_c)$ is

$$Z_t(M_c) = \left\langle\, s_{c-(t-1)}, \dots, s_c \,\right\rangle \gamma_{c+1}(M)/\gamma_{c+1}(M),$$

where $t = \min\{c-1, p-1\}$.

Consider also the following split metacyclic groups, for $c \geq 2$:

$$D_c = \begin{cases} \left\langle\, x, y : x^{p^c}, y^{p^c}, [x,y] = x^p \,\right\rangle, & \text{for } p > 2; \\ \left\langle\, x, y : x^{2^c}, y^{2^{c-1}}, [x,y] = x^2 \,\right\rangle, & \text{for } p = 2. \end{cases} \tag{2.2}$$

Since D_c is generated by x, y, its commutator subgroup is the smallest normal subgroup containing the commutator $[x, y] = x^p$, and thus the commutator subgroup of D_c is $\langle\, x^p \,\rangle$. Since x commutes with $[x, y] = x^p$, we have $[x^{p^i}, y] = [x, y]^{p^i} = x^{p^{i+1}}$ for $i \geq 0$, so that D_c is a group of nilpotence class c. Moreover $\Omega_1(D_c) = \{g \in D_c : g^p = 1\}$ has order p^2 and coincides with $Z(D_c)$, where $\Omega_1(D_c) = \left\langle\, x^{p^{c-1}}, y^{p^{c-1}} \,\right\rangle$ for p odd, and $\Omega_1(D_c) = \left\langle\, x^{2^{c-1}}, y^{2^{c-2}} \,\right\rangle$ for $p = 2$.

In the following, we will make use of a couple of elementary facts.

Lemma 2.2. *Let G_1 and G_2 be non-trivial finite p-groups with nilpotence classes c_1, c_2 and p-spectra S_1, S_2.*

Let $G_1 \times G_2$ be their direct product.

Then

(1) $G_1 \times G_2$ has class $\max\{c_1, c_2\}$ and

(2) $G_1 \times G_2$ has p-spectrum $S_1 \cup S_2$.

Proof. For the terms of the upper central series we have

$$Z_i(G_1 \times G_2) = Z_i(G_1) \times Z_i(G_2), \tag{2.3}$$

hence (1).

As to (2), if g is an element of order p in $Z_i(G_1) \setminus Z_{i-1}(G_1)$, say, then $(g, 1)$ is an element of order p in $Z_i(G_1 \times G_2) \setminus Z_{i-1}(G_1 \times G_2)$, according to (2.3).

Conversely if $(g, h) \in Z_i(G_1 \times G_2) = Z_i(G_1) \times Z_i(G_2)$ has order p, then $g \in Z_i(G_1)$, $h \in Z_i(G_2)$, and $g^p = 1 = h^p$. If $(g, h) \notin Z_{i-1}(G_1 \times G_2) = Z_{i-1}(G_1) \times Z_{i-1}(G_2)$, then either $g \notin Z_{i-1}(G_1)$, so that g is an element of order p in $Z_i(G_1) \setminus Z_{i-1}(G_1)$, or $h \notin Z_{i-1}(G_2)$, so that h is an element of order p in $Z_i(G_2) \setminus Z_{i-1}(G_2)$. □

We need some further preliminary work in order to produce some indecomposable examples later. Given two non-trivial finite p-groups G_1, G_2, take elements z_1, z_2 of order p in $Z(G_1), Z(G_2)$ and consider the group $Q = (G_1 \times G_2)/\langle z_1 z_2 \rangle$, where we treat $G_1 \times G_2$ as an internal direct product. As the images $\overline{G_1}, \overline{G_2}$ of G_1, G_2 in Q are isomorphic to G_1, G_2 it follows that the class of Q is the same as that of $G_1 \times G_2$.

Lemma 2.3. *Let $x_i \in G_i$. The following are equivalent*

(1) $\overline{x_1 x_2} \in Z_n(Q)$, and
(2) $x_1 \in Z_n(G_1)$ and $x_2 \in Z_n(G_2)$.

