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SOME REMARKS ON UNIPOTENT AUTOMORPHISMS

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Abstract. An utomorphism α of teh group G is said to be n-unipotent if $[g,_n \alpha] = 1$ for all $g \in G$. In this paper we obtain some results related to nilpotency of groups of n-unipotent automorphisms of solvable groups. We also show that, assuming the truth of a conjecture about the representation theory of solvable groups raised by P. Neumann, it is possible to produce, for a suitable prime p, an example of a f.g. solvable group possessing a group of p-unipotent automorphisms which is isomorphic to an infinite Burnside group. Conversely we show that, if there exists a f.g. solvable group G with a non nilpotent p-group H of n-automorphisms, then there is such a counterexample where n is a prime power and H has finite exponent.

1. Introduction

If G is a group and a is an automorphism of G, we say that a is a *nil-automorphism* if, for every $g \in G$ there exists $n = n(g)$ such that $[g,_n a] = 1$, where commutators are taken in the holomorph $G\mathrm{Aut}(G)$ and, as usual, the n-fold commutator $[g,_n a]$ is defined recursively by $[g,_0 a] = g$ and, for $n > 0$, $[g,_n a] = [[g,_{n-1} a], a]$. When n can be chosen independently from g, we say that a is *n-unipotent* or simply *unipotent* if the integer n is understood. The notion of nil and unipotent automorphism was defined by Plotkin in [11] and, in recent years, some papers have been devoted to the study of these kind of automorphisms.

Nil and unipotent automorphisms can be regarded as a natural generalization of the concept of an Engel element, since a nil-automorphism a of G is just a left-Engel element in $G\langle a\rangle$. Another way to look at nil-automorphisms, is to consider them as a generalization of unipotent automorphisms of vector spaces. For these reasons there are several natural questions that can be asked about nil-automorphisms, which are suggested by known facts about Engel groups or unipotent linear groups. Maybe the first question one could ask is whether a group of automorphisms whose elements are

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n-unipotent, needs to be locally nilpotent. This problem has been investigated, and some results were obtained for groups in particular classes. In [2] the authors study unipotent automorphisms of groups with residual properties. Notably they prove that, if G is a finitely generated residually finite group (or a finitely generated profinite group) , every group of n-unipotent automorphisms of G is locally nilpotent. In [4] the author considers unipotent automorphisms of finitely generated solvable groups. When the group acted upon is an extension of an abelian group by a polycyclic group (in particular this happens for finitely generated metabelian groups), it is shown that groups of n-unipotent automorphisms are locally nilpotent. The same results are obtained for solvable groups of finite Prüfer rank. Both papers [2] and [4] rely heavily on results by Crosby and Traustason on normal right-Engel subgroups (see [3]). In turn, in their work, Crosby and Traustason use some deep results on Lie algebras due to Zel'manov. In [12] the authors study the structure of a group of n-unipotent automorphisms of a group G under the additional assumption that both G and the group of automorphisms are solvable. Another issue discussed in that paper, is the existence of a series stabilized by a given group of n-unipotent automorphisms. Some examples are given to show that, even in some easy cases, n-unipotent automophisms may be very far from stabilizing a series. As far as automorphisms of solvable groups are concerned, the most interesting open question seems to be the following:

Question *Let G be a finitely generated solvable group and $H \leq \mathrm{Aut}(G)$ a group of n-unipotent automorphisms. Is H locally-nilpotent?.*

When G is not finitely generated the question has a negative answer. To see this fix a prime p and consider the 2-generator Burnside group H of exponent p. When p is large enough H is infinite and (clearly) not locally nilpotent. Let G be the additive group of $\mathbb{F}H$ with $\mathbb{F}$ a finite field of characteristic p. The group H acts by right multiplication on G and, for every $g \in G$ and $h \in H$ we have

$$[g, h] = g(h - 1)$$

Thus

$$[g,_p h] = g(h - 1)^p = g(h^p - 1) = 0$$

and therefore H acts on G as a group of p-unipotent automorphisms of G.

One fact that we want to stress is that no such examples are known (at least to us) when the group G is torsion-free. In fact, while the above question may have negative answer, it might be the case that, restricting to the class of torsion-free groups, the answer would be positive, even if one drops the assumption that G is finitely generated.

In this paper we consider groups of n-unipotent automorphisms of solvable groups, and prove some facts concerning the question about their local nilpotence. In the last chapter we show that, assuming the truth of a conjecture by P. Neumann about modules over solvable groups, it is possible to produce an example that shows the above question has negative answer. In this example the automorphism group H is a non-nilpotent finitely generated p-group of finite exponent.

