

Left 3-Engel elements in groups: A survey

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Abstract. We survey left 3-Engel elements in groups.

1 Introduction

An element a of a group G is left Engel if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$\underbrace{[[[x, a], a], \dots, a]}_{n(x)} = 1.$$

If $n(x)$ is bounded above, then a is bounded left Engel. More precisely, a is left n -Engel if $n(x) \leq n$ for all $x \in G$.

Recall that the Hirsch-Plotkin radical, $\text{HP}(G)$, of G is the product of its locally nilpotent normal subgroups. It is straightforward to see that every element of $\text{HP}(G)$ is left Engel in G and the converse is known for some classes of groups, including solvable groups and finite groups (more generally, groups satisfying the maximal condition on subgroups) [3, 6]. The converse is not true in general, not even for bounded left Engel elements. In fact, whereas it is clear that a left 2-Engel element is always in $\text{HP}(G)$, this remains an open question for left 3-Engel elements. There have been breakthroughs in recent years. In [17] it is shown that every left 3-Engel element of odd order is contained in $\text{HP}(G)$. In [7] this result is generalised to include elements of every order, by replacing left 3-Engel with a stronger condition that we call strong left 3-Engel. Left 3-Engel elements of odd order are strong left 3-Engel.

The results of [24] imply that, in order to show that every left 3-Engel element of finite order is in $\text{HP}(G)$, it suffices to consider elements of order 2. By looking at a similar setting for Lie algebras, there are reasons to doubt that elements of order 2 must be in $\text{HP}(G)$ however. Little is known about left 4-Engel elements, although there are some interesting results in [2].

Groups of prime power exponent satisfy some Engel type conditions and the solution to the restricted Burnside problem also uses the fact that the associated Lie ring satisfies certain Engel type identities [29, 30]. Burnside [5] observed that every element of a group of exponent 3 is a left 2-Engel element, and so the fact that every

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left 2-Engel element lies in the Hirsch-Plotkin radical can be seen as the underlying reason why groups of exponent 3 are locally finite. For groups of 2-power exponent there is a close link with left Engel elements. If a is an involution in a finitely generated group G of exponent 2^n , then

$$\underbrace{[[[x, a], a], \dots, a]}_{n+1} = [x, a]^{(-2)^n} = 1.$$

Thus a is a left $(n + 1)$ -Engel element of G . Hence, if $G/G^{2^{n-1}}$ is finite and the left $(n + 1)$ -Engel elements of G are in $\text{HP}(G)$, then G is finite. For sufficiently large n , the variety of groups of exponent 2^n is not locally finite [13, 18], so for sufficiently large n there are left n -Engel elements not contained in $\text{HP}(G)$. Since groups of exponent 4 are locally finite [23], if all left 4-Engel elements of a group G of exponent 8 are in $\text{HP}(G)$, then G is locally finite.

Is a left 3-Engel element of prime power order in a group G contained in $\text{HP}(G)$? We reduce this question to elements of prime order using the following result by Abdollahi [1]: for every prime p and every left 3-Engel element x of finite p -power, x^p is in the Baer radical of G and, in particular, $\langle x^p \rangle^G$ is locally nilpotent. For the case $p = 2$ this implies the local finiteness of groups of exponent 4, originally proved by Sanov [23]. It is also proved in [1] that two left 3-Engel elements generate a nilpotent group of class at most 4, and this bound is achieved.

We now swap the role of a and x in the definition of a left Engel element. Thus $a \in G$ is a *right Engel element* if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[a,_{n(x)} x] = 1.$$

If $n(x)$ is bounded above by n , then a is a right n -Engel element. By a result of Heineken [11], if a is a right n -Engel element of G , then a^{-1} is a left $(n + 1)$ -Engel element.

In [19] Newell proved that if a is a right 3-Engel element of G , then $a \in \text{HP}(G)$; in fact he proved the stronger result that $\langle a \rangle^G$ is nilpotent of class at most 3. A natural question arises whether $\langle a \rangle^G$ is nilpotent when a is a left 3-Engel element of G . In [22] it was shown that this is not the case by giving an example of a locally finite 2-group with a left 3-Engel element a such that $\langle a \rangle^G$ is not nilpotent and in [8] this was generalised to an infinite family of examples. In [9] this was extended to include every odd prime. Thus for each prime p there is a locally finite p -group G containing a left 3-Engel element a such that $\langle a \rangle^G$ is not nilpotent.

The structure of this survey is as follows. In Section 2 we discuss (strong) sandwich elements in Lie algebras and groups. We give a consistent power-conjugate presentation [12, Section 9.4] of the largest 3-generator sandwich group, which is nilpotent of class 5 and at most 3 if all elements have odd order. We then present one of our main results: every finitely generated strong sandwich group is nilpotent. It follows that a strong left 3-Engel element of a group G always lies in $\text{HP}(G)$; we mention applications for groups of exponents 5, 9 and 8. We do not know if all sandwich groups of rank 4 are nilpotent, but in Section 3 we give partial results. We show that there is a largest finite sandwich group of rank 4 generated by involutions. In

Section 3.1 we obtain a new reduction theorem for sandwich groups of rank 4 with generators that have two commuting pairs. In Section 3.2 we report some results of computations in residually nilpotent sandwich groups of rank 5 generated by involutions. In particular, we show that these are finite if there are at least 3 commuting pairs among the generators. Finally, in Sections 4 and 5 we consider the global nilpotence problem for the normal closure of left 3-Engel elements in locally finite p -groups.

2 Local nilpotence of (strong) sandwich groups and strong left 3-Engel elements

The approach of [7, 17] is by working with sandwich groups; these are group-theoretic analogues of sandwich Lie algebras introduced by Kostrikin [14]. Kostrikin and Zel'manov [16] proved that sandwich Lie algebras are locally nilpotent and this is a key ingredient to both Kostrikin's solution to the restricted Burnside problem for groups of prime exponent [15], and Zel'manov's general solution [29, 30].

2.1 Sandwich Lie algebras

As for group commutators, we use the left normed convention for Lie products.

Definition. An element a of a Lie algebra L is a *sandwich element* if $axa = 0$ and $axya = 0$ for all $x, y \in L$. A Lie algebra is a *sandwich Lie algebra* if it can be generated (as Lie algebra) by sandwich elements.

The second condition is superfluous in odd characteristic: $0 = x(yaa) = xyaa - 2xaya + xaay = 2axya$. In characteristic 2 it is needed as this example shows.

Example. Consider the largest vector space $L = \langle a, b, c \rangle$ over $\text{GF}(2)$ subject to $\text{Id}(c)$ being abelian, $bc = 0$ and $bx b = axa = cxc = 0$, for all $x \in L$. Then L is a Lie algebra generated by $a, b, ab, c(ab)^n$, and $c(ab)^n a$ for $n \geq 0$. To show that L is non-nilpotent (or, equivalently, infinite dimensional), we give a concrete example where these generators are basis elements. Let $y = ab, u_n = c(ab)^n$ and $v_{n+1} = c(ab)^n a$. We have the following multiplication table for these basis elements of L .

$$\begin{aligned} ab &= y, \\ ya &= yb = 0, \\ u_n a &= v_{n+1}, & u_n b &= 0, & u_n y &= u_{n+1}, \\ v_n a &= 0, & v_n b &= u_n, & v_n y &= v_{n+1}, \\ u_n v_m &= 0, & v_n v_m &= 0, & u_n u_m &= 0. \end{aligned}$$

It suffices to show that the Jacobi identity holds for these generators. Since the characteristic is 2, we deduce that:

$$\begin{aligned} aby + bya + yab &= 0, \\ u_n ab + abu_n + bu_n a &= v_{n+1} b + yu_n = 2u_{n+1} = 0, \\ v_n ab + av_n + bv_n a &= yv_n + u_n a = 2u_{n+1} = 0, \\ u_n ay + ayu_n + yu_n a &= v_{n+1} y + u_{n+1} a = 2v_{n+2} = 0, \\ v_n by + byv_n + yv_n b &= u_n y + v_{n+1} b = 2u_{n+1} = 0. \end{aligned}$$

2.2 3-generator sandwich groups

In [26] the notion of a sandwich group was introduced.

Definition. A subset X of a group G is a *sandwich set* if $\langle a, b^g \rangle$ is nilpotent of class at most 2 for all $a, b \in X$ and $g \in \langle X \rangle$. If G is generated by a sandwich set, then G is a *sandwich group*.

The connection with left 3-Engel elements arises because the following are equivalent.

- (1) If a is a left 3-Engel element of a group G , then a is in the locally nilpotent radical of G .
- (2) Every finitely generated sandwich group is nilpotent.

The equivalence is a consequence of the following: if a is a left 3-Engel element in G , then $\langle a \rangle^G$ is a sandwich group; every element of a sandwich set X is left 3-Engel in $\langle X \rangle$.

A significant feature of sandwich groups is that there is a largest sandwich group of any given rank [26].

Theorem 2.1 *Let $R = \langle x, y, z \rangle$ be the free sandwich group of rank 3. If the generators have odd order, then R has class at most 3, else R has class 5.*

We obtained in [26] the following consistent power-conjugate presentation for R .

$$\text{Let } e_1(z, z^x, y) = [z, x, y, y].$$

Generators

$$x_1 = e_1(z, z^x, y), \quad x_2 = e_1(x, x^y, z), \quad x_3 = e_1(y, y^z, x),$$

$$x_4 = [z, x, [z, y]], \quad x_5 = [x, y, [x, z]], \quad x_6 = [y, z, [y, x]], \\ x_7 = [z, x, y], \quad x_8 = [z, y, x]$$

$$x_9 = [z, x], \quad x_{10} = [z, y], \quad x_{11} = [x, y]$$

$$x_{12} = x, \quad x_{13} = y, \quad x_{14} = z.$$

Relations

$$x_1^2 = x_2^2 = x_3^2 = 1, \quad x_4^2 = x_5^2 = x_6^2 = x_3x_2x_1,$$

$$x_4^{x_{12}} = x_4x_3, \quad x_4^{x_{13}} = x_4x_1, \quad x_5^{x_{13}} = x_5x_1, \quad x_5^{x_{14}} = x_5x_2, \\ x_6^{x_{12}} = x_6x_3, \quad x_6^{x_{14}} = x_6x_2, \\ x_7^{x_9} = x_7x_2x_3, \quad x_7^{x_{10}} = x_7x_1, \quad x_7^{x_{11}} = x_7x_1, \quad x_7^{x_{12}} = x_7x_5x_3x_2, \\ x_7^{x_{13}} = x_7x_1, \quad x_7^{x_{14}} = x_7x_4x_3x_2, \\ x_8^{x_9} = x_8x_3, \quad x_8^{x_{10}} = x_8x_2x_1, \quad x_8^{x_{11}} = x_8x_3, \quad x_8^{x_{12}} = x_8x_3, \\ x_8^{x_{13}} = x_8x_6x_3, \quad x_8^{x_{14}} = x_8x_4x_3,$$

$$\begin{aligned} x_9^{x_{10}} &= x_9x_4, & x_9^{x_{11}} &= x_9x_5, & x_9^{x_{13}} &= x_9x_7, & x_{10}^{x_{11}} &= x_{10}x_6, & x_{10}^{x_{12}} &= x_{10}x_8, \\ x_{11}^{x_{14}} &= x_{11}x_8x_7^{-1}x_6x_5x_4x_3, & x_{12}^{x_{13}} &= x_{12}x_{11}, & x_{12}^{x_{14}} &= x_{12}x_9^{-1}, & x_{13}^{x_{14}} &= x_{13}x_{10}^{-1}. \end{aligned}$$

2.3 (Strong) sandwich groups and strong left 3-Engel elements

The results of [17] imply that a group G generated by a finite sandwich set consisting of elements of odd order is nilpotent. Thus every left 3-Engel element of odd order in G is in $\text{HP}(G)$.

We wish to extend this result to include groups generated by elements of any order. We take a cue from the definition of sandwich algebras. In [7] we introduced the notion of a strong sandwich set.

