GROUPS SATISFYING A STRONG COMPLEMENT PROPERTY

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Abstract. Let $G = NH$ be a finite group where N is normal in G and H is a complement of N in G. For a given generating sequence $(h_1, \dots, h_d)$ of H we say that $(N, (h_1, \dots, h_d))$ satisfies the strong complement property, if $\langle h_1^{x_1}, \dots, h_d^{x_d} \rangle$ is a complement of N in G for all $x_1, \dots, x_d \in N$. When d is the minimal number of elements needed to generate H, and $(N, (h_1, \dots, h_d)))$ satisfies the strong complement property for every generating sequence $(h_1, \dots, h_d)$ with length d, then we say that (N, H) satisfies the strong complement property. In the case when $|N|$ and $|H|$ are coprime, we show that (N, H) can only satisfy the strong complement property if H is cyclic or if H acts trivially on N. We give on the other hand a number of examples that show this does not need to be the case when considering the strong complement property of $(N, (h_1, \dots, h_d))$ for a given fixed generating sequence. In the case when N and H are not of coprime order, we give examples where (N, H) satisfies the strong complement property and where H is not cyclic and does not act trivially on N.

1. Introduction

Let $G = NH$ be a finite group where N is normal in G and H is a complement of N in G. Consider a given generating sequence $(h_1, \dots, h_d)$ of H, that is $\langle h_1, \dots, h_d \rangle = H$. We say that $(N, (h_1, \dots, h_d))$ satisfies the *strong complement property* if $\langle h_1^{x_1}, \dots, h_d^{x_d} \rangle$ is a complement of N in G for all $x_1, \dots, x_d \in N$.

It is easy to see that if $d > d(H)$ and $(N, (h_1, \dots, h_d))$ satisfies the strong complement property for any generating sequence $(h_1, \dots, h_d)$ of H with length d, then H acts trivially on N (see Remark 5).

When d is precisely the minimal number of elements $d(H)$ needed to generate H, and $(N, (h_1, \dots, h_d))$ satisfies the strong complement property for any generating sequence $(h_1, \dots, h_d)$ with length d, then we say that (N, H) satisfies the *strong complement property.*

Note that (N, H) satisfies the strong complement property when H acts trivially on N or H is a cyclic group. So, in the following we will only deal with the case where H is a non-cyclic group.

The natural question that arises is whether there are any examples apart from those two trivial ones. We will show that this is not the case when $|N|$ and $|H|$ are coprime.

2010 *Mathematics Subject Classification.* 20E22, 20D99.

Key words and phrases. Complements, Finite groups.

This work originates from the third author's visit to Padua. He thanks the university of Padua for the hospitality and financial support. The visit was also partially financed through a grant from EPSRC.

Theorem 1. *Let $G = H \ltimes N$ be a semidirect product of finite groups N and H where H is not cyclic and where H and N have coprime orders. If (N, H) satisfies the strong complement property, then H acts trivially on N.*

As usual in this setting, the main case is when N is a minimal normal subgroup of G. The bulk of the work is dealing with the case when N is abelian, in which case it can be viewed as a faithful and irreducible H-module. Consider that setting in full generality, that is without assuming that $|N|$ and $|H|$ are coprime. By a result of Aschbacher and Guralnick [1], if H is not cyclic and $H = \langle h_1, \ldots, h_d\rangle$, then there exist $v_1, \ldots, v_d \in N$ such that $G = \langle h_1v_1, \ldots, h_dv_d\rangle$. Note that $(N, (h_1, \ldots, h_d))$ satisfies the strong complement property if and only if $G \neq \langle h_1^{x_1}, \ldots, h_d^{x_d}\rangle$ for every choice of $x_1, \ldots, x_d \in N$. Since $h_i^{x_i} = h_i[h_i, x_i]$, this holds precisely when the vectors $v_1, \ldots, v_d \in N$ such that $G = \langle h_1v_1, \ldots, h_dv_d\rangle$ can never be chosen with the further restriction that $v_i \in [h_i, N]$ for every $1 \leq i \leq d$. This leads to a condition on the size of the centralizers of the elements h_i in N (see Theorem 2). Combining this result with a theorem of Scott [13, Theorem 1], we prove in Theorem 7 that (N, H) only satisfies the strong complement property when H is either cyclic or acts trivially on N.