2. The results

The question raised in in the introduction has positive answer when G is a finitely generated metabelian group, as shown in [4], so it seems natural to try to attack this problem using induction on the derived length of the group$G$. However this approach doesn't seem to work well. Hence, as a first step, one could try to see what happens for groups which are *central-by-metabelian.* This class is of considerable interest and, when a certain property of metabelian groups fails for groups of larger derived length, it often happens that it already fails for central-by-metabelian groups.

Instead of considering our question for central-by-metabelian groups, we prove a slightly more general fact.

Proposition 2.1. *Let G be a finitely generated solvable group, H a group of n-unipotent automorphisms of G and N an H-invariant normal subgroup of G. Assume that*

(1) $H/C_H(G/N')$ is nilpotent;
(2) N is nilpotent.

Then H is nilpotent.

Proof. Let k be an integer such that $K = \gamma_k(H)$ acts trivially on G/N'. In particular K acts trivially on N/N' and, by a standard argument, also on the lower central factors $\gamma_i(N)/\gamma_{i+1}(N)$. Thus K stabilizes a finite series, and is therefore nilpotent. The group H is therefore metanilpotent, hence solvable, and we can apply [12, Theorem 2.6] to deduce that H stabilizes a finite series in G and is, therefore, nilpotent. □

From this we get the following result.

Theorem 2.2. *Let G be a finitely generated group that is metanilpotent and $H \leq Aut(G)$ a group of n-unipotent automorphisms. Then H is nilpotent.*

Proof. As G is metanilpotent we know that $\gamma_k(G)$ is nilpotent for some $k \geq 2$. Let $N = \gamma_k(G)$ then G/N' is abelian-by-polycyclic and, by [4] $H/C_H(G/N')$ is nilpotent. Thus Proposition 2.1 applies. □

Remark. Notice that Theorem 2.2 applies in particular to groups that are centre-by-metabelian as in that case $[G, G]$ is nilpotent of class at most 2.

Proposition 2.1 can be combined with some bounds obtained in [4], to produce a stronger result.

Theorem 2.3. *Let G be a finitely generated solvable group, H a finitely generated group of n-unipotent automorphisms of G and N a H-invariant normal subgroup of G. Assume that*

(1) $H/C_H(G/N')$ is nilpotent;

(2) N is residually nilpotent.

Then H is nilpotent.

Proof. By Proposition 2.1, each $H_k = H/C_H(G/\gamma_k(N))$ is nilpotent. By [4, Theorem A] the class of H_k is bounded by a function $c = c(l,d)$ of l, the derived length of G, and d, the minimal number of generators of H. Thus

$$\gamma_{c+1}(H) \leq \bigcap_{k\geq 2} C_H(G/\gamma_k(N))$$

hence $[G, \gamma_{c+1}(H)] \leq \bigcap_{k\geq 2} \gamma_k(N) = 1$, proving that $\gamma_{c+1}(H) = 1$. □

Theorem 2.3 holds for example whenever G' is residually nilpotent, taking $N = G'$ and recalling that n-unipotent groups of finitely generated metabelian groups are locally nilpotent.

In questions dealing with unipotent actions, the assumption that the acted group is torsion-free has often a strong influence. Perhaps the most striking example of this fact is the proof that torsion-free profinite n-Engel groups are nilpotent ([13]).

In [12] the authors prove that, if the solvable group G has a characteristic series with torsion-free abelian factors, and H is a solvable group of n-unipotent automorphisms of G, then H stabilizes a finite series in G and it is therefore nilpotent. This fact is of particular interest, because it does not need G or H to be finitely generated. The group G must, of course, be torsion-free, but this is not enough to ensure the existence of a characteristic series with torsion-free abelian factors. For example if we let

$$G = \langle a, b, z, x \mid [a,b] = z,\ [a,z] = [b,z] = 1,\ a^x = b^{-1},\ b^x = a,\ x^2 = z\rangle$$

The subgroup $N = \langle a, b\rangle$ is normal, nilpotent of class 2 and torsion-free, and G/N has order 2. It is easy to deduce that G itself is torsion-free. If $A \neq 1$ is a normal abelian subgroup of G, then $AN/N \leq G/N$, so it is finite of order at most 2. The isomorphism $AN/N \simeq A/A\cap N$ implies that $A \cap N \neq 1$. Therefore $A \cap N$ must intersect $\zeta(N) = \langle z\rangle$ in a non trivial subgroup, say $\langle z^k\rangle$. Then the element xA has order $2k$ and G/A is not torsion-free. As a consequence we have that G can not possess a finite series with abelian torsion-free factors.