Definition. A subset X of a group G is a *strong sandwich set* if:

- (1) $\langle a, b^g \rangle$ is nilpotent of class at most 2 for all $a, b \in X$ and $g \in \langle X \rangle$;
- (2) $\langle a, b^f, c^g \rangle$ is nilpotent of class at most 3 for all $a, b, c \in X$ and $f, g \in \langle X \rangle$.

If G is generated by a strong sandwich set, then it is a *strong sandwich group*.

Remark. A strong sandwich group is somewhat analogous to a sandwich Lie algebra: if all the elements of X have odd order, then condition (2) is superfluous. This follows from Theorem 2.1.

There is also a connection to left 3-Engel elements. We first need a definition.

Definition. An element a of a group G is a *strong left 3-Engel element* if:

- (1) $\langle a, a^g \rangle$ is nilpotent of class at most 2 for all $g \in G$.
- (2) $\langle a, a^f, a^g \rangle$ is nilpotent of class at most 3 for all $f, g \in G$.

Remark. Notice that a is left 3-Engel in G if and only if it satisfies (1). Condition (2) is superfluous when a has odd order. Therefore a left 3-Engel element of odd order is a strong left 3-Engel element.

Strong left 3-Engel elements and strong sandwich groups are related because the following are equivalent.

- (1) If a is a strong left 3-Engel element of G , then $\langle a \rangle^G$ is locally nilpotent.
- (2) Every finitely generated strong sandwich group is nilpotent.

The following results obtained in [7] about finitely generated strong sandwich groups generalise work of Jabara and Traustason [17].

Theorem 2.2 *Every finitely generated strong sandwich group is nilpotent.*

Theorem 2.3 *If a is a strong left 3-Engel element of a group G , then $\langle a \rangle^G$ is locally nilpotent.*

A key ingredient in proving these results was the following [7].

Proposition 2.4 *Let X be a strong sandwich set in a group G and let $a, b \in X$. Then $X \cup \{[a, b]\}$ is also a strong sandwich set in G .*

We discuss briefly how Theorem 2.2 was proved using this proposition. Let $X = \{x_1, x_2, \dots, x_r\}$ be a strong sandwich set and let \overline{X} consist of all commutators in X (in any order and with any bracketing). Iterated use of Proposition 2.4 shows that \overline{X} is a strong sandwich set. As every 3-generator strong sandwich group is nilpotent of class at most 3, the Hall-Witt identity essentially reduces to the Jacobi identity. So for every $u, v, w \in \overline{X}$

$$[u, [v, w]] = [u, v, w][u, w, v]^{-1}.$$

As a result, we applied the same basic approach as used in Chanyshv's proof of the local nilpotence of sandwich Lie algebras [27, Section 3.2]. We mention applications to groups of exponents 5, 9 and 8.

Theorem A [26]. *A group of exponent 5 is locally finite if and only if it satisfies the law*

$$[z, [y, x, x, x], [y, x, x, x], [y, x, x, x]] = 1.$$

Remark. This result implies in particular that a group of exponent 5 is locally finite if and only if all of its 3-generator subgroups are finite; it was originally proved by Vaughan-Lee [28].

Theorem B [17]. *Let w be a word in n variables x_1, \dots, x_n where the variety of groups satisfying the law $w^3 = 1$ is a locally finite variety of groups of exponent 9. Then the same is true for the variety of groups on $n + 1$ variables satisfying the law $(x_{n+1}^3 w^3)^3 = 1$.*

Remark. We can use Theorem B to construct an explicit sequence of words. Define the word $w_n = w_n(x_1, \dots, x_n)$ in n variables recursively by $w_1 = x_1$ and $w_{n+1} = x_{n+1}^3 w_n^3$. The variety of groups satisfying the law $x_1^3 = 1$ is locally finite by Burnside and by repeated application of Theorem B we see that, for each $n \geq 1$, the variety of groups satisfying the law w_n^3 is a locally finite variety of groups of exponent 9.

Theorem C [7]. *Assume all groups of exponent 8 satisfying a law $w = 1$ in n variables x_1, \dots, x_n are locally nilpotent. Let V be the variety on $n + 3$ variables satisfying both $[x_{n+1}, w, w, w] = 1$ and $\langle w, w^{x_{n+2}}, w^{x_{n+3}} \rangle$ is nilpotent of class at most 3. Then V is locally nilpotent.*

Remark. Starting for example with $w = x_1^4$, Theorem C gives us a sequence of locally nilpotent varieties.

3 Left 3-Engel involutions

Theorem 2.3 implies that if a left 3-Engel element has odd order, then its normal closure is locally nilpotent. This observation and [24] imply that, to generalise this to left 3-Engel elements of arbitrary finite order, it suffices to consider elements of order 2. Despite Theorems 2.2 and 2.3, the question whether the normal closure of

a left 3-Engel involution is locally nilpotent remains open. The best general result is the following.

Theorem 3.1 *Let x be a left 3-Engel involution in a group G . If $\langle x \rangle^G$ has no elements of order 8, then $\langle x \rangle^G$ is locally nilpotent.*

Even this apparently slight progress was surprisingly difficult to establish. The difficulty comes from the fact that we do not know initially that $\langle x \rangle^G$ is a 2-group. Once we have a 2-group, it is clear that $\langle x \rangle^G$ has exponent 4 and thus it is locally finite by Sanov [23].

Consider sandwich groups generated by involutions. From [26] we know that such a group is nilpotent if its rank is at most 3. The largest sandwich group of rank 3 generated by involutions has order 2^{13} and class 5.

3.1 Rank 4 sandwich groups generated by involutions

It remains unknown whether sandwich groups of rank 4 generated by involutions are nilpotent. Nickel's nilpotent quotient algorithm [21] is available in both GAP [10] and MAGMA [4]. Using these implementations, we proved that such a group has a largest nilpotent quotient of order 2^{776} and class 13. The calculation took 18 days of CPU time using MAGMA 2.27-3 on a computer with a 2.6 GHz processor.

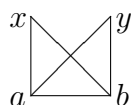
We have some partial results on the nilpotence of such groups. Before presenting them, we need the following definition.

Definition. Let G be a sandwich group generated by a finite set $X = \{a_1, \dots, a_r\}$ of sandwich elements. The *commutativity graph*, $V(G)$, of G is an (undirected) graph whose vertices are the generators X , and a pair of distinct vertices a_i and a_j are joined by an edge if and only if a_i and a_j commute.

Remarks. (1) The commutativity graph of the free r -generator sandwich group has no edges. The largest r -generator sandwich group with commutativity graph the complete graph on r vertices is the free abelian group of rank r .

(2) Let H and K be the largest r -generator sandwich groups with commutativity graphs $V(H)$ and $V(K)$ respectively. If $V(H) \subseteq V(K)$, then K is isomorphic to a quotient of H .

Assume for the remainder of the section that G is a sandwich group generated by 4 involutions. If $V(G)$ is the complete graph, then G is elementary abelian of order 16. There is just one type of commutativity graph with 5 edges, namely



and the largest sandwich group $\langle x, y, a, b \rangle$ with this commutativity graph is $\langle x, y \rangle \times \langle a, b \rangle = D_8 \times C_2^2$, a group of order 32. If the commutativity graph has 4 edges, then

there are two types of graphs to consider: either the two removed edges are adjacent or not.

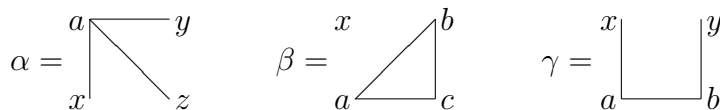


The largest sandwich group with the first commutativity graph is $\langle x, y \rangle \times \langle a, b \rangle = D_8 \times D_8$, it has order 64. For the second group, observe that a and b^c commute with each of b and a^c . Thus

$$\langle a, b, c \rangle = \langle a, b^c \rangle \wr \langle c \rangle = D_8 \wr C_2,$$

the standard wreath product of D_8 by C_2 . Thus the largest sandwich group with the second commutativity graph is $\langle a, b, c \rangle \times \langle x \rangle = (D_8 \wr C_2) \times C_2$, a group of order 256.

Next, we consider the case when the commutativity graph has 3 edges. This is much more difficult. There are three types of commutativity graphs:



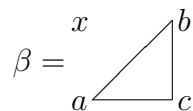
The largest sandwich group with the first commutativity graph is

$$G_\alpha = \langle x, y, z \rangle \times \langle a \rangle = R \times C$$

where R is the largest sandwich group generated by involutions x, y, z . From [26] we know that R is nilpotent of class at most 5. The results of [24] show that the others are finite.

3.1.1 Sandwich groups with commutativity graph β

Let $G_\beta = \langle x, a, b, c \rangle$ be a sandwich group where x, a, b, c are involutions and G_β has commutativity graph



In [24] it was shown that G_β has order 2^{28} and class 9. It has the following consistent power-conjugate presentation.

Let

$$t(a) = [[x^c, x^a], [x^a, x^b]], t(b) = [[x^a, x^b], [x^b, x^c]], t(c) = [[x^b, x^c], [x^c, x^a]]$$

and

$$y(a) = [x, x^{bc}, x^a], y(b) = [x, x^{ca}, x^b], y(c) = [x, x^{ab}, x^c].$$

Generators

$$b_1 = [t(a), b], b_2 = t(a), b_3 = t(b) \\ b_4 = y(b), b_5 = y(b)^a, b_6 = y(a), b_7 = y(a)^b, b_8 = y(a)^c,$$

$$b_9 = [x, x^{ab}][x^c, x^{abc}], b_{10} = [x, x^{ab}], b_{11} = [x, x^{bc}][x^a, x^{abc}], \\ b_{12} = [x, x^{bc}], b_{13} = [x, x^{ac}][x^b, x^{abc}], b_{14} = [x, x^{ac}], b_{15} = [x, x^{abc}][x^c, x^{ab}], \\ b_{16} = [x, x^{abc}][x^a, x^{bc}], b_{17} = [x, x^{abc}],$$

$$b_{18} = x, b_{19} = x^a, b_{20} = x^b, b_{21} = x^c, b_{22} = x^{ab}, b_{23} = x^{ca}, \\ b_{24} = x^{bc}, b_{25} = x^{abc},$$

$$b_{26} = a, b_{27} = b, b_{28} = c.$$

Relations

$$b_1^2 = \dots = b_{28}^2 = 1.$$

$$b_2^{b_{27}} = b_2 b_1, b_2^{b_{28}} = b_2 b_1, b_3^{b_{26}} = b_3 b_1, b_3^{b_{28}} = b_3 b_1 \\ b_4^{b_{19}} = b_4 b_2, b_4^{b_{21}} = b_4 b_3 b_2, b_4^{b_{22}} = b_4 b_2 b_1, b_4^{b_{23}} = b_4 b_3, b_4^{b_{24}} = b_4 b_3 b_2 b_1, \\ b_4^{b_{25}} = b_4 b_3, b_4^{b_{26}} = b_5, b_4^{b_{28}} = b_8 b_6 b_4 \\ b_5^{b_{18}} = b_5 b_2, b_5^{b_{20}} = b_5 b_2 b_1, b_5^{b_{21}} = b_5 b_3 b_1, b_5^{b_{23}} = b_5 b_3 b_2 b_1, \\ b_5^{b_{24}} = b_5 b_3 b_1, b_5^{b_{25}} = b_5 b_3 b_2, b_5^{b_{26}} = b_4, b_5^{b_{28}} = b_8 b_6 b_5, \\ b_6^{b_{20}} = b_6 b_3, b_6^{b_{21}} = b_6 b_3 b_2, b_6^{b_{22}} = b_6 b_3 b_1, b_6^{b_{23}} = b_6 b_3 b_2 b_1, b_6^{b_{24}} = b_6 b_2, \\ b_6^{b_{25}} = b_6 b_2, b_6^{b_{27}} = b_7, b_6^{b_{28}} = b_8, \\ b_7^{b_{18}} = b_7 b_3, b_7^{b_{19}} = b_7 b_3 b_1, b_7^{b_{21}} = b_7 b_2 b_1, b_7^{b_{23}} = b_7 b_2 b_1 \\ b_7^{b_{24}} = b_7 b_3 b_2 b_1, b_7^{b_{25}} = b_7 b_3 b_2, b_7^{b_{27}} = b_6, b_7^{b_{28}} = b_8 b_7 b_6, \\ b_8^{b_{18}} = b_8 b_3 b_2, b_8^{b_{19}} = b_8 b_3 b_2 b_1, b_8^{b_{20}} = b_8 b_2 b_1, b_8^{b_{22}} = b_8 b_2 b_1, \\ b_8^{b_{24}} = b_8 b_3 b_1, b_8^{b_{25}} = b_8 b_3, b_8^{b_{27}} = b_8 b_7 b_6, b_8^{b_{28}} = b_6,$$