The above mentioned theorem of Scott generalizes a theorem of Ree about permutations [12, Theorem 1]. Ree's formula allows us to generalize the arguments used in the abelian case to the case where N is a minimal nonabelian group. The almost simple case requires the classification of finite simple groups.

The structure of the paper is as follows. In section 2, we do some preliminary work regarding the case when N is a faithful and irreducible H-module. We also present examples that show that $(N, (h_1, \ldots, h_d))$ can satisfy the strong complement property for a given fixed sequence although $|N|$ and $|H|$ are coprime and $d \geq 2$. In section 3, we will continue our investigation into the case when N is a faithful irreducible H-module and prove our main result in this direction, that is Theorem 7. In Section 4, we handle the case when N is non-abelian and in Section 5 we consider the coprime case and we prove Theorem 1. In the 6th and final section we present some more examples.

2. Some preliminaries.

This section is devoted to analyze the case where N is an abelian minimal normal subgroup of $G = NH$ and H is a not cyclic group acting non-trivially on N. So we set $N = V$ and we view V as a faithful and irreducible H-module.

The study of the first cohomology group $\mathrm{H}^1(G, V)$ is intimately connected with questions about the generation of the semidirect product $G = V \rtimes H$. In [1], Aschbacher and Guralnick proved that $|\,\mathrm{H}^1(G, V)| < |V|$ and deduce from this that if $d \geq 2$ and H can be generated by d elements, then G too can be generated by d elements. More precisely if $H = \langle h_1, \ldots, h_d\rangle$, then there exist $v_1, \ldots, v_d \in V$ such that $G = \langle h_1v_1, \ldots, h_dv_d\rangle$.

Note that $(V, (h_1, \ldots, h_d))$ satisfies the strong complement property, that is $\langle h_1^{w_1}, \ldots, h_d^{w_d}\rangle$ is a complement of V in G for every choice of $w_1, \ldots, w_d \in V$, precisely when there is no $w_1, \ldots, w_d \in V$ such that $G = \langle h_1^{w_1}, \ldots, h_d^{w_d}\rangle$. Since $h_i^{w_i} = h_i[h_i, w_i]$, this holds when the vectors $v_1, \ldots, v_d \in V$ such that $G = \langle h_1v_1, \ldots, h_dv_d\rangle$ can not be chosen with the further restriction that $v_i \in [h_i, V]$ for every $1 \leq i \leq d$.

Before proving a criterion to decide whether $(V,(h_1,\dots,h_d))$ satisfies the strong complement property for a generating sequence $(h_1,\dots,h_d)$ of H, we need to introduce some definitions and notations. Let $F=\operatorname{End}_H V$ and let $n=\dim_F V$. We will denote by $\operatorname{Der}(H,V)$ the set of the derivations from H to V (i.e. the maps $\delta : H\to V$ with the property that $\delta(h_1h_2)=\delta(h_1)^{h_2}+\delta(h_2)$ for every $h_1,h_2\in H$). If $v\in V$ then the map $\delta_v : H\to V$ defined by $\delta_v(h)=[h,v]$ is a derivation. The set $\operatorname{InnDer}(H,V)=\{\delta_v \mid v\in V\}$ of the inner derivations from H to V is a subgroup of $\operatorname{Der}(V,H)$ and the factor group $\mathrm{H}^1(H,V)=\operatorname{Der}(H,V)/\operatorname{InnDer}(H,V)$ is the first cohomology group of H with coefficients in V. It is clear that V, $\operatorname{Der}(H,V)$, $\operatorname{InnDer}(H,V)$ and $\mathrm{H}^1(H,V)$ are vector spaces of F. Let

$$n=\dim_F V \text{ and } r=\dim_F \mathrm{H}^1(H,V).$$

It follows from [1, Theorem A] and [6, Theorem 1] that

$$r\le \lfloor n/2\rfloor \le n-1. \tag{2.1}$$

For our purpose it is also important to consider the F-subspace

$$\Delta_V(h_1,\dots,h_d)$$

of $\operatorname{Der}(H,V)$ consisting of the derivations δ with the property that, for every $i\in\{1,\dots,n\}$, there exists $v_i\in V$ such that $\delta(h_i)=\delta_{v_i}(h_i)$. Clearly $\operatorname{InnDer}(H,V)\le\Delta_V(h_1,\dots,h_d)$ and so

$$n=\dim_F \operatorname{InnDer}(H,V)\le \dim_F\Delta_V(h_1,\dots,h_d)\le\dim_F\operatorname{Der}(H,V)=n+r.$$