Proof. It is immediate that (2) implies (1)

Now assume (1) In particular

$$[x_1 x_2, g_1, \cdots, g_n] = [x_1, g_1, \cdots, g_n] \in \langle\, z_1 z_2 \,\rangle \cap G_1 = 1$$

for all $g_1, \ldots, g_n \in G_1$ and therefore $x_1 \in Z_n(G_1)$. Similarly we see that $x_2 \in Z_n(G_2)$. □

Note in particular that $\overline{x_1 x_2} \notin Z_n(Q)$ iff either $x_1 \notin Z_n(G_1)$ or $x_2 \notin Z_n(G_2)$. It follows that $\overline{x_1 x_2} \in Z_n(Q) \setminus Z_{n-1}(Q)$ iff either $x_1 \in Z_n(G_1) \setminus Z_{n-1}(G_1)$ or $x_2 \in Z_n(G_2) \setminus Z_{n-1}(G_2)$. Now assume furthermore that $z_1 z_2$ is not a pth power in G. Then $\overline{(x_1 x_2)}^p = 1$ if and only if $(x_1 x_2)^p \in \langle z_1 z_2 \rangle$ iff $(x_1 x_2)^p = 1$ iff $x_1^p = 1$ and $x_2^p = 1$. From this we see that

Proposition 2.4. *Suppose $z_1 z_2$ is not a pth power. Then Q and G have the same nilpotence class and the same p-spectrum.*

We cannot drop the requirement that $z_1 z_2$ is not a pth power as the following example shows.

Example 2.5. Consider the group $H = D_2 \times \langle\, d \,\rangle$ where d is of order p^2. Notice that the p-spectrum of H is $\{1\}$. Now let $K = H/\langle\, x^p d^p \,\rangle$. Then $\overline{xd}$ is of order p in K and $\overline{xd} \in Z_2(K) \setminus Z(K)$ and thus the p-spectrum of K is $\{1, 2\}$.

2.3. **Proof of part** (2a)**.** For $k = 1$ an example is provided by D_c, the group defined in (2.2).

For $k \geq 2$ a decomposable example is provided by $M_k \times D_c$, according to Lemma 2.2.

We now provide two different indecomposable examples.

2.3.1. *First example.* Consider first the case when $c = ke$ is a multiple of k. Let

$$A = \langle\, a_1 \,\rangle \times \cdots \times \langle\, a_k \,\rangle$$

be a homocyclic group of exponent p^e, for some $e \geq 1$. The assignment

$$a_1 \mapsto a_1 a_2, a_2 \mapsto a_2 a_3, \ldots, a_{k-1} \mapsto a_{k-1} a_k, a_k \mapsto a_k a_1^p$$

defines an endomorphism β of A, which is actually an automorphism of order a power of p, as it is such an automorphism modulo the Frattini subgroup of A. Let p^t be the order of β.

Consider the group G which is the semidirect product of A by a cyclic group $\langle\, b \,\rangle$ of order p^{t+1}, with b acting on A via β, so that $b^{p^t} \in Z(G)$. (G coincides with D_e for $k = 1$.)

If $ha \in G$ is an element of order dividing p (that is, it is either the identity or has order p), for some $h \in \langle\, b \,\rangle$ and $a \in A$, then its projection h on $\langle\, b \,\rangle$ has also order dividing p, and thus $h \in \left\langle\, b^{p^t} \,\right\rangle \leq Z(G)$, so that $a \in \Omega_1(A)$. It follows that the elements of G of order dividing p form the set $\left\langle\, \Omega_1(A), b^{p^t} \,\right\rangle = \left\langle\, a_1^{p^{e-1}}, \ldots, a_k^{p^{e-1}}, b^{p^t} \,\right\rangle$. Now the element $a_{k-i+1}^{p^{e-1}}$ of order p lies in $Z_i(G) \setminus Z_{i-1}(G)$, for $1 \leq i \leq k$.

For the general case when c is not a multiple of k, if $c = ke - s$ for some $e \geq 1$ and $1 \leq s < k$, it suffices to take the subgroup of the G we just constructed given by

$$\langle\, a_1^p, \ldots, a_s^p, a_{s+1}, \ldots, a_k, b \,\rangle\,.$$

2.3.2. *Second example.* Let $2 \leq k \leq p-1$ and $c \geq k$. As an application of Proposition 2.4, we produce an example of an indecomposable finite p-group $G_{(k,c)}$ that has class c and p-spectrum $\{1, \cdots, k\}$.