$$b_9^{b_{12}} = b_9 b_1, b_9^{b_{14}} = b_9 b_1, b_9^{b_{18}} = b_9 b_6 b_4, \\ b_9^{b_{19}} = b_9 b_6 b_5, b_9^{b_{20}} = b_9 b_7 b_4, b_9^{b_{21}} = b_9 b_6 b_4, \\ b_9^{b_{22}} = b_9 b_7 b_5, b_9^{b_{23}} = b_9 b_6 b_5, b_9^{b_{24}} = b_9 b_7 b_4, \\ b_9^{b_{25}} = b_9 b_7 b_5, \\ b_{10}^{b_{11}} = b_{10} b_1, b_{10}^{b_{12}} = b_{10} b_3, b_{10}^{b_{13}} = b_{10} b_1, b_{10}^{b_{14}} = b_{10} b_2, \\ b_{10}^{b_{16}} = b_{10} b_1, b_{10}^{b_{17}} = b_{10} b_3 b_2, b_{10}^{b_{21}} = b_{10} b_6 b_4, \\ b_{10}^{b_{23}} = b_{10} b_6 b_5, b_{10}^{b_{24}} = b_{10} b_7 b_4, b_{10}^{b_{25}} = b_{10} b_7 b_5, \\ b_{10}^{b_{28}} = b_{10} b_9, \\ b_{11}^{b_{14}} = b_{11} b_1, b_{11}^{b_{18}} = b_{11} b_6, b_{11}^{b_{19}} = b_{11} b_6, \\ b_{11}^{b_{20}} = b_{11} b_7, b_{11}^{b_{21}} = b_{11} b_8, b_{11}^{b_{22}} = b_{11} b_7, b_{11}^{b_{23}} = b_{11} b_8, b_{11}^{b_{24}} = b_{11} b_8 b_7 b_6, \\ b_{11}^{b_{25}} = b_{11} b_8 b_7 b_6, \\ b_{12}^{b_{13}} = b_{12} b_1, b_{12}^{b_{14}} = b_{12} b_3 b_2, b_{12}^{b_{15}} = b_{12} b_1, \\ b_{12}^{b_{17}} = b_{12} b_2, b_{12}^{b_{19}} = b_{12} b_6, b_{12}^{b_{22}} = b_{12} b_7, b_{12}^{b_{23}} = b_{12} b_8, b_{12}^{b_{25}} = b_{12} b_8 b_7 b_6, \\ b_{12}^{b_{26}} = b_{12} b_{11}, \\ b_{13}^{b_{18}} = b_{13} b_4, b_{13}^{b_{19}} = b_{13} b_5, b_{13}^{b_{20}} = b_{13} b_4 \\ b_{13}^{b_{21}} = b_{13} b_8 b_6 b_4, b_{13}^{b_{23}} = b_{13} b_5, b_{13}^{b_{23}} = b_{13} b_8 b_6 b_5, b_{13}^{b_{24}} = b_{13} b_8 b_6 b_4 \\ b_{13}^{b_{25}} = b_{13} b_8 b_6 b_5, \\ b_{14}^{b_{15}} = b_{14} b_1, b_{14}^{b_{16}} = b_{14} b_1, b_{14}^{b_{17}} = b_{14} b_3, b_{14}^{b_{20}} = b_{14} b_4,$$

$$\begin{aligned}
b_{14}^{b_{22}} &= b_{14}b_5, & b_{14}^{b_{24}} &= b_{14}b_8b_6b_4, & b_{14}^{b_{25}} &= b_{14}b_8b_6b_5, & b_{14}^{b_{27}} &= b_{14}b_{13}, \\
b_{15}^{b_{18}} &= b_{15}b_6b_4b_3b_2, & b_{15}^{b_{19}} &= b_{15}b_6b_5b_3b_2b_1, \\
b_{15}^{b_{20}} &= b_{15}b_7b_4b_3b_2b_1, & b_{15}^{b_{21}} &= b_{15}b_6b_4b_3b_2, & b_{15}^{b_{22}} &= b_{15}b_7b_5b_3b_2, & b_{15}^{b_{23}} &= b_{15}b_6b_5b_3b_2b_1, \\
b_{15}^{b_{24}} &= b_{15}b_7b_4b_3b_2b_1, & b_{15}^{b_{25}} &= b_{15}b_7b_5b_3b_2, \\
b_{16}^{b_{18}} &= b_{16}b_6b_2, & b_{16}^{b_{19}} &= b_{16}b_6b_2, & b_{16}^{b_{20}} &= b_{16}b_7b_2b_1, \\
b_{16}^{b_{21}} &= b_{16}b_8b_2b_1, & b_{16}^{b_{22}} &= b_{16}b_7b_2b_1, & b_{16}^{b_{23}} &= b_{16}b_8b_2b_1, & b_{16}^{b_{24}} &= b_{16}b_8b_7b_6b_2, \\
b_{16}^{b_{25}} &= b_{16}b_8b_7b_6b_2, \\
b_{17}^{b_{19}} &= b_{17}b_6b_2, & b_{17}^{b_{20}} &= b_{17}b_4b_3, & b_{17}^{b_{21}} &= b_{17}b_6b_4b_3b_2, & b_{17}^{b_{22}} &= b_{17}b_7b_5b_3b_2, \\
b_{17}^{b_{23}} &= b_{17}b_8b_6b_5b_3, & b_{17}^{b_{24}} &= b_{17}b_8b_7b_6b_2, & b_{17}^{b_{26}} &= b_{17}b_{16}, & b_{17}^{b_{27}} &= b_{17}b_{16}b_{15}, \\
b_{17}^{b_{28}} &= b_{17}b_{15},
\end{aligned}$$

$$\begin{aligned}
b_{18}^{b_{22}} &= b_{18}b_{10}, & b_{18}^{b_{23}} &= b_{18}b_{14}, & b_{18}^{b_{24}} &= b_{18}b_{12}, & b_{18}^{b_{25}} &= b_{18}b_{17}, & b_{18}^{b_{26}} &= b_{19}, \\
b_{18}^{b_{27}} &= b_{20}, & b_{18}^{b_{28}} &= b_{21}, \\
b_{19}^{b_{20}} &= b_{19}b_{10}, & b_{19}^{b_{21}} &= b_{19}b_{14}, & b_{19}^{b_{24}} &= b_{19}b_{17}b_{16}, & b_{19}^{b_{25}} &= b_{19}b_{12}b_{11}, & b_{19}^{b_{26}} &= b_{18}, \\
b_{19}^{b_{27}} &= b_{22}, & b_{19}^{b_{28}} &= b_{23}, \\
b_{20}^{b_{21}} &= b_{20}b_{12}, & b_{20}^{b_{23}} &= b_{20}b_{17}b_{16}b_{15}, & b_{20}^{b_{25}} &= b_{20}b_{14}b_{13}, & b_{20}^{b_{26}} &= b_{22}, & b_{20}^{b_{27}} &= b_{18}, \\
b_{20}^{b_{28}} &= b_{24}, \\
b_{21}^{b_{22}} &= b_{21}b_{17}b_{15}, & b_{21}^{b_{25}} &= b_{21}b_{10}b_9, & b_{21}^{b_{26}} &= b_{23}, & b_{21}^{b_{27}} &= b_{24}, & b_{21}^{b_{28}} &= b_{18}, \\
b_{22}^{b_{23}} &= b_{22}b_{12}b_{11}, & b_{22}^{b_{24}} &= b_{22}b_{14}b_{13}, & b_{22}^{b_{26}} &= b_{20}, & b_{22}^{b_{27}} &= b_{19}, & b_{22}^{b_{28}} &= b_{25}, \\
b_{23}^{b_{24}} &= b_{23}b_{10}b_9, & b_{23}^{b_{26}} &= b_{21}, & b_{23}^{b_{27}} &= b_{25}, & b_{23}^{b_{28}} &= b_{19}, \\
b_{24}^{b_{26}} &= b_{25}, & b_{24}^{b_{27}} &= b_{21}, & b_{24}^{b_{28}} &= b_{20}, \\
b_{25}^{b_{26}} &= b_{24}, & b_{25}^{b_{27}} &= b_{23}, & b_{25}^{b_{28}} &= b_{22}.
\end{aligned}$$

3.1.2 Sandwich groups with commutativity graph γ

Let $G_\gamma = \langle a, b, x, y \rangle$ be a sandwich group where a, b, x, y are involutions and G_γ has commutativity graph

$$\gamma = \begin{array}{ccc} x & & y \\ | & & | \\ a & \text{---} & b \end{array}$$

In [24], it was shown that G_γ has order 2^{20} and class 9. It has the following consistent power-conjugate presentation.

Generators

$$\begin{aligned}
e_1 &= [x, b, [y, a], x, y, [x, b]], & e_2 &= [x, b, [y, a], y, x, [y, a]], \\
e_3 &= [x, b, [y, a], x, y, x], & e_4 &= [x, b, [y, a], y, x, y], \\
e_5 &= [x, b, [y, a], x, y], & e_6 &= [x, b, [y, a], y, x], \\
e_7 &= [x, b, [y, a], x], & e_8 &= [x, b, [y, a], y], \\
e_9 &= [x, b, y, x], & e_{10} &= [x, [y, a], y]
\end{aligned}$$

$$\begin{aligned}
e_{11} &= [x, b, [y, a]], & e_{12} &= [x, b, y], & e_{13} &= [x, [y, a]], \\
e_{14} &= [x, y]
\end{aligned}$$

$$e_{15} = [x, b], \quad e_{16} = [y, a], \quad e_{17} = x, \quad e_{18} = y$$

$$e_{19} = a, \quad e_{20} = b.$$

Relations

$$e_1^2 = e_2^2 = \dots = e_{20}^2 = 1.$$

$$\begin{aligned} e_3^{e_{20}} &= e_3e_1, \quad e_4^{e_{19}} = e_4e_2, \quad e_5^{e_{15}} = e_5e_1, \quad e_5^{e_{17}} = e_5e_3, \\ e_6^{e_{16}} &= e_6e_2, \quad e_6^{e_{18}} = e_6e_4, \quad e_7^{e_{12}} = e_7e_1, \quad e_7^{e_{14}} = e_7e_3, \quad e_7^{e_{18}} = e_7e_5, \\ e_8^{e_{13}} &= e_8e_2, \quad e_8^{e_{14}} = e_8e_4, \quad e_8^{e_{17}} = e_8e_6 \\ e_9^{e_{11}} &= e_9e_1, \quad e_9^{e_{13}} = e_9e_3, \quad e_9^{e_{16}} = e_9e_5e_4e_2, \quad e_9^{e_{19}} = e_9e_7e_6e_1, \\ e_{10}^{e_{11}} &= e_{10}e_2, \quad e_{10}^{e_{12}} = e_{10}e_4, \quad e_{10}^{e_{15}} = e_{10}e_6e_3e_1, \quad e_{10}^{e_{20}} = e_{10}e_8e_5e_2, \\ e_{11}^{e_{14}} &= e_{11}e_6e_5e_4e_3, \quad e_{11}^{e_{17}} = e_{11}e_7, \quad e_{11}^{e_{18}} = e_{11}e_8, \\ e_{12}^{e_{13}} &= e_{12}e_6e_5e_4e_3, \quad e_{12}^{e_{16}} = e_{12}e_8, \quad e_{12}^{e_{17}} = e_{12}e_9, \quad e_{12}^{e_{19}} = e_{12}e_{11}e_8, \\ e_{13}^{e_{15}} &= e_{13}e_7, \quad e_{13}^{e_{18}} = e_{13}e_{10}, \quad e_{13}^{e_{20}} = e_{13}e_{11}e_7, \\ e_{14}^{e_{15}} &= e_{14}e_9, \quad e_{14}^{e_{16}} = e_{14}e_{10}, \quad e_{14}^{e_{19}} = e_{14}e_{13}e_{10}, \quad e_{14}^{e_{20}} = e_{14}e_{12}e_9, \\ e_{15}^{e_{16}} &= e_{15}e_{11}, \quad e_{15}^{e_{18}} = e_{15}e_{12}, \quad e_{16}^{e_{17}} = e_{16}e_{13}, \\ e_{17}^{e_{18}} &= e_{17}e_{14}, \quad e_{17}^{e_{20}} = e_{17}e_{15}, \quad e_{18}^{e_{19}} = e_{18}e_{16}. \end{aligned}$$

3.1.3 Sandwich groups whose commutativity graph have 2 edges

Let $G = \langle a, x, y, z \rangle$ be a sandwich group where a, x, y, z are involutions and G has

$$\delta = \begin{array}{ccc} & x & z \\ & | & \\ \delta = & a & \text{---} y \end{array}$$

We know little about such groups. In the remainder of this section we prove the following ‘‘reduction’’ theorem.