In particular there exists an integer $s_V(h_1,\dots,h_d)$, with $0\le s_V(h_1,\dots,h_d)\le r$, such that

$$\dim_F\Delta_V(h_1,\dots,h_d)=n+s_V(h_1,\dots,h_d).$$

Theorem 2. *Let $H=\langle h_1,\dots,h_d\rangle$ be a finite non-cyclic group and V a faithful and irreducible H-module. Then $(V,(h_1,\dots,h_d))$ satisfies the strong complement property if and only if*

$$\sum_{1\le i\le d}\dim_F C_V(h_i)=n(d-1)-s_V(h_1,\dots,h_d).$$

Proof. It is well known that if $\delta\in\operatorname{Der}(H,V)$, then

$$H_\delta=\{h\delta(h)\mid h\in H\}=\langle h_1\delta(h_1),\dots,h_d\delta(h_d)\rangle$$

is a complement of V in G and the maps $\delta\mapsto H_\delta$ is a bijection from $\operatorname{Der}(H,V)$ to the set of the complements of V in G. Notice that there exists $v_1,\dots,v_d\in V$ such that $H_\delta=\langle h_1^{v_1},\dots,h_d^{v_d}\rangle$ if and only if $\delta\in\Delta_V(h_1,\dots,h_d)$. Moreover, since $H=\langle h_1,\dots,h_d\rangle$ and V is a faithful module, the map $\alpha:\operatorname{Der}(H,V)\to V^d$ defined via $\delta\mapsto(\delta(h_1),\dots,\delta(h_d))$ is F-linear and injective. Now consider the F-endomorphism $\beta: V^d\to V^d$ defined by setting $\beta(v_1,\dots,v_d)=([h_1,v_1],\dots,[h_d,v_v])$. If $v_1,\dots,v_d\in V$, then $\langle h_1^{v_1},\dots,h_d^{v_d}\rangle$ either coincides with G or is a complement of V in G. The second case occurs if and only if there exists $\delta\in\Delta_V(h_1,\dots,h_d)$ such that $\langle h_1^{v_1},\dots,h_d^{v_d}\rangle=H_\delta$ and in such a case $\beta(v_1,\dots,v_d)=\alpha(\delta)$. But then $(V,(h_1,\dots,h_d))$ satisfies the strong complement property if and only if $\beta(V^d)=\alpha(\Delta_V(h_1,\dots,h_d))$.

Since $\alpha(\Delta_V(h_1,\dots,h_d)) \le \beta(V^d)$ we have that $\beta(V^d) = \alpha(\Delta_V(h_1,\dots,h_d))$ if and only if

$$\begin{aligned} n + s_V(h_1,\dots,h_d) &= \dim_F(\Delta_V(h_1,\dots,h_d)) = \dim_F(\alpha(\Delta_V(h_1,\dots,h_d))) \\ &= \dim_F(\beta(V^d)) = d\cdot n - \sum_{1\le i\le d} \dim_F(C_V(h_i)), \end{aligned}$$

i.e. if and only if $\sum_{1\le i\le d} \dim_F(C_V(h_i)) = n(d-1) - s_V(h_1,\dots,h_d)$. □

Corollary 3. *Suppose that* $\mathrm{H}^1(H,V)=0$. *Then* $(V,(h_1,\dots,h_d))$ *satisfies the strong complement property if and only if*

$$\sum_{1\le i\le d} \dim_F C_V(h_i) = n(d-1).$$

Recall that the assumption $\mathrm{H}^1(G,V)=0$ in the case of soluble groups is assured by the following unpublished result by Gaschütz (see [14, Lemma 1]).

Lemma 4. *Let* $G \ne 1$ *be a finite soluble group and let* V *be a faithful and irreducible* G*-module. Then* $\mathrm{H}^1(G,V)=0$.