Let $G_1 = D_c$ and $G_2 = \langle s, t \rangle$ be the largest 2-generator group of exponent p and class k. Then $G = G_1 \times G_2$ has the class and p-spectrum we want. We want a group that is furthermore indecomposable. Pick a non-trivial $d \in \gamma_k(G_2)$ and consider the group $Q = G/\left\langle\, x^{p^{c-1}} d \,\right\rangle$. By Proposition 2.4 we know that Q has the same class and p-spectrum as G. It remains to show that Q is indecomposable. We argue by contradiction and suppose that

$$Q = A \times B,$$

where A and B are non-trivial. The group Q is of rank 4. Pick some generators for A and some for B. Suppose these are g_1, g_2, g_3, g_4. Consider first the case when $k \geq 3$. Here $x^{p^{c-1}} \in [G_1, G_1]G_1^p$ and $d \in \gamma_3(G_2)$. There must be two elements among g_1, g_2, g_3, g_4, that are linearly independent modulo $G_2[G_1, G_1]G_1^p$. Without loss of generality we can assume that these are g_1 and g_2 and replacing these with suitable products of their powers, we can assume that

$$g_1 \in \overline{xG_2[G_1, G_1]G_1^p}$$

and

$$g_2 \in \overline{yG_2[G_1, G_1]G_1^p}.$$

As $[x, y] \notin [G_1, G_1]^p\gamma_3(G_1)$, we have that $[g_1, g_2] \neq 1$ and g_1, g_2 must belong to the same component, say A. Again as $[x, y] \notin [G_1, G_1]^p\gamma_3(G_1)$ all the generators in B must be in $\overline{G_2[G_1, G_1]G_1^p}$. Without loss of generality, we can assume that $g_3 \in B$ and that $g_3 = se$ with $e \in [G_1, G_1]G_1^p$. If $g_4 \in B$ we can assume w.l.o.g. that $g_4 = tf$ with $f \in [G_1, G_1]G_1^p$. If $g_4 \in A$, then replacing it by a suitable $g_4 g_1^\alpha g_2^\beta$ we can again assume that g_4 is on the same form $g_4 = tf$. As $[s, t] \notin \gamma_3(G_2)$ we however see that $[g_3, g_4] \neq 1$ and therefore $g_3, g_4 \in B$.

Using the fact that A and B are normal in Q, we see that

$$A \geq [A, \underbrace{\overline{G_1}, \ldots, \overline{G_1}}_{k-1}] = \gamma_k(\overline{G_1})$$

and

$$B \geq [B, \underbrace{\overline{G_2}, \ldots, \overline{G_2}}_{c-1}] = \gamma_c(\overline{G_2}).$$

But then A contains $\overline{x^{p^{c-1}}}$ and B contains $\overline{d^{-1}}$. As these elements are the same we get the contradiction that $A \cap B \neq 1$.

We need some extra care when treating the case $k = 2$. We deal separately with two subcases. Suppose first that $c \geq 3$. The same argument as above works as before in that we get two generators for A, $g_1 \in \overline{xG_2[G_1, G_1]G_1^p}$ and $g_2 \in \overline{yG_2[G_1, G_1]G_1^p}$. The same argument as in the $k \geq 3$ case, also shows that all the generators of B must be in $G_2[G_1, G_1]G_1^p$ and w.l.o.g. we can assume that $g_3 = se$ with $e \in [G_1, G_1]G_1^p$. Here we specifically choose $d = [s, t]$. We have

$$A \geq [A, \underbrace{\overline{G_1}, \ldots, \overline{G_1}}_{k-1}] = \gamma_k(\overline{G_1})$$

and

$$B \geq [B, \overline{G_2}] = \gamma_2(\overline{G_2})$$

as $\gamma_2(\overline{G_2}) = \overline{\langle [s, t] \rangle}$. As before we get the contradiction that $A \cap B \neq 1$.

Finally we deal with the remaining case $c = k = 2$. Here the Frattini subgroup, $F(Q)$, of Q is contained in $Z(Q)$. Again we let $d = [s, t]$. As $[Q, Q]$ is of rank 1, generated by $\overline{d}^{-1} = \overline{x}^p$, we must have that one of A, B is abelian. In particular there is some generator of the form

$g = \overline{x^{\alpha}y^{\beta}s^{\gamma}t^{\delta}e}$, where not all $\alpha, \beta, \gamma, \delta$ are divisible by p and $\bar{e} \in Z(Q)$, where $g \in Z(Q)$. However

$$\begin{aligned} 1 &= [g, \bar{y}] = \overline{[x, y]}^{\alpha} = \bar{d}^{-\alpha} \\ 1 &= [\bar{x}, g] = \overline{[x, y]}^{\beta} = \bar{d}^{-\beta} \\ 1 &= [g, \bar{t}] = \overline{[s, t]}^{\gamma} = \bar{d}^{\gamma} \\ 1 &= [\bar{s}, g] = \overline{[s, t]}^{\delta} = \bar{d}^{\delta}. \end{aligned}$$

that gives the contradiction that p divides all of α, β, γ and δ.

2.4. **Proof of part** (2b)**.** Note that the assumptions imply $c \geq p$. In the case when $n = 1$ and $c_1 = c = p$, an indecomposable example is given by M_p. So from now on we may assume $c > p \geq 2$.