Theorem 3.2 *If $[[z, x], [z, a]] = 1$ and $[[z, y], [z, a]] = 1$, then G is nilpotent.*

Remark. We used the MAGMA implementation of the nilpotent quotient algorithm to determine that the largest nilpotent quotient Q of G has order 2^{71} and class 10. If we impose the two relations of Theorem 3.2, then Q has order 2^{38} and class 9.

Let $N = \bigcup_{i=0}^{\infty} Z_i(G)$. As G is finitely generated, it suffices to show that G/N is nilpotent. Since G is finitely generated, G/N has trivial centre. Without loss of generality, we can thus assume that $Z(G) = 1$. Under this assumption, we will prove that $G = 1$. Our lengthy proof of this claim concludes in Proposition 3.21 where we prove that G is nilpotent.

The next two results do not rely on this assumption. The first is particularly useful.

Proposition 3.3 *Let $u \in G$ where u commutes with each of $x, y, [x, z], [y, z]$, and $[a, z]$. Then $u = 1$.*

Proof We first show that if $v \in G$ commutes with each of x, y, z and $[z, a]$, then $v = 1$. As a first step, we prove that $[v, a]$ commutes with x, y, z and $[z, a]$. Since

each of a and v commutes with x , y and $[z, a]$, it is clear that $[v, a]$ commutes with each, so it suffices to show that $[v, a]$ commutes with z . This follows from

$$[v, a]^z = [v, a^z] = [v, a[a, z]] = [v, [z, a]a] = [v, a][v, [z, a]]^a = [v, a].$$

Thus each of v , $[v, a]$ and $[v, a]a$ commutes with each of x, y and z . However, $[v, a]a$ commutes also with a . Thus $[v, a]a \in Z(G)$ so $[v, a]a = 1$. It follows that $[v, a] = 1$, so $v = 1$.

Next, notice that $[u, z]$ commutes with $x, y, [x, z], [y, z]$ and $[a, z]$. Note that z commutes with $[x, z], [y, z]$ and $[a, z]$, so $[u, z]$ commutes with each of these elements. It thus suffices to show that $[u, z]$ commutes with x and y . This follows from

$$\begin{aligned} [u, z]^x &= [u, z[x, z]] = [u, [z, x]z] = [u, z][u, [z, x]]^z = [u, z], \\ [u, z]^y &= [u, z[y, z]] = [u, [z, y]z] = [u, z][u, [z, y]]^z = [u, z]. \end{aligned}$$

Thus $[u, z]$ commutes with x, y and $[z, a]$, but $[u, z, z]$ also commutes with z . From the first paragraph we deduce that $[u, z, z] = 1$. Hence $[u, z]$ commutes with each of x, y, z and $[z, a]$ and thus it follows again from the first paragraph that $[u, z] = 1$. The same argument shows that $u = 1$. \square

Below we use the following power-conjugate presentation for the largest sandwich group of rank 3 generated by involutions; it is an immediate consequence of that given in Section 2.2.

$$\text{Let } e_1(z, z^x, y) = [z, x, y, y].$$

Generators

$$x_1 = e_1(z, z^x, y), \quad x_2 = e_1(x, x^y, z), \quad x_3 = e_1(y, y^z, x),$$

$$\begin{aligned} x_4 &= [z, x, [z, y]], & x_5 &= [x, y, [x, z]], & x_6 &= [y, z, [y, x]], \\ x_7 &= [z, x, y], & x_8 &= [z, y, x] \end{aligned}$$

$$x_9 = [z, x], \quad x_{10} = [z, y], \quad x_{11} = [x, y]$$

$$x_{12} = x, \quad x_{13} = y, \quad x_{14} = z.$$

Relations

$$\begin{aligned} x_1^2 &= \cdots = x_6^2 = 1, & x_3 &= x_2x_1, & x_7^2 &= x_1, & x_8^2 &= x_3, & x_9^2 &= \cdots = x_{14}^2 = 1, \\ x_4^{x_{12}} &= x_4x_2x_1, & x_4^{x_{13}} &= x_4x_1, & x_5^{x_{13}} &= x_5x_1, & x_5^{x_{14}} &= x_5x_2, \\ x_6^{x_{12}} &= x_6x_2x_1, & x_6^{x_{14}} &= x_6x_2, \\ x_7^{x_9} &= x_7x_1, & x_7^{x_{10}} &= x_7x_1, & x_7^{x_{11}} &= x_7x_1, & x_7^{x_{12}} &= x_7x_5x_1, \\ x_7^{x_{13}} &= x_7x_1, & x_7^{x_{14}} &= x_7x_4x_1, \\ x_8^{x_9} &= x_8x_2x_1, & x_8^{x_{10}} &= x_8x_2x_1, & x_8^{x_{11}} &= x_8x_2x_1, & x_8^{x_{12}} &= x_8x_2x_1, \\ x_8^{x_{13}} &= x_8x_6x_2x_1, & x_8^{x_{14}} &= x_8x_4x_2x_1, \\ x_9^{x_{10}} &= x_9x_4, & x_9^{x_{11}} &= x_9x_5, & x_9^{x_{13}} &= x_9x_7, & x_{10}^{x_{11}} &= x_{10}x_6, & x_{10}^{x_{12}} &= x_{10}x_8, \\ x_{11}^{x_{14}} &= x_{11}x_8x_7x_6x_5x_4x_2, & x_{12}^{x_{13}} &= x_{12}x_{11}, & x_{12}^{x_{14}} &= x_{12}x_9, & x_{13}^{x_{14}} &= x_{13}x_{10}. \end{aligned}$$

Lemma 3.4 $\gamma_3(\langle [x, y], [z, a] \rangle) = 1$.

Proof Notice that $1 = [x, y^2] = [x, y]^2$ and $1 = [z, a^2] = [z, a]^2$. So

$$[[x, y], [a, z], [a, z]] = [[x, y], [z, a], [x, y]] = ([a, z][x, y])^4$$

and thus it suffices to show that $[[x, y], [z, a], [z, a]] \in Z_2(G) = 1$. Calculating in the sandwich group $\langle x, y, a^z \rangle$, using the presentation above, we see that

$$[[x, y], [a, z], [a, z]] = [x, y, a^z, a^z] = [x_{11}, x_{14}, x_{14}] = [x_8 x_7 x_6 x_5 x_4 x_2, x_{14}] = x_2.$$

Observe that x_2 commutes with x, y and a^z , and also with a . We finish the proof by showing that $[[x, y], [a, z], [a, z], z]$ commutes with each of a, x, y and z . We will calculate in the sandwich group $\langle \bar{a}, \bar{b}, \bar{c}, \bar{x} \rangle$, where $\bar{a} = a$, $\bar{b} = x$, $\bar{c} = x^y$ and $\bar{x} = z$. Since it has commutativity graph β , we can use the relations satisfied by such a group. We have already seen that $[[x, y], [a, z], [a, z]]$ commutes with $[z, a] = [\bar{x}, \bar{a}]$ and $x = \bar{b}$. We will use this in the following calculation:

$$\begin{aligned} [[x, y], [z, a], [z, a]] &= [\bar{b}\bar{c}, [\bar{x}, \bar{a}], [\bar{x}, \bar{a}]] \\ &= [[\bar{x}, \bar{a}]^{\bar{b}\bar{c}}, [\bar{x}, \bar{a}]] \quad (\text{commutes with } \bar{b} = x) \\ &= [[\bar{x}, \bar{a}]^{\bar{c}}, [\bar{x}, \bar{a}]^{\bar{b}}] \\ &= [\bar{x}^{\bar{c}} \bar{x}^{\bar{a}\bar{c}}, \bar{x}^{\bar{b}} \bar{x}^{\bar{a}\bar{b}}] \\ &= [\bar{x}^{\bar{c}}, \bar{x}^{\bar{b}} \bar{x}^{\bar{a}\bar{b}}]^{\bar{x}^{\bar{a}\bar{c}}} [\bar{x}^{\bar{a}\bar{c}}, \bar{x}^{\bar{b}} \bar{x}^{\bar{a}\bar{b}}] \\ &= [\bar{x}^{\bar{c}}, \bar{x}^{\bar{a}\bar{b}}]^{\bar{x}^{\bar{a}\bar{c}}} [\bar{x}^{\bar{c}}, \bar{x}^{\bar{b}}]^{\bar{x}^{\bar{a}\bar{b}} \bar{x}^{\bar{a}\bar{c}}} [\bar{x}^{\bar{a}\bar{c}}, \bar{x}^{\bar{a}\bar{b}}] [\bar{x}^{\bar{a}\bar{c}}, \bar{x}^{\bar{b}}]^{\bar{x}^{\bar{a}\bar{b}}} \\ &= (b_{17} b_{15})^{b_{23}} b_{12}^{b_{22} b_{23}} b_{12}^{b_{26}} (b_{17} b_{16} b_{15})^{b_{22}} \\ &= b_{16} b_{11} b_2 b_1. \end{aligned}$$

This element commutes with $\bar{b} = x$ and thus

$$b_{16} b_{11} b_2 b_1 = (b_{16} b_{11} b_2 b_1)^{b_{27}} = b_{16} b_{11} b_2 b_1 b_1,$$

so $b_1 = 1$. Hence

$$[[x, y], [z, a], [z, a], z] = [b_{16} b_{11} b_2, b_{18}] = b_2$$

commutes with each of x, z and a . Since $[x, y] = [y, x]$, we see by symmetry that b_2 also commutes with y . \square

We deduce that $b_2 = 1$. We analyse some consequences. Consider the elements:

$$\begin{aligned} b_{11} &= [\bar{x}^{\bar{b}\bar{c}}, \bar{x}] [\bar{x}^{\bar{a}}, \bar{x}^{\bar{a}\bar{b}\bar{c}}] = [z, z^{[x, y]}] [z^a, z^{a[x, y]}]; \\ b_6 &= [b_{11}, b_{18}] = [b_{11}, \bar{x}] = [b_{11}, z]; \\ b_8 b_7 &= [b_6, b_{27} b_{28}] = [b_6, [x, y]]; \end{aligned}$$

all are symmetric in x, y .

Since $b_1 = b_2 = 1$, we deduce that $b_8 b_7$ commutes with each of x, a and z , and by symmetry also with y ; thus $b_8 b_7 = 1$. Recall that $1 = [x, y, a^z, a^z] = x_2$ in $\langle x, y, a^z \rangle$. In summary:

Lemma 3.5

- (1) $x_2 = 1$ in $\langle x, y, a^z \rangle$.
- (2) $b_1 = b_2 = 1$, $b_8 = b_7$ and $b_{16} = b_{11}$ in $\langle a, x, x^y, z \rangle$ and $\langle a, y, y^x, z \rangle$.