We want to present now some examples in which $(V,(h_1,\dots,h_d))$ satisfies the strong complement property for at least one generating sequence $(h_1,\dots,h_d)$ of H.

Example. Let p be a fixed prime. By the Dirichlet's theorem on arithmetic progression, there exists a prime q with the property that p divides $q-1$. Let F be the field with q elements and let $C=\langle\lambda\rangle$ be the subgroup of order p of the multiplicative group of F. For every positive integer t, we fix a Sylow p-subgroup P_t of $\mathrm{Sym}(p^t)$ and consider the wreath product $H_t = C \wr P_t$. Notice that H_t is isomorphic to the iterated wreath product $C_p \wr \cdots \wr C_p$, where C_p is the cyclic group of order p and the number of factors is $t+1$. In particular H_t can be generated by $t+1$ elements. Let $n=p^t$ and let $V_t = F^n$ be an n-dimensional vector space over F. The group H_t has an irreducible action on V_t defined as follows: if $v = (f_1,\dots,f_n) \in V_t$ and $h=(c_1,\dots,c_n)\sigma \in H_t$, where $c_i \in C$ and $\sigma \in P_t$, then $v^h = (f_{1\sigma^{-1}}c_{1\sigma^{-1}},\dots,f_{n\sigma^{-1}}c_{n\sigma^{-1}})$.

We claim that for every positive integer t, there exist $t+1$ generators $(h_1,\dots,h_{t+1})$ of H_t such that $(V_t,(h_1,\dots,h_{t+1}))$ satisfies the strong complement property. By Corollary 3 and Lemma 4, this is equivalent to find a generating sequence $(h_1,\dots,h_{t+1})$ of H_t with the property that

$$\sum_{1\le i\le t+1} \dim_F C_{V_t}(h_i) = p^t\cdot t. \tag{2.2}$$

First consider the case $t=1$. Consider the elements $h_1=(\lambda,1,\dots,1)$ and $h_2=(1,2,\dots,p)\in P_1$, and note that $H_1=\langle h_1,h_2\rangle$. Since

$$C_{V_1}(h_1)=\{(0,f_2,\dots,f_p)\mid f_2,\dots,f_p\in F\},\quad C_{V_1}(h_2)=\{(f,\dots,f)\mid f\in F\},$$

we have that $\dim_F C_{V_1}(h_1)+\dim_F C_{V_2}(h_2) = p-1+1=p$, and (2.2) holds. Now suppose that $(h_1,\dots,h_{t+1})$ is a generating sequence for H_t and that $(V_t,(h_1,\dots,h_{t+1}))$ satisfies the strong complement property. We may identify H_{t+1} with the wreath product $H_t \wr P_1$ and V_{t+1} with V_t^p. With this identification, if $v=(v_1,\dots,v_p)\in V_{t+1}$ and $h=(x_1,\dots,x_p)\sigma \in H_t\wr P_1$, then $v^h = (v_{1\sigma^{-1}}^{x_{1\sigma^{-1}}},\dots,v_{p\sigma^{-1}}^{x_{p\sigma^{-1}}})$. For $1\le i\le t+1$, let $k_i = (h_i, 1, \dots, 1) \in H_t^p$, moreover let $k_{t+2} = (1, 2, \dots, p) \in P_1$. We have that $H_{t+1} = \langle k_1, \dots, k_{t+1}, k_{t+2} \rangle$. Moreover if $1 \le i \le t+1$, then

$$C_{V_{t+1}}(k_i) = \{(v_1, \dots, v_p) \mid v_1 \in C_{V_t}(h_i), v_2, \dots, v_p \in V_t\}$$

so

$$\dim_F C_{V_{t+1}}(k_i) = \dim_F C_{V_t}(h_i) + (p-1)p^t.$$

On the other hand $C_{V_{t+1}}(k_{t+2}) = \{(v, \dots, v) \mid v \in V_t\}$ so

$$\dim_F C_{V_{t+1}}(k_{t+2}) = \dim_F V_t = p^t.$$

Since $(V_t, (h_1, \dots, h_{t+1}))$ satisfies the strong complement property, by (2.2) we have

$$\sum_{1 \le i \le t+1} \dim_F C_{V_t}(h_i) = p^t \cdot t.$$

But then

$$\begin{aligned} \sum_{1 \le i \le t+2} \dim_F C_{V_{t+1}}(k_i) &= \sum_{1 \le i \le t+1} \dim_F C_{V_t}(h_i) + (t+1)(p-1)p^t + p^t \\ &= p^t \cdot t + (t+1)(p-1)p^t + p^t = (t+1)p^{t+1}. \end{aligned}$$

Hence (2.2) holds and $(V_{t+1}, (k_1, \dots, k_{t+2}))$ satisfies the strong complement property.