We can now improve the result cited above, showing that it is enough to assume that G is torsion-free.

Theorem 2.4. *Let G be a torsion-free solvable group and H a group of n-unipotent automorphisms of G. Then H is nilpotent.*

Proof. We use [7] to see that the group G has a characteristic subgroup K such that

(1) K is nilpotent of class at most 2;
(2) $C_G(K) \leq K$.

The upper central series of K has torsion-free factors hence, by [12], $H/C_H(K)$ is nilpotent. Set

$A = C_H(K)$. Since $[G, K, A] = 1$ and $[K, A, G] = 1$, the Three-subgroups-lemma gives $[[G, A], K] = 1$, which means $[G, A] \leq C_G(K) \leq K$. Therefore $[G, A, A] = 1$ and A stabilizes the series $1 \leq [G, A] \leq G$. Thus A is abelian and, since G is torsion-free, A is torsion-free as well. Choose any $h \in H$ and consider the group $A\langle h\rangle$. We apply [2, Lemma 2.14] to see that $A\langle h\rangle$ is nilpotent and an inspection of the proof shows that each h acts on A has an n^2-unipotent automorphism. It is now possible to apply the result from [12] to H acting on A, to see that A has a finite series stabilized by H. As A is abelian and H/A is nilpotent, the series of A that is centralised by H can then be extended to obtain a central series for H. Thus H is nilpotent, as claimed. □

3. Final remarks

In this section we want to point out some facts which may indicate that, in general, n-unipotent groups of automorphisms of finitely generated solvable groups, may not be locally nilpotent.

In [9]. Peter Neumann discusses several *pathological* behaviours arising in the representation theory of infinite solvable groups. He proposes four conjectures of increasing strength, the third of which could be relevant in the study of unipotent automorphisms. The conjecture is the following

Conjecture *For every countable ring R there exist a finitely generated solvable group S and a finitely generated $\mathbb{Z}S$-module M such that R is isomorphic to a subring of of* $\mathrm{End}_{\mathbb{Z}S}(M)$.

Assuming the conjecture true, we can construct a finitely generated solvable group G with a group of unipotent automorphisms which is not locally nilpotent.

Choose a prime p such that the free 2-generator Burnside group $B = B(2, p)$ is infinite.

Let $\mathbb{F}$ be the field with p elements, $R = \mathbb{F}B$ the group ring of B. If Neumann's conjecture is true we can find a f.g. solvable group S and a f.g. $\mathbb{Z}S$-module M such that R is (isomorphic to) a subring of $\mathrm{End}_{\mathbb{Z}S}(M)$. Therefore $\mathrm{Aut}_{\mathbb{Z}S}(M)$ contains a subgroup H isomorphic to B. Since H is a group of automorphisms of M, it acts faithfully on M.

Let $G = MS$ be the semidirect product of M by S. The group G is clearly a finitely generated solvable group. For each $h \in H$ let $\phi_h : G \longrightarrow G$ be the map defined by $(ms)^{\phi_h} = m^h g$. This map is well defined and we check that it is indeed an automorphism of G. If $x = ms$ and $y = nt$, we have $xy = (ms)(nt) = mn^{s^{-1}}st$, so that

$$(xy)^{\phi_h} = (mn^{s^{-1}}st)^{\phi_h} = (mn^{s^{-1}})^{\phi_h}st = (mn^{s^{-1}})^h st = m^h n^{s^{-1}h} st.$$

Since h is an S-automorphism of M we obtain

$$m^h n^{s^{-1}h} st = m^h n^{hs^{-1}} st = (m^h s)(n^h t) = x^{\phi_h} y^{\phi_h}$$

proving that each ϕ_h is an homomorphism. A routine calculation shows that each ϕ_h is a bijection. We therefore have a map $\Phi : H \longrightarrow \mathrm{Aut}(G)$ sending each h to ϕ_h. It is readily seen that Φ is an homomorphism and the kernel of Φ is $\ker(\Phi) = \{h \mid m^h = m \quad \forall m \in M\} = C_H(M) = 1$. In particular the image of Φ is not nilpotent. Identify H with $\mathrm{Im}(\Phi)$. Let $\alpha\phi_h$ be any element of H. By definition of ϕ_h we see that $[G, \alpha] \leq M$. Pick any $m \in M$ and consider the commutator $[m, \alpha]$. To simplify calculations, let us switch to additive notation. It is readily seen that $[m,\alpha]=m(h-1)$ and, in general $[m,_k \alpha]=m(h-1)^k$. Thus

$$[m,_p \alpha]=m(h-1)^p=m(h^p-1)=m(1-1)=0$$

hence $[g,_{p+1}\alpha]=1$ for all $g\in G$. This means that H is a f.g. group of $(p+1)$-unipotent automorphisms of G but, at the same time, it is quite far from being nilpotent, geing isomorphic to an infinite f.g. free Burnside group.