From now on, we use the conditions given in the statement of Theorem 3.2. Since $[z^y, z^a] = [z, y, [z, a]] = 1$, we deduce that $\langle z, z^a, z^y, x \rangle$ is a β -group. We will later work with this subgroup. First we prove a crucial lemma.

Lemma 3.6 $[z, a, x] = [z, x, a]$ and $[z, a, y] = [z, y, a]$.

Proof Since $[z, x, [z, a]] = 1$, it follows that $[z, x, a] = [x^z, a]$ commutes with z . Hence

$$[z, x, a] = [x^z, a] = [x^z, a]^z = [x, a^z] = [a^z, x] = [z, a, x].$$

The latter claim is proved similarly. \square

Lemma 3.7 $[x, y, z, z] = 1$.

Proof We calculate in the β -group $\langle \bar{a}, \bar{b}, \bar{c}, \bar{x} \rangle = \langle a, x, x^y, z \rangle$. We know that

$$b_1 = b_2 = 1, \quad b_8 = b_7, \quad b_{16} = b_{11}.$$

Furthermore

$$1 = [z^a, z^x] = [\bar{x}^{\bar{a}}, \bar{x}^{\bar{b}}] = b_{10}.$$

As a consequence,

$$\begin{aligned} 1 = [b_{10}, b_{12}] &= b_3 \\ 1 = [b_{10}, b_{28}] &= b_9 \\ 1 = [b_9, b_{19}] &= b_6 b_5 \\ 1 = [b_9, b_{22}] &= b_7 b_5 \\ 1 = [b_9, b_{20}] &= b_7 b_4. \end{aligned}$$

The last three identities imply that $b_4 = b_5 = b_6 = b_7 = b_8$. We show that this common element is trivial. We consider

$$[z, z^{[x,y]}, z^a] = [\bar{x}, \bar{x}^{\bar{b}\bar{c}}, \bar{x}^{\bar{a}}] = [b_{12}, b_{19}] = b_6.$$

From the presentation, $b_6^x = b_6^{\bar{b}} = b_6^{b_{27}} = b_7 = b_6$, so b_6 commutes with x . Also b_6 commutes with $[z, x] = b_{18}b_{20}$ and $[z, a] = b_{18}b_{19}$. As b_6 is symmetric in x and y , it commutes with y and $[z, y]$. Proposition 3.3 implies that $b_6 = 1$. Hence

$$b_4 = b_5 = b_6 = b_7 = b_8 = 1.$$

Next, consider $[z, z^{[x,y]}][z^a, z^{a[x,y]}] = b_{12}b_{12}^{b_{26}} = b_{11}$. Observe that b_{11} commutes with each of $x = b_{27}$, $[z, x] = b_{18}b_{20}$ and $[z, a] = b_{18}b_{19}$. As b_{11} is symmetric in x and y , it commutes with y and $[z, y]$. Proposition 3.3 implies that $b_{11} = 1$. Hence

$$b_{16} = b_{11} = 1.$$

Finally, we consider $[x, y, z, z] = [z, z^{[x,y]}] = b_{12}$. Observe that b_{12} commutes with each of $x = b_{27}$, $[z, x] = b_{18}b_{20}$ and $[z, a] = b_{18}b_{19}$. As b_{12} is symmetric in x and y , it commutes with y and $[z, y]$. Proposition 3.3 implies that $b_{12} = [x, y, z, z] = 1$. \square

Remark. We know from the presentation for $\langle x, y, z \rangle$ that $\gamma_5(\langle x, y, z \rangle)$ is generated by $[x, y, z, z]$, $[z, x, y, y]$ and $[z, y, x, x]$, and their product is trivial. Also $[x, y, z, z] = 1$. Hence $\gamma_5(\langle x, y, z \rangle)$ is cyclic, and generated by $[z, x, y, y] = [z, y, x, x]$.

Lemma 3.8 $\langle x, y, a^z \rangle$ is nilpotent of class at most 4 and $[[a^z, x], [a^z, y]] = 1$.

Proof As $\langle x, y, a^z \rangle$ is a sandwich group, we know from the presentation of the largest such group that $\gamma_5(\langle x, y, a^z \rangle)$ is generated by $[x, y, a^z, a^z]$, $[a^z, x, y, y]$ and $[a^z, y, x, x]$. The product of these elements is trivial and Lemma 3.5 implies that $[x, y, a^z, a^z] = 1$. Using Lemma 3.6 and the presentation for the γ -group $\langle \bar{x}, \bar{a}, \bar{b}, \bar{y} \rangle = \langle x^z, x, a, y \rangle$, we deduce that

$$[a^z, x, y, y] = [x^z, a, y, y] = [\bar{x}, \bar{b}, \bar{y}, \bar{y}] = [e_{12}, e_{18}] = 1.$$

Hence $\langle x, y, a^z \rangle$ is nilpotent of class at most 4. For the last claim, notice that $[a^z, x, [a^z, y]]$ commutes with z by the assumptions of Theorem 3.2 and, as $\langle x, y, a^z \rangle$ is nilpotent of class at most 4, it commutes with x and y , and it obviously commutes with a . Hence $[a^z, x, [a^z, y]] \in Z(G) = 1$. \square

Lemma 3.9 In the β -group $\langle \bar{a}, \bar{b}, \bar{c}, x \rangle = \langle z, z^a, z^y, x \rangle$, the following relations hold:

$$b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = b_9 = b_{10} = 1.$$

Proof Observe that $1 = [z^x, z^a] = [\bar{a}^x, \bar{b}] = [x, \bar{a}, \bar{b}] = xx^{\bar{a}}x^{\bar{b}}x^{\bar{a}\bar{b}}$ from which it follows that $x^{\bar{a}\bar{b}} = xx^{\bar{a}}x^{\bar{b}}$. In particular,

$$1 = [x, x^{\bar{a}\bar{b}}] = b_{10}.$$

As a consequence,

$$\begin{aligned} 1 &= [b_{10}, b_{11}] &= b_1 \\ 1 &= [b_{10}, b_{14}] &= b_2 \\ 1 &= [b_{10}, b_{12}] &= b_3 \\ 1 &= [b_{10}, b_{28}] &= b_9 \\ 1 &= [b_9, b_{18}] &= b_6 b_4 \\ 1 &= [b_9, b_{19}] &= b_6 b_5 \\ 1 &= [b_9, b_{20}] &= b_7 b_4. \end{aligned}$$

From the last three identities, $b_4 = b_5 = b_6 = b_7$. We show that this common element is trivial. We consider $t = [z, a, x, [z, y, x, x]]$. Calculating first in $\langle z, z^a, z^y, x \rangle$,

$$t = [\bar{a}\bar{b}, x, [\bar{a}\bar{c}, x, x]] = [b_{26}b_{27}, b_{18}, b_{14}] = [b_{18}^{b_{26}b_{27}}b_{18}, b_{14}] = [b_{22}b_{18}, b_{14}] = b_5.$$

From the presentation, $[b_5, [z, a]] = [b_5, b_{26}b_{27}] = b_4b_5 = 1$.

We next consider t in the γ -group $\langle \bar{x}, \bar{a}, \bar{b}, \bar{y} \rangle = \langle x^z, x, a, y \rangle$. We first find some relations in this group. Using the fact that $\langle x, y, a^z \rangle$ is nilpotent of class at most 4 and Lemma 3.6, we deduce that

$$1 = [a^z, x, [x, y], y] = [x^z, a, [x, y], y] = [\bar{x}, \bar{b}, [\bar{a}, \bar{y}], \bar{y}] = [e_{11}, e_{18}] = e_8.$$

As a consequence,

$$\begin{aligned} 1 &= [e_8, e_{13}] &= e_2 \\ 1 &= [e_8, e_{14}] &= e_4 \\ 1 &= [e_8, e_{17}] &= e_6. \end{aligned}$$

We use the facts that $[z, a, x] = [z, x, a]$ from Lemma 3.6 and $[z, y, x, x] = [z, x, y, y]$. Hence

$$\begin{aligned}
t &= [[z, x, a], [z, x, y, y]] \\
&= [[x^z, a], [x^z x, y, y]] \\
&= [[\bar{x}, \bar{b}], [\bar{a}\bar{x}, \bar{y}, \bar{y}]] \\
&= [\bar{x}, \bar{b}, [[\bar{a}, \bar{y}][\bar{a}, \bar{y}, \bar{x}][\bar{x}, \bar{y}], \bar{y}]] \\
&= [\bar{x}, \bar{b}, [\bar{a}, \bar{y}, \bar{x}, \bar{y}]^{\bar{x}, \bar{y}}] \\
&= [e_{15}, e_{10}^{e_{14}}] \\
&= [e_{15}, e_{10}] \\
&= e_3 e_1.
\end{aligned}$$

From the presentation, we deduce that t commutes with $y = \bar{y} = e_{18}$ and $x = \bar{a} = e_{19}$. Both $[z, x, a]$ and $[z, x, y, y]$ commute with z . Hence t commutes with z and thus with $[z, x]$ and $[z, y]$. Thus t commutes with each of $x, y, [z, x], [z, y]$ and $[z, a]$. Proposition 3.3 implies that $t = 1$. Hence $b_5 = 1$. \square

Lemma 3.10 $\langle x, y, z \rangle$ is nilpotent of class at most 4 and $[[z, x], [z, y]] = 1$.

Proof In the β -group $\langle \bar{a}, \bar{b}, \bar{c}, x \rangle = \langle z, z^a, z^y, x \rangle$

$$[z, y, x, x] = [x^{zz^y}, x] = [\bar{x}^{\bar{a}\bar{c}}, x] = b_{14}.$$

The expressions for both b_{14} and $b_{13} = [z, y, x, x, [z, a]] = [b_{14}, b_{26}b_{27}]$ are symmetric in x and y . We read from the presentation that:

$$\begin{aligned}
[b_{13}, [z, a]] &= [b_{13}, b_{26}b_{27}] = 1; \\
[b_{13}, x] &= [b_{13}, b_{18}] = b_4 = 1; \\
[b_{13}, [z, x]] &= [b_{13}, [\bar{a}, x]] = [b_{13}, b_{18}b_{19}] = b_5 b_4 = 1.
\end{aligned}$$

As b_{13} is symmetric in x and y , it commutes with y and $[z, y]$, so by Proposition 3.3 it is trivial. Hence $[z, y, x, x] = b_{14}$ commutes with $[z, a]$. It clearly commutes with each of $x, y, [z, x]$ and $[z, y]$, so by Proposition 3.3 it is trivial. Therefore $\langle x, y, z \rangle$ is nilpotent of class at most 4. Hence $[z, x, [z, y]]$ commutes with each of $x, y, [z, x]$ and $[z, y]$. As $[z, a]$ commutes with both $[z, x]$ and $[z, y]$ it follows that $[z, x, [z, y]]$ commutes with $[z, a]$. Proposition 3.3 implies that $[z, x, [z, y]] = 1$. \square

Lemma 3.11 In the γ -group $\langle \bar{x}, \bar{a}, \bar{b}, \bar{y} \rangle = \langle x^z, x, a, y \rangle$ the following relations hold:

$$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = e_{10} = 1, \quad e_{11} = e_9.$$

Proof We have seen in the proof of Lemma 3.9 that $e_2 = e_4 = e_6 = e_8 = 1$. As $\langle x, y, z \rangle$ is nilpotent of class at most 4,

$$\begin{aligned}
1 &= [z, x, y, y] \\
&= [x^z x, y, y] \\
&= [\bar{x}\bar{a}, \bar{y}, \bar{y}] \\
&= [[\bar{x}, \bar{y}][\bar{x}, \bar{y}, \bar{a}][\bar{a}, \bar{y}], y] \\
&= [\bar{x}, \bar{y}, \bar{a}, \bar{y}]^{\bar{a}, \bar{y}} \\
&= [e_{14}, e_{19}, e_{18}]^{e_{16}} \\
&= [e_{13}e_{10}, e_{18}]^{e_{16}} \\
&= e_{10}^{e_{16}} \\
&= e_{10}.
\end{aligned}$$

Also

$$\begin{aligned} 1 &= [e_{10}, e_{15}, e_{20}] = [e_3 e_1, e_{20}] = e_1 \\ 1 &= [e_{10}, e_{15}] = e_3 \\ 1 &= [e_{10}, e_{20}] = e_5. \end{aligned}$$

We consider $[z, x, a, [z, x, [x, y]]]$. Both $[z, x, a]$ and $[z, x, [x, y]]$ commute with each of z and x so $[z, x, a, [z, x, [x, y]]]$ commutes with each. We compute in the γ -group $\langle x^z, x, a, y \rangle$. Note that

$$[z, x, a, [z, x, [x, y]]] = [x^z, a, [x^z, [x, y]]] = [\bar{x}, \bar{b}, [\bar{x}, [\bar{a}, \bar{y}]]] = [e_{15}, e_{13}] = e_7.$$

From the presentation, e_7 commutes with $e_{20} = \bar{b} = a$ and $e_{18} = \bar{y} = y$. It follows that $e_7 \in Z(G) = 1$.