Example. Another example of generators satisfying the strong complement property can be obtained considering the action of $\mathrm{Sym}(n)$ on the full delete module V over the field F with p element, where p is a prime which does not divide n. We have that $V = \{(f_1, \dots, f_n) \in F^n \mid f_1 + \dots + f_n = 0\}$ and $\mathrm{Sym}(n)$ acts on V by permuting the n entries. Let $\alpha = (1, 2)$ and $\beta = (2, \dots, n)$. We have

$$\begin{aligned} C_V(\alpha) &= \{(x, x, y_3, \dots, y_n) \mid x + x + y_3 + \dots + y_n = 0\} \\ C_V(\beta) &= \{(x, y, \dots, y) \mid x + (p-1)y = 0\} \end{aligned}$$

so $\dim_F C_V(\alpha) + \dim_F C_V(\beta) = n-2+1 = n-1 = \dim_F V$. Since $\mathrm{H}^1(\mathrm{Sym}(n), V) = 0$ (see for example [9, (5.8)]), it follows from Corollary 3 that $(V, (\alpha, \beta))$ satisfies the strong complement property.

3. The case where N is a faithful and irreducible H-module

Now we investigate the following question: is it possible that $(N, (h_1, \dots, h_d))$ satisfies the strong complement property for all the generating sequences $(h_1, \dots, h_d)$ of length d?

Remark 5. *Note that if $(N, (h_1, \dots, h_d))$ satisfies the strong complement property and $h_d \in \langle h_1, \dots, h_{d-1} \rangle$, then h_d centralizes N. Indeed, given an element $x \in N$, we have that $K = \langle h_1^x, \dots, h_{d-1}^x, h_d \rangle$ is a complement of N in G. Therefore, as $H^x \le K$ and $h_d \in K$, we conclude that $[h_d, x] \le K \cap N = 1$.*

It easily follows that if $d > d(H)$ and $(N, (h_1, \dots, h_d))$ satisfies the strong complement property for any generating sequence $(h_1, \dots, h_d)$ of H with length d, then H acts trivially on N.

In this section we analyze the case where $N = V$ is minimal and abelian, so in the following V will denote a faithful and irreducible H-module. Moreover we set $F = \mathrm{End}_H V$ and $n = \dim_F V$.

Lemma 6. *Let H be a finite non-cyclic group and V a faithful and irreducible H-module. If (V,H) satisfies the strong complement property, then*

$$\dim_F \mathrm{H}^1(H,V) \geq \frac{d(H)-1}{d(H)+1}\cdot n.$$

Proof. Let $d=d(H)$ and $r=\dim_F \mathrm{H}^1(H,V)$. By Theorem 2 and the hypothesis, for every generating sequence $(y_1,\ldots,y_d)$ of H we have that

$$\sum_{1\leq i\leq d} \dim_F C_V(y_i) \geq n(d-1)-r. \tag{3.1}$$

Choose a generating sequence $(h_1,\ldots,h_d)$ of H and consider the elements

$$x_1 = h_1h_2\cdots h_{d-1}h_d, \quad x_2=h_2h_3\cdots h_dh_1, \quad \ldots, \quad x_d=h_dh_1\cdots h_{d-2}h_{d-1}.$$