So, assuming Neumann's conjecture true, we can show that the question we asked in the introduction, has negative answer. Conversely, suppose there exists a f.g. solvable group G with a non nilpotent p-group H of unipotent automorphisms which is not locally nilpotent. We show that it is possible to choose G and H in such a way that

(1) H is finitely generated of finite exponent;
(2) $[G,H]$ is elementary abelian of exponent p;
(3) $[g,_{p+1} h]=1$ for all $g\in G$ and $h\in H$.

This will indicate that the example we constructed above, would be somehow typical. Let $\mathcal{C}$ be the class of all pairs (G,H) where G, H are f.g. groups, G is solvable, $H\le \mathrm{Aut}(G)$ is a non nilpotent p-group acting n-unipotently for some n, and assume that $\mathcal{C}$ is not empty. Choose, among all possible pairs (G,H) of $\mathcal{C}$, one for which G has minimal derived length. If A is the last term of the derived series of G, H induces a group of n-unipotent automorphisms of G/A. By our choice of (G,H), the group $H/C_H(G/A)$ must be nilpotent. Being a finitely generated p-group $H/C_H(G/A)$ is finite. Hence $C_H(G/A)$ is still finitely generated and infinite, so we may take it as our H. In other words we assume $[G,H]\le A$. If T is the torsion subgroup of A,H induces a group of unipotent automorphisms of A/T. On the other hand A/T is torsion-free so a nontrivial unipotent automorphism must have infinite order. Therefore $[A,H]\le T$ and $[G,H,H]\le T$. This implies that $H/C_H(G/T)$ is abelian, hence finite. Thus $C_H(G/T)$ is finitely generated and infinite so, taking G/T instead of G and $C_H(G/T)$ instead of H, we may assume that $[G,H]$ is a torsion abelian group. If B is the p'-part of $[G,H]$ we have $[B,H]=1$ so that, by the same argument we used above, we may assume that $B=1$, just changing G with G/B and H with $C_H(G/B)$. What we have now is that $C=[G,H]$ is an abelian p-group. Let h be any element of H whose order is denoted by p^k. For every $c\in C$ the commutator $[c,_n h]$ is 1 and an easy induction on n shows that $[a^{p^{(n-1)k}},h]=1$. Fix F a finite set of generators for H and let p^k the maximum order of the elements of F. If $m=(n-1)k$ then $D=C^{p^m}$ is centralized by F, hence it is centralized by $\langle F\rangle=H$. It follows that $[G,C_H(G/D),C_H(G/D)]=1$ and $C_H(G/D)$ is abelian. The group $H/C_H(G/D)$ is therefore still not nilpotent and we have $(G/D,H/C_H(G/D))\in\mathcal{C}$. So it is possible to choose $(G,H)\in\mathcal{C}$ in such a way that $[G,H]$ is an abelian p-group of finite exponent, and, among all such pairs, we select (G,H) so that the exponent of $[G,H]$ is minimal. If $[G,H]$ has exponent bigger than p, the same kind of argument used in the previous paragraph would show that, if $N=[G,H]^p$, the pair $(G/N,H/C_H(G/N))$ is in $\mathcal{C}$, against the fact that the exponent of $[G,H]$ was minimal. Let p^m the smallest power of p such that $n\le p^m$. Since $[G,H]$ is an elementary abelian

p-group we have, for all $h \in H$ and $a \in [G,H]$ (using additive notation)

$$0 = [a,_{p^m} h] = a(h-1)^{p^m} = a(h^{p^m} - 1).$$

Hence, $h^{p^m} = 1$ for all h, proving that H has finite exponent. For every $g \in G$ and $h \in H$, we get $[g,_{p^m+1} h] = 1$, so that each element of H is a (p^m+1)-unipotent automorphism.

Our last remark concerns the torsion-free case. Even assuming Neumann's conjecture true, we have been unable to contruct an example of a finitely generated solvable group G, admitting a finitely generated group of n-unipotent automorphisms which is not a torsion group. However, as pointed out before, we suspect that such example does not exists.

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