Finally, we consider $[z, x, a, [z, x, y]]$. By Lemma 3.10 we know that $[z, x, y]$ commutes with z and as $[z, x, a]$ also commutes with z we see that $[z, x, a, [z, x, y]]$ commutes with z . From the γ -group presentation, we deduce that

$$\begin{aligned} [z, x, a, [z, x, y]] &= [x^z, a, [x^z x, y]] \\ &= [\bar{x}, \bar{b}, [\bar{x}\bar{a}, \bar{y}]] \\ &= [e_{15}, e_{14}^{e_{19}} e_{16}] \\ &= [e_{15}, e_{14} e_{13} e_{16}] \\ &= e_9 e_{11}. \end{aligned}$$

We also see that $e_9 e_{11}$ commutes with $e_{20} = \bar{b} = a$, $e_{18} = \bar{y} = y$ and $e_{17} = x$. Hence $e_9 e_{11} \in Z(G) = 1$. \square

Lemma 3.12 *In the β -group $\langle \bar{a}, \bar{b}, \bar{c}, \bar{x} \rangle = \langle a, x, x^y, z \rangle$ the following relations hold:*

$$b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = b_{10} = b_{11} = b_{12} = b_{16} = 1, \quad b_{15} = b_{13}.$$

Proof All equalities were established in the proof of Lemma 3.7 apart from $b_{15} = b_{13}$. In the proof of Lemma 3.11 we show that $[z, x, a, [z, x, [x, y]]] = e_7 = 1$. We now view it as an element of $\langle a, x, x^y, z \rangle$. Our relations already imply that $\gamma_3(\langle \bar{x} \rangle^{\langle \bar{a}, \bar{b}, \bar{c} \rangle}) = 1$. This simplifies our computations:

$$\begin{aligned} 1 &= [z, x, a, [z, x, [x, y]]] \\ &= [\bar{x}\bar{x}^{\bar{a}}\bar{x}^{\bar{b}}\bar{x}^{\bar{a}\bar{b}}, \bar{x}\bar{x}^{\bar{b}}\bar{x}^{\bar{b}\bar{c}}\bar{x}^{\bar{c}}] \\ &= [\bar{x}^{\bar{a}}, \bar{x}^{\bar{b}\bar{c}}][\bar{x}^{\bar{a}}, \bar{x}^{\bar{c}}][\bar{x}^{\bar{a}\bar{b}}, \bar{x}^{\bar{b}\bar{c}}][\bar{x}^{\bar{a}\bar{b}}, \bar{x}^{\bar{c}}] \\ &= b_{17} b_{14} b_{14} b_{13} b_{17} b_{15} \\ &= b_{13} b_{15}. \end{aligned}$$

This finishes the proof. \square

Armed with the last two lemmas, we finish the proof of the reduction theorem via a series of technical lemmas.

Lemma 3.13

- (a) $[z, a, x, y] = [z, x, a, y] = [z, x, y, a]$.
- (b) $[z, a, x, y, x] = [z, x, a, y, x] = [z, x, y, a, x] = [z, x, y, x, a]$.

Proof (a) As $[z, x, a] = [z, a, x]$, it suffices to show that $[z, x, a, y] = [z, x, y, a]$. Calculating in $\langle \bar{x}, \bar{a}, \bar{b}, \bar{y} \rangle = \langle x^z, x, a, y \rangle$, we deduce that

$$[z, x, y, a] = [\bar{x}\bar{a}, \bar{y}, \bar{b}] = [[\bar{x}, \bar{y}][\bar{x}, \bar{y}, \bar{a}][\bar{a}, \bar{y}], \bar{b}] = e_{12},$$

and

$$[z, x, a, y] = [\bar{x}, \bar{b}, \bar{y}] = [e_{15}, e_{18}] = e_{12}.$$

(b) By (a) the first three equalities hold, and thus it only remains to show that $[z, x, a, y, x] = [z, x, y, x, a]$. Now $[z, x, a, y, x] = [e_{12}, e_{19}] = e_{11}$. Also

$$\begin{aligned} [z, x, y, x, a] &= [\bar{x}\bar{a}, \bar{y}, \bar{a}, \bar{b}] \\ &= [[\bar{x}, \bar{y}][\bar{x}, \bar{y}, \bar{a}][\bar{a}, \bar{y}], \bar{a}, \bar{b}] \\ &= [\bar{x}, \bar{y}, \bar{a}, \bar{b}] \\ &= [e_{14}, e_{19}, e_{20}] \\ &= [e_{13}, e_{20}] \\ &= e_{11}. \end{aligned}$$

This finishes the proof. \square

Lemma 3.14 $[z, x, y, x, [z, a, y]] = [z, x, y, x, [z, a], y] = [z, a, x, y, x, [z, y]] = [z, a, x, y, x, z, y]$.

Proof Notice that $[z, x, y, x] = [z, x, [y, x]]$ and $[z, a, x, y, x] = [z, a, x, [y, x]]$. Calculating in $\langle a, x, x^y, z \rangle$, we deduce that

$$[z, x, [y, x], [z, a]] = [z, a, x, [y, x], z] = b_{17}b_{14}.$$

Hence the 2nd and 4th terms are equal. The Hall-Witt identity implies that

$$\begin{aligned} 1 &= [z, a, y, [z, x, y, x]]^y [y, [z, x, y, x], [z, a]]^{[z, x, y, x]} [z, x, y, x, [z, a], y]^{[z, a]} \\ &= [z, a, y, [z, x, y, x]] [z, x, y, x, [z, a], y]^{[z, a]}. \end{aligned}$$

Thus

$$[z, x, y, x, [z, a, y]] = [z, x, y, x, [z, a], y]^{[z, a]}. \quad (1)$$

Notice that $[z, a, y] = [z, y, a]$. Using the Hall-Witt identity again, we get

$$\begin{aligned} 1 &= [z, y, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y]]^{[z, x, y, x]} [z, x, y, x, [z, y], a]^{[z, y]} \\ &= [z, y, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y]], \end{aligned}$$

where we use the fact that $[z, x, y, x, a]$ commutes with $[z, x, y, x]$, which can for example be read from the presentation for $\langle x^z, x, a, y \rangle$. Thus

$$[z, x, y, x, [z, y, a]] = [z, x, y, x, a, [z, y]]^a. \quad (2)$$

As $[z, x, [y, x], [z, a]] = b_{17}b_{14}$ commutes with a , and the same is true for y and $[z, a]$, we see from (1) that $[z, x, y, x, [z, a, y]]$ commutes with a . As this element also commutes with z , the conjugation by a and $[z, a]$ in (1) and (2) can be dropped and the claimed equalities hold. \square

Lemma 3.15 $[z, x, y, x, [z, a, y, x]] = [z, x, y, x, [z, a, y], x] = [z, x, y, x, [z, a], y, x] = [z, a, x, y, x, [z, y, x]]$.

Proof It follows from Lemma 3.14 that the 2nd and 3rd terms are equal. We use the Hall-Witt identity to deduce that

$$\begin{aligned} 1 &= [z, a, y, x, [z, x, y, x]]^x [x, [z, x, y, x], [z, a, y, x]]^{[z, x, y, x]} [z, x, y, x, [z, a, y, x]]^{[z, a, y]} \\ &= [z, a, y, x, [z, x, y, x]] [z, x, y, x, [z, a, y, x]]^{[z, a, y]}. \end{aligned}$$

Thus

$$[z, x, y, x, [z, a, y, x]] = [z, x, y, x, [z, a, y, x]]^{[z, a, y]}. \quad (3)$$

We also deduce that

$$\begin{aligned} 1 &= [z, y, x, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y, x]]^{[z, x, y, x]} [z, x, y, x, [z, y, x], a]^{[z, y, x]} \\ &= [z, y, x, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y, x]], \end{aligned}$$

since $[a, [z, x, [y, x]]]$ commutes with $[z, x, [y, x]]$ in $\langle a, x, x^y, z \rangle$. Thus

$$[z, x, y, x, [z, y, x, a]] = [z, x, y, x, a, [z, y, x]]^a. \quad (4)$$

From the proof of Lemma 3.14, we see that $[z, x, y, x, [z, a, y]]$ commutes with a . The same is true for x and $[z, a, y]$ and thus it follows from (3) that $[z, x, y, x, [z, a, y, x]]$ commutes with a . Calculating in the γ -group $\langle \bar{x}, \bar{a}, \bar{b}, \bar{y} \rangle = \langle y^z, y, a, x \rangle$, we see that

$$[z, y, a, x, [z, y]] = [\bar{x}, \bar{b}, \bar{y}, \bar{x}\bar{a}] = e_{11}e_9 = 1.$$

Thus $[z, x, y, x, [z, a, y, x]]$ commutes with $[z, y]$ and hence also with $[z, y, a] = [z, a, y]$. Therefore the conjugation by a and $[z, a, y]$ can be dropped and the lemma follows from (3) and (4) and Lemma 3.14. \square

Lemma 3.16 $[z, x, y, x, [z, a, y, x, y]] = [z, x, y, x, [z, a, y, x], y] = [z, a, x, y, x, [z, y, x, y]]$.

Proof The Hall-Witt identity implies that

$$\begin{aligned} 1 &= [z, a, y, x, y, [z, x, y, x]]^y [y, [z, x, y, x], [z, a, y, x]]^{[z, x, y, x]} [z, x, y, x, [z, a, y, x], y]^{[z, a, y, x]} \\ &= [z, a, y, x, y, [z, x, y, x]] [z, x, y, x, [z, a, y, x], y]^{[z, a, y, x]}, \end{aligned}$$

so

$$[z, x, y, x, [z, a, y, x, y]] = [z, x, y, x, [z, a, y, x], y]^{[z, a, y, x]}. \quad (5)$$

From the proof of Lemma 3.15, we see that $[z, x, y, x, [z, a, y, x]]$ commutes with $[z, a, y]$ and clearly y commutes with $[z, a, y]$. It can be read from the presentation for $\langle y^z, y, a, x \rangle$ that $[z, y, a, x] = [z, a, y, x]$ commutes with $[z, a, y] = [z, y, a]$. It follows that the RHS, and thus LHS, of (5) commutes with $[z, a, y]$. Clearly the LHS also commutes with x . Hence it commutes with $[z, a, y, x]$ and

$$[z, x, y, x, [z, a, y, x, y]] = [z, x, y, x, [z, a, y, x], y]. \quad (6)$$

Also

$$\begin{aligned} 1 &= [z, y, x, y, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y, x, y]]^{[z, x, y, x]} [z, x, y, x, [z, y, x, y], a]^{[z, y, x, y]} \\ &= [z, y, x, y, a, [z, x, y, x]]^a [a, [z, x, y, x], [z, y, x, y]]. \end{aligned}$$

Thus

$$[z, x, y, x, [z, y, x, y, a]] = [z, x, y, x, a, [z, y, x, y]]^a.$$

By the proof of Lemma 3.15, the RHS of (6) commutes with a . Hence

$$[z, x, y, x, [z, y, x, y, a]] = [z, x, y, x, a, [z, y, x, y]].$$

This finishes the proof. \square

Lemma 3.17 $[z, x, y, x, [z, a, y, x, y]] = [z, a, x, y, x, [z, y, x, y]] = 1$.