2.4.1. *A decomposable example.* By Lemma 2.2, such an example is given by

$$H = M_{c_1} \times M_{c_2} \times \cdots \times M_{c_n} \times D_c, \tag{2.4}$$

where the factor D_c is redundant when $c = c_n$.

This example also shows that the layers of the upper central series that contain elements of order p may well contain also elements of order greater than p.

2.4.2. *An indecomposable example.* We will construct an indecomposable example as a subgroup G of the decomposable one H of (2.4).

Denote by $a_i, t_{i,1}, \dots, t_{i,c_i}$ the images in M_{c_i} of $\alpha, s_1, \dots, s_{c_i}$.

For each i, consider the element $x_i = a_i t_{1,i}$ of order p of M_{c_i}, which together with a_i generates M_{c_i}. Consider for each i the non-abelian maximal subgroup

$$X_i = \langle\, x_i, \gamma_2(M_{c_i}) \,\rangle$$

of M_i, and the maximal subgroup

$$E = \langle\, x, y^p \,\rangle$$

of $D_c = \langle\, x, y \,\rangle$. Write

$$\widehat{X_i} = \{1\} \times \cdots \times \underbrace{X_i}_{i\text{–th place}} \times \dots \{1\}$$

and

$$\widehat{E} = \{1\} \times \cdots \times \{1\} \times E.$$

For $z \in M_{c_i}$, write

$$\widehat{z} = (1, \dots, 1, \underbrace{z}_{i\text{–th place}}, 1, \dots, 1) \in \widehat{X_i},$$

and for $z \in E$ write

$$\widehat{z} = (1, \dots, 1, z) \in \widehat{E}.$$

Let also

$$a = (a_1, \dots, a_n, y) \in H.$$

Our example will be the subgroup of H given by

$$G = \left\langle a, \widehat{X}_1, \dots, \widehat{X}_n, \widehat{E} \right\rangle.$$

For $z \in X_i$ one has

$$[\widehat{z}, a] = \widehat{[z, a_i]} \in \widehat{X}_i,$$

and for $z \in E$ one has

$$[\widehat{z}, a] = \widehat{[z, y]} \in \widehat{E}.$$

It follows that

$$\left\langle a, \widehat{X}_i \right\rangle \cong M_{c_i}, \quad \left\langle a, \widehat{E} \right\rangle \cong D_c.$$

and for each $k \geq 2$ we have the equality

$$\gamma_k(M_{c_1}) \times \cdots \times \gamma_k(M_{c_n}) \times \gamma_k(D_c) = \gamma_k(G). \tag{2.5}$$

The centre of H is

$$Z(H) = \left\langle \widehat{t_{1,c_1}}, \dots, \widehat{t_{n,c_n}}, \widehat{x^{p^{c-1}}}, \widehat{y^{p^{c-1}}}. \right\rangle$$

for p odd, and

$$Z(H) = \left\langle \widehat{t_{1,c_1}}, \dots, \widehat{t_{n,c_n}}, \widehat{x^{2^{c-1}}}, \widehat{y^{2^{c-2}}}. \right\rangle$$

for $p = 2$. We have $Z(H) \leq G$, and

$$Z(H) = C_H(a) \cap C_H(\widehat{x}),$$

so that $Z(G) = Z(H)$.

Moreover $Z(G)$ is contained in the Frattini subgroup $\mathrm{Frat}(G)$ of G, as

$$\widehat{t_{i,c_i}} \in \widehat{\gamma_{c_i}(M_{c_i})}, \quad \widehat{x^{p^{c-1}}} \in \left\langle \widehat{x^p} \right\rangle$$

and

$$a^p = \widehat{y^p} \tag{2.6}$$

(recall $c \geq 3$). Therefore G has no non-trivial abelian direct factor.

Now $H/Z(H)$ and $G/Z(G)$ have the same structure as H and G, for parameters

$$c_1 - 1, \dots, c_n - 1, c - 1$$

(dropping $c_1 - 1$ if it equals $p - 1$). Therefore $Z_c(G) = G$, and for all $j < c$ the j-th of the upper central series of G coincides with

$$(X_1 \cap Z_j(M_{c_1})) \times \cdots \times (X_n \cap Z_j(M_{c_n})) \times (E \cap Z_j(D_c)),$$

so that the class and the p-spectrum of G are as required.

Suppose now, by way of contradiction, that G admits a non-trivial decomposition $G = G_1 \times G_2$. Since $U = \left\langle \widehat{X_1}, \dots, \widehat{X_n}, \widehat{E} \right\rangle$ is a maximal subgroup of G, there will be an element of G_1, say, of the form au, for some $u \in U$. Consider a fixed set L of minimal generators of G_2; these will be part of a set of minimal generators of G.