Since $x_1,\ldots,x_d$ are conjugate in H, we have $\dim_F C_V(x_1)=\cdots=\dim_F C_V(x_d)$. Set

$$\gamma_0=\dim_F C_V(x_1), \quad \gamma_1=\dim_F C_V(h_1),\ldots, \quad \gamma_d=\dim_F C_V(h_d)$$

and, for $i\in\{0,\ldots,d\}$,

$$\kappa_i=\sum_{0\leq j\leq d, j\neq i}\gamma_j.$$

Notice that

$$\sum_{0\leq i\leq d}\kappa_i = d(\gamma_0+\gamma_1+\cdots+\gamma_d).$$

By applying (3.1) to the generating sequence $(h_1,\ldots,h_d)$ we get $\kappa_0\geq n(d-1)-r$; similarly considering the generating sequence $(h_1,\ldots,h_{i-1},x_i,h_{i+1},\ldots,h_d)$ we obtain $\kappa_i\geq n(d-1)-r$ for every $i\neq 0$. This implies

$$d(\gamma_0+\gamma_1+\cdots+\gamma_d)=\sum_{0\leq i\leq d}\kappa_i \geq (d+1)(n(d-1)-r). \tag{3.2}$$

On the other hand, by a theorem of Scott [13, Theorem 1],

$$\gamma_0+\gamma_1+\cdots+\gamma_d\leq n(d-1). \tag{3.3}$$

Comparing (3.2) and (3.3) we deduce

$$(d+1)(n(d-1)-r)\leq n(d-1)d$$

and thus $r\geq (d-1)n/(d+1)$, as desired. □

Theorem 7. *Assume that V is a faithful irreducible H module. If (V,H) satisfies the strong complement property when H is either cyclic or acts trivially on V.*

Proof. Assume that H is a finite non-cyclic group and V a faithful and irreducible H-module and suppose, by contradiction, that (V,H) satisfies the strong complement property. Let $F=\mathrm{End}_H V$, $n=\dim_F V$ and $r=\dim_F \mathrm{H}^1(H,V)$. From (2.1) and Lemma 6 we have that

$$\frac{d(H)-1}{d(H)+1}\cdot n\leq r\leq \frac{n}{2}. \tag{3.4}$$

Therefore

$$\frac{d(H)-1}{d(H)+1}\leq\frac{1}{2}$$

and consequently $d(H)\leq 3$. Suppose $d(H)=3$. Then by (3.4) we deduce that $r=n/2$. From [6, Theorem 1] it follows that either $F^*(H)=L_2(2^n)$, $n>1$, or

$F^*(H) = \mathrm{Alt}(6)$, but this would imply $d(H) = 2$, a contradiction. This proves that $d(H) = 2$.

For a generating set $x_1, \dots, x_d$ of H let us define

$$\tilde{\mathrm{H}}^1(x_1, \dots, x_d) = \frac{\Delta(x_1, \dots, x_d)}{\mathrm{InnDer}(H, V)} \leq \mathrm{H}^1(H, V).$$

According to the definition given in Section 2

$$\dim_F \tilde{\mathrm{H}}^1(x_1, \dots, x_d) = s_V(x_1, \dots, x_d).$$

Assume $H = \langle h_1, h_2 \rangle$, let $h_0 = h_1 h_2$, $\gamma_i = \dim_F C_V(h_i)$ for $0 \leq i \leq 2$ and set $h(V) = n - \gamma_1 - \gamma_2 - \gamma_3$. By Theorem 2

$$2h(V) = s_V(h_0, h_1) + s_V(h_0, h_2) + s_V(h_1, h_2) - n$$

while by [13, Proposition 1.a]

$$h(V) \geq s_V(h_0, h_1, h_2)$$

so we have

$$s_V(h_0, h_1) + s_V(h_0, h_2) + s_V(h_1, h_2) \geq 2s_V(h_0, h_1, h_2) - n. \tag{3.5}$$

From the definition, we have

$$\begin{aligned} \tilde{\mathrm{H}}^1(h_0, h_1, h_2) &= \tilde{\mathrm{H}}^1(h_0, h_1) \cap \tilde{\mathrm{H}}^1(h_0, h_2) \\ &= \tilde{\mathrm{H}}^1(h_0, h_1) \cap \tilde{\mathrm{H}}^1(h_1, h_2) \\ &= \tilde{\mathrm{H}}^1(h_0, h_2) \cap \tilde{\mathrm{H}}^1(h_1, h_2). \end{aligned}$$

So in particular

$$\begin{aligned} s_V(h_0, h_1) + s_V(h_0, h_2) - s_V(h_0, h_1, h_2) &= \