Proof From Lemma 3.16,

$$u := [z, x, y, x, [z, a, y, x, y]] = [z, a, x, y, x, [z, y, x, y]].$$

Each of $[z, x, y, x]$ and $[z, a, y, x, y]$ commutes with each of x, y and $[z, y]$. That $[z, a, y, x, y] = [z, y, a, x, y]$ commutes with $[z, y]$ can be read from the presentation for $\langle y^z, y, a, x \rangle$. Thus the LHS commutes with $x, y, [z, y]$ and by symmetry the RHS commutes with $[z, x]$. Thus u commutes with each of $x, y, [z, x]$ and $[z, y]$. We now show that u commutes with a :

$$\begin{aligned} u^a &= [z, a, x, y, x, [z, y, x, y]^a] \\ &= [z, a, x, y, x, [z, y, x, y][z, a, y, x, y]] \\ &= [z, a, x, y, x, [z, a, y, x, y][z, y, x, y]] \quad (\text{working in } \langle y^z, y, a, x \rangle) \\ &= [z, a, x, y, x, [z, y, x, y]][z, a, x, y, x, [z, a, y, x, y]]^{[z, y, x, y]} \\ &= [z, a, x, y, x, [z, y, x, y]] \\ &= u. \end{aligned}$$

In the 3rd last identity we use the fact that $\langle a^z, x, y \rangle$ is nilpotent of class at most 4.

We prove the following claim: if v commutes with each of $a, x, y, [z, x]$ and $[z, y]$, then the same is true for $[v, [z, a]]$. To see this, notice first that as $[z, a]$ commutes with $[z, x], [z, y]$ and a , the same is true for $[v, [z, a]]$. It follows that v commutes with $[z, a, x] = [z, x, a]$ and thus

$$[v, [z, a]]^x = [v, [z, a][z, a, x]] = [v, [z, a, x][z, a]] = [v, [z, a]][v, [z, a, x]]^{[z, a]} = [v, [z, a]]$$

and $[v, [z, a]]$ commutes with x . Likewise $[v, [z, a]]$ commutes with y . Thus $[v, [z, a]]$ commutes with each of $a, x, y, [z, x]$ and $[z, y]$.

Next, notice that $\langle [z, a]^u, [z, a] \rangle \leq \langle a, z, z^u \rangle$ and thus $\langle [z, a]^u, [z, a] \rangle$ is nilpotent, as every sandwich group of rank 3 is nilpotent. Using this fact, let m be the smallest positive integer such that $[u, m [z, a]] = 1$. We claim that $m = 1$. Notice that $[u, m-1 [z, a]]$ commutes with each of $x, y, [z, a], [z, x]$ and $[z, y]$. Proposition 3.3 implies that $[u, m-1 [z, a]] = 1$. Hence $m = 1$ and $u = 1$. \square

Lemma 3.18 $v := [z, x, y, x, [z, a, y, x]] = 1$.

Proof By Lemmas 3.17 and 3.16 we know that v commutes with y . It also commutes with x (as $[a^z, y, x, x] = 1$). Observe that $[z, x, y, x, a]$ commutes with $[z, x, y, x]$ in $\langle x^z, x, a, y \rangle$ and $[z, x, y, x, a] = [z, a, x, y, x]$ commutes with $[z, a, y, x]$ in $\langle a^z, x, y \rangle$. Hence

$$v^a = [[z, x, y, x]^a, [z, a, y, x]] = [[z, x, y, x, a][z, x, y, x], [z, a, y, x]] = [z, x, y, x, [z, a, y, x]] = v,$$

so v commutes with a . It remains to show that v commutes with z . Lemmas 3.14 and 3.15 imply that

$$v = [z, x, y, x, [z, a, y, x]] = [z, a, x, y, x, [z, y, x]].$$

We read from the presentation for $\langle x^z, x, a, y \rangle$ that

$$[z, a, x, y, x] = [z, x, a, y, x]$$

commutes with $[z, x]$ and by Lemma 3.10 this is also true for $[z, y]$. Hence

$$[z, a, x, y, x, [z, y]] = [z, x, y, x, [z, a, y]]$$

commutes with $[z, x]$. Clearly $[z, x, y, x]$ and $[z, a, y]$ commute with z . Thus

$$v^z = [z, x, y, x, [z, a, y], x[x, z]] = [z, x, y, x, [z, a, y], [z, x]x] = v[z, x, y, x, [z, a, y], [z, x]]^x = v.$$

Thus $v \in Z(G) = 1$. \square

Lemma 3.19 $v := [z, x, y, x, [z, a, y]] = 1$.

Proof Lemmas 3.15 and 3.18 imply that v commutes with x . It also commutes with y and z . By the proof of Lemma 3.14, it commutes with a . Thus $v \in Z(G) = 1$. \square

Lemma 3.20 $v := [z, x, y, x, [z, a]] = 1$.

Proof In $\langle a, x, x^y, z \rangle$ we read that $v = b_{17}b_{14}$ commutes with a, x and z . From Lemma 3.19 and 3.14, it commutes also with y . Thus $v \in Z(G) = 1$. \square

Proposition 3.21 G is nilpotent.

Proof As $\langle x, y, z \rangle$ is nilpotent of class at most 4, it follows from Lemma 3.20 and Proposition 3.3 that $[z, x, y, x] = 1$. By symmetry in x and y , we deduce that $[z, y, x, y] = 1$. We saw earlier that $[z, x, [z, y]] = 1$. It follows that $\langle x, y, z \rangle$ is nilpotent of class at most 3. In particular, calculating in $\langle a, x, x^y, z \rangle$, we see that

$$1 = [z, x, [y, x], [z, a]] = b_{17}b_{14}.$$

Hence $b_{17} = b_{14}$. From the presentation for a β -group, we see that b_{14} commutes with b_{28} . Hence $1 = [b_{17}, b_{28}] = b_{15}$. But $b_{15} = b_{13}$ so $b_{13} = 1$. Also b_{17} commutes with $a = b_{26}$ and $z = b_{18}$, and $x = b_{27}$. Recall that $b_{17} = [z, z^{a[x, y]}]$, so it is symmetric in x and y ; thus it commutes with y . Hence $b_{17} \in Z(G) = \{1\}$.

We now consider $[x, y, z]$. Calculating in $\langle a, x, x^y, z \rangle$, we see that $[x, y, z, [z, a]] = b_{17} = 1$. As $\langle x, y, z \rangle$ is nilpotent of class at most 3, Proposition 3.3 implies that $[x, y, z] = 1$. But now $[x, y]$ commutes with a, x, y, z and thus $[x, y] = 1$. Hence G is a β -group and thus nilpotent. \square

As a corollary to Theorem 3.2, we deduce the following.

Theorem 3.22 If $G = \langle a, x, y, z \rangle$ is a δ -group, then $\langle z, z^x, z^y, a \rangle$ is nilpotent.

Proof Observe that $\langle a_1, x_1, y_1, z_1 \rangle = \langle z, z^x, z^y, a \rangle$ is also a δ -group. Now

$$[z_1, a_1, x_1, z_1] = [a, z, z^x, a] = [a^{zz^x}, a] = [z, x, a, a] = [x^z, a, a] = 1.$$

Similarly we see that $[z_1, a_1, y_1, z_1] = 1$. It follows from Theorem 3.2 that $\langle a_1, x_1, y_1, z_1 \rangle$ is nilpotent. \square

3.1.4 Sandwich groups whose commutativity graph has 1 edge

Let G be the largest sandwich group generated by 4 involutions where precisely one pair commutes. We used the MAGMA implementation of the nilpotent quotient algorithm to determine that the largest nilpotent quotient of G has order 2^{235} and class 11.

3.2 Rank 5 sandwich groups generated by involutions

Let G be a residually nilpotent sandwich group of rank 5 generated by involutions. It remains an open question whether G is finite. As a step towards its resolution, we report the following.

Theorem 3.23 *A residually nilpotent sandwich group of rank 5 generated by involutions is finite if there are least three commuting pairs among its generators.*

We consider all 10 commutativity graphs for such a group G that have 3 or 4 edges. In Table 1 we list the graphs and upper bounds for the class and orders of the largest 2-quotient of each group.

We proved these claims computationally by studying each of the 10 groups using our implementation in MAGMA of the p -quotient algorithm [20]. Let Q be the group generated by the involutions a, b, c, d, e which satisfy one of these commutativity graphs. It required too much CPU time to impose the sandwich condition directly. Instead, we added up to 2500 random instances of this condition as explicit relations to the presentation for Q , and constructed its largest finite 2-quotient. The most expensive of these computations took 7 days of CPU time using MAGMA 2.27-3 on a computer with a 2.6 GHz processor. Computational evidence suggests that the largest 2-quotient of G and Q coincide.

Commutativity graph	Class bound	$\log_2 G $
$a - b, a - c, a - d, a - e$	13	777
$a - b - c - d - e$	17	2643
$a - b - c - d - a$	16	2831
$a - b - c - a, d - e$	16	3145
$a - b - c - a - d$	16	3324
$a - b, a - c, a - d - e$	16	2636
$a - b - c - d$	17	10354
$a - b - d - a$	16	12598
$a - b, a - c, a - d$	16	10906
$a - b - c, d - e$	16	9987

Table 1: Class and order bounds for the 2-quotients

4 Global nilpotence question for $\langle a \rangle^G$ in locally finite 2-groups

In this section we give an example from [22] of a locally finite 2-group G with a left 3-Engel element a such that $\langle a \rangle^G$ is not nilpotent. The construction is based on a

Lie algebra given in [25]. This result was generalised in [8] to an infinite family of examples.

4.1 The Lie algebra

Let \mathbb{F} be the field of order 2 and consider the 4-dimensional vector space $V = \mathbb{F}x + \mathbb{F}u + \mathbb{F}v + \mathbb{F}w$. We equip L with a binary product where

$$u \cdot v = u, \quad v \cdot w = w, \quad w \cdot u = v, \quad u \cdot x = 0, \quad v \cdot x = 0, \quad w \cdot x = u.$$

We then extend the product linearly on V . Observe that V is a Lie algebra with a trivial center and $W = \mathbb{F}u + \mathbb{F}v + \mathbb{F}w$ is a simple ideal.

Let $E = \langle \text{ad}(x), \text{ad}(u), \text{ad}(v), \text{ad}(w) \rangle \leq \text{End}(V)$ be the associative enveloping algebra of V . Now E is 12-dimensional with basis

$$\begin{aligned} e_1 &= \text{ad}(w), & e_2 &= \text{ad}(w)^2, & e_3 &= \text{ad}(w)^3, & e_4 &= \text{ad}(v), \\ e_5 &= \text{ad}(v)\text{ad}(w), & e_6 &= \text{ad}(v)\text{ad}(w)^2, & e_7 &= \text{ad}(u), & e_8 &= \text{ad}(u)\text{ad}(w), \\ e_9 &= \text{ad}(u)\text{ad}(w)^2, & e_{10} &= \text{ad}(x)\text{ad}(v), & e_{11} &= \text{ad}(x)\text{ad}(w), & e_{12} &= \text{ad}(x)\text{ad}(w)^2. \end{aligned}$$

We construct a certain locally nilpotent Lie algebra over \mathbb{F} of countably infinite dimension. For ease of notation, we introduce the following modified union of subsets of \mathbb{N} :

$$A \sqcup B = \begin{cases} A \cup B, & \text{if } A \cap B = \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

For each non-empty subset A of \mathbb{N} , let W_A be a copy of W . That is, $W_A = \{z_A : z \in W\}$ with addition $z_A + t_A = (z + t)_A$. We take the direct sum

$$W^* = \bigoplus_{\emptyset \neq A \subseteq \mathbb{N}} W_A.$$

We view W^* as a Lie algebra by defining a multiplication

$$z_A \cdot t_B = (zt)_{A \sqcup B},$$

for $z_A \in W_A$ and $t_B \in W_B$, and then extend this product linearly on W^* . The interpretation here is that $z_\emptyset = 0$. Finally, we extend this to a semidirect product with $\mathbb{F}x$

$$V^* = W^* \oplus \mathbb{F}x$$

induced from the action $z_A \cdot x = (zx)_A$.