Assume there is an element $v \in L$ which is contained in U. Then

$$1 = [au, v] = [a, v]^u [u, v].$$

Since $[u, v] \in \gamma_2(U) \leq \gamma_3(G)$, we obtain $[a, v] \in \gamma_3(G)$. Write

$$v = (v_1, \dots, v_n, d),$$

so that

$$[a, v] = ([a_1, v_1], \dots, [a_n, v_n], [y, d]).$$

Since the groups M_{c_i} are of maximal class, and $[a_i, v_i] \in \gamma_3(M_{c_i})$, we obtain $v_i \in \gamma_2(M_{c_i})$. Also, $[y, d] \in \gamma_3(D_c) = \left\langle x^{p^2} \right\rangle$ implies $d \in \langle x^p, y^p \rangle$.

From (2.5) and (2.6) it follows that $v \in \mathrm{Frat}(G)$, contradicting the fact that v is in a minimal set of generators of G.

Therefore all elements of L can be assumed, taking suitable powers, to be of the form aw, for some $w \in U$. Since the product of such an element by the inverse of another of the same form is in U, we obtain that L has only one element, so that G_2 is a non-trivial cyclic direct factor of G, a contradiction.

2.5. **Not every central series will do.** In a group $G = D_c$, for $p > 2$ and $c \geq 2$, we have $\gamma_2(G) = \langle x^p \rangle$, and $y^{p^{c-1}}$ is an element of order p in the first layer $G \setminus \gamma_2(G)$ of the lower central series. But then the only other layer of the lower central series of G that contains elements of order p is the last non-empty one $\gamma_c(G) \setminus \{1\} = \left\langle x^{p^{c-1}} \right\rangle \setminus \{1\}$.

In fact, in the proof of Subsection 2.1 we have used the implication (2) $\Longrightarrow$ (1) from the following characterisation of the upper central series, which is presumably well-known.

Lemma 2.6. *Let G be a group, let $c \geq 1$, and let*

$$\{1\} = G_0 < G_1 < \dots < G_c = G \tag{2.7}$$

be a central series. The following are equivalent:

(1) *for every $2 \leq m \leq c$ and every $x \in G_m \setminus G_{m-1}$, there is $y \in G$ such that $[x, y] \in G_{m-1} \setminus G_{m-2}$;*
(2) *the series* (2.7) *coincides with the upper central series, that is, $G_i = Z_i(G)$ for all $1 \leq i \leq c$.*

Proof. When $c = 1$ (1) is vacuously true, and (2) holds true as $G = G_1$ is abelian, that is, $G_1 = Z(G)$. So from now on we are assuming $c \geq 2$. The upper central series $G_i = Z_i(G)$ clearly satisfies (1). In fact, if $x \in Z_m(G)$, for some $m \geq 2$, is such that $[x, y] \in Z_{m-2}(G)$ for all $y \in G$, then $x \in Z_{m-1}(G)$, as $Z_{m-1}(G)/Z_{m-2}(G) = Z(G/Z_{m-2}(G))$.

Conversely, assume the series (2.7) satisfies (1), and proceed by induction on $c \geq 2$. We have $G_1 \leq Z(G)$. If by way of contradiction $G_1 < Z(G)$, pick $z \in Z(G) \setminus G_1$. There will be an $m \geq 2$ such that $z \in G_m \setminus G_{m-1}$ but $[z, y] = 1 \in G_{m-2}$ for all $y \in G$, defeating (1). Thus $G_1 = Z(G) = Z_1(G)$.

Now if $xG_1 \in (G_m/G_1) \setminus (G_{m-1}/G_1)$, for some $m \geq 3$, we have $x \in G_m \setminus G_{m-1}$. By (1) there is $y \in G$ for which $[x, y] \in G_{m-1} \setminus G_{m-2}$, so that $[xG_1, yG_1] \in (G_{m-1}/G_1) \setminus (G_{m-2}/G_1)$. We can thus apply the inductive hypothesis to G/G_1 to obtain (2). □

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(A. Caranti) Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38123 Trento, Italy

Email address: andrea.caranti@unitn.it

URL: https://caranti.maths.unitn.it/

(C. M. Scoppola) Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio (Coppito 1), I-67100 Coppito, Italy

Email address: carlo.scoppola@univaq.it

(G. Traustason) Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK

Email address: gt223@bath.ac.uk

URL: https://people.bath.ac.uk/gt223/