Notice that V^* has basis

$$\{x\} \cup \{u_A, v_A, w_A : \emptyset \neq A \subseteq \mathbb{N}\}$$

and

$$\begin{aligned} u_A \cdot u_B &= v_A \cdot v_B = w_A \cdot w_B = 0, \\ u_A \cdot x &= 0, \quad v_A \cdot x = 0, \quad w_A \cdot x = u_A \end{aligned}$$

and

$$u_A \cdot v_B = u_{A \sqcup B}, \quad v_A \cdot w_B = w_{A \sqcup B}, \quad w_A \cdot u_B = v_{A \sqcup B}.$$

Every finitely generated subalgebra of V^* is contained in some

$$S = \langle x, u_{A_1}, \dots, u_{A_r}, v_{B_1}, \dots, v_{B_s}, w_{C_1}, \dots, w_{C_t} \rangle.$$

Since $zxx = 0$ for all $z \in V^*$, it follows that S is nilpotent of class at most $2(r+s+t)$. Hence V^* is locally nilpotent.

We now construct a group $G \leq \text{GL}(V^*)$ containing $a := 1 + \text{ad}(x)$ where a is a left 3-Engel element of G but $\langle a \rangle^G$ is not nilpotent. Let y be one of x, u_A, v_A, w_A . Observe that $\text{ad}(y)^2 = 0$ so

$$(1 + \text{ad}(y))^2 = 1 + 2\text{ad}(y) + \text{ad}(y)^2 = 1.$$

Thus $1 + \text{ad}(y)$ is an involution in $\text{GL}(V^*)$. The subgroups

$$\mathcal{U} = \langle 1 + \text{ad}(u_A) : A \subseteq \mathbb{N} \rangle, \mathcal{V} = \langle 1 + \text{ad}(v_A) : A \subseteq \mathbb{N} \rangle, \mathcal{W} = \langle 1 + \text{ad}(w_A) : A \subseteq \mathbb{N} \rangle$$

are elementary abelian of countably infinite rank. Analysis of $G := \langle a, \mathcal{U}, \mathcal{V}, \mathcal{W} \rangle$ establishes the following.

Theorem 4.1 *The element a is a left 3-Engel element of G . However $\langle a \rangle^G$ is not nilpotent.*

5 Global nilpotence question for $\langle x \rangle^G$ in locally finite p -groups, for odd p

In Section 4, following [8, 22], we gave an example of a locally finite 2-group G with a left 3-Engel element a such that $\langle a \rangle^G$ is not nilpotent. We now provide such an example for locally finite p -groups where p is any odd prime [9]. The odd case is more involved and the construction quite different from the $p = 2$ case. We first describe the pair (G, x) that will provide our example and we show directly that x is a left 3-Engel element of G . The description of G is not as transparent as in Section 4. To show that $\langle x \rangle^G$ is not nilpotent requires more work. We construct first a pair (L, z) where L is a Lie algebra over \mathbb{F}_p , the field of p elements, and $\text{Id}(z)$ is not nilpotent. The pair (L, z) can be seen as the Lie algebra analogue of our group construction. We then build a group H within $\text{End}(L)$ containing $1 + \text{ad}(z)$ where $(H, 1 + \text{ad}(z))$ is a homomorphic image of (G, x) . Since $(1 + \text{ad}(z))^H$ is not nilpotent, $\langle x \rangle^G$ is not nilpotent.

5.1 The group G and Lie algebra L

Let x, a_1, a_2, \dots be an infinite list of group variables. Recall that a simple commutator in x, a_1, a_2, \dots is a group word defined recursively as follows: x, a_1, a_2, \dots are simple commutators; if u and v are simple commutators, then $[u, v]$ is a simple commutator. A simple commutator s has multi-weight (m, e_1, e_2, \dots) in x, a_1, a_2, \dots , if x occurs m times and a_i occurs e_i times in s . The weight of s is $m + e_1 + e_2 + \dots$. The following definition is critical.

Definition. Let s be a simple commutator of multi-weight (m, e_1, e_2, \dots) in x, a_1, a_2, \dots . The *type* of s is $t(s) = e_1 + e_2 + \dots - 2m$.

Remark. If u, v are simple commutators in x, a_1, a_2, \dots , then $t([u, v]) = t(u) + t(v)$. In particular, $t([u, a_j]) = t(u) + 1$ and $t([u, x]) = t(u) - 2$.

For a fixed odd prime p , let $G = \langle x, a_1, a_2, \dots \rangle$ be the largest group satisfying the following conditions:

- (1) $\langle a_i \rangle^G$ is abelian for all $i \geq 1$;
- (2) $\langle x \rangle^G$ is metabelian;
- (3) $x^p = a_1^p = a_2^p = \dots = 1$;
- (4) if $s \neq x$ is a simple commutator in x, a_1, a_2, \dots and $|t(s)| \geq 2$, then $s = 1$.

It is not difficult to see that G is a locally finite p -group. From (4), it is clear that $s = 1$ unless it is of the form $[x, a_{j_1}, a_{j_2}, a_{j_3}, x, a_{j_4}, a_{j_5}, \dots, x, a_{j_{2m}}, a_{j_{2m+1}}]$.

In [9] it was proved directly that x is a left 3-Engel element of G . To show that $\langle x \rangle^G$ is not nilpotent, it suffices to show that the special commutators above, which are not trivial by (4), are non-trivial. The structure of G is not transparent enough for us to prove this directly. Instead we look at the analogous Lie algebra setting.

We first consider the largest Lie algebra $F = \langle z, c_1, c_2, \dots \rangle$ over \mathbb{F}_p such that:

1. $\text{Id}(c_i)$ is abelian for $i = 1, 2, \dots$;
2. $\text{Id}(z)$ is metabelian.

Let B be the following basis for $\text{Id}_F(z)$:

$$[[z, c_{I_1}], [z, c_{I_2}], \dots, [z, c_{I_m}]]$$

where $m \geq 1$, I_1, \dots, I_m are pairwise disjoint and $[z, c_{I_1}] > [z, c_{I_2}] \leq \dots \leq [z, c_{I_m}]$.

We define the type of a Lie commutator by analogy with that of a group commutator.

Definition. Let s be a simple commutator of multi-weight (m, e_1, \dots, e_r) in z, c_1, \dots, c_r . The *type* of s is $t(s) = e_1 + \dots + e_r - 2m$.

Remark. If c and d are simple commutators, then $t([c, d]) = t(c) + t(d)$. In particular, $t([c, x]) = t(c) - 2$ and $t([c, a_1]) = t(c) + 1$.

We construct a Lie algebra L that is a quotient of F by a certain multi-homogeneous ideal J . Since F is multi-graded, L is also multi-graded. Now J is the smallest ideal containing the following:

3. $c \in B$ is in J if $c \neq z$ is a commutator of type having absolute value greater than 1;
4. $c \in B$ is in J if one of I_1, I_3, \dots, I_m has size greater than 2.

By careful analysis, we determine the structure of J and prove that $\text{Id}_L(z)$ is not nilpotent.

From L , we obtain a group $H = \langle 1 + \text{ad}(z), 1 + \text{ad}(c_1), 1 + \text{ad}(c_2), \dots \rangle$. We deduce that H is a homomorphic image of G by showing that relations (1)–(4) of G hold in H where x, a_1, a_2, \dots are replaced by $1 + \text{ad}(z), 1 + \text{ad}(c_1), 1 + \text{ad}(c_2), \dots$. Finally, we show that $\langle 1 + \text{ad}(z) \rangle^H$ is non-nilpotent, so $\langle x \rangle^G$ is non-nilpotent.

Theorem 5.1 *The element x is a left 3-Engel element of G . However $\langle x \rangle^G$ is not nilpotent.*

References

- [1] A. Abdollahi, Left 3-Engel elements in groups, *J. Pure Appl. Algebra* **188** (2004), 1-6.
- [2] A. Abdollahi and H. Khosravi, On the right and left 4-Engel elements, *Comm. Algebra* **38** (2010), 993-943.
- [3] R. Baer, Engelsche Elemente Noetherscher Gruppen, *Math. Ann.* **133** (1957), 256-270.
- [4] W. Bosma, J. Cannon and C. Playoust, The Magma Algebra System I: The User Language, *J. Symbolic Comput.*, **24** (1997), 235-265.
- [5] W. Burnside, On an unsettled question in the theory of discontinuous groups, *Quart. J. Pure Appl. Math.* **37** (1901), 230-238.
- [6] K. W. Gruenberg, The Engel elements of a soluble group, *Illinois J. Math.* **3** (1959), 151-169.
- [7] A. Hadjievangelou and G. Traustason, Sandwich groups and (strong) left 3-Engel elements in groups, submitted.
- [8] A. Hadjievangelou and G. Traustason, Left 3-Engel elements in locally finite 2-groups, *Comm. Algebra* **49** (2021), 4869-4882.
- [9] A. Hadjievangelou, M. Noce and G. Traustason, Locally finite p -groups with a left 3-Engel element whose normal closure is not nilpotent, *Internat. J. Algebra Comput.* **31** (2021), 135-160.
- [10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.12.1; 2022.
- [11] H. Heineken, Eine Bemerkung über engelsche Elemente, *Arch. Math.* **11** (1960), 321.
- [12] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Chapman and Hall/CRC, London, 2005

- [13] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, *Internat. J. Algebra Comput.* **4** (1994), 1-308.
- [14] A. I. Kostrikin, *Around Burnside*, Moscow, Nauka (1986).
- [15] A. I. Kostrikin, The Burnside Problem, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **23** (1959), 3-34.
- [16] A. I. Kostrikin and E. I. Zel'manov, A Theorem on Sandwich Algebras, *Trudy Mat. Inst. Steklov*, **183** (1988), 142-149.
- [17] E. Jabara and G. Traustason, Left 3-Engel elements of odd order in groups, *Proc. Amer. Math. Soc.* **147** (2019), 1921-1927.
- [18] I. G. Lysenok, Infinite Burnside groups of even period, *Izv. Ross. Akad. Nauk. Ser. Math.* **60** (1996), 453-654.
- [19] M. Newell, On right-Engel elements of length three, *Proc. Roy. Irish Acad.* **96A**, No.1 (1996), 17-24.
- [20] M. F. Newman and E. A. O'Brien, Application of computers to questions like those of Burnside, II, *Internat. J. Algebra Comput.* **6**, 593-605, 1996.
- [21] W. Nickel, Computing nilpotent quotients of finitely presented groups. Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), 175-191, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 25, Amer. Math. Soc., Providence, RI, 1996.
- [22] M. Noce, G. Tracey and G. Traustason, A left 3-Engel element whose normal closure is not nilpotent, *J. Pure Appl. Algebra* **224**, (2020), 1092-1101.
- [23] I. N. Sanov, Solution of Burnside's Problem for Exponent Four, Leningrad State Univ., *Ann. Maths. Ser.* **10** (1940), 166-170.
- [24] G. Tracey and G. Traustason, Left 3-Engel elements in groups of exponent 60, *Internat. J. Algebra Comput.* **28** (2018), 673-695.
- [25] G. Traustason, Engel Lie-algebras, *Quart. J. Math. Oxford. Ser. (2)*, **44** (1993), 355-384.
- [26] G. Traustason, Left 3-Engel elements in groups of exponent 5, *J. Algebra*, **414** (2014), 41-71.
- [27] M. Vaughan-Lee, *The Restricted Burnside Problem*, Clarendon Press Oxford (2nd edition) (1993).
- [28] M. Vaughan-Lee, Engel-4 groups of exponent 5, *Proc. Lond. Math. Soc.*, **74** (1997), 306-334.
- [29] E. I. Zel'manov, The solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR Izvestia* **36** (1991), 41-60.
- [30] E. I. Zel'manov, The solution of the restricted Burnside problem for 2-groups, *Mat. Sb.* **182** (1991), 568-592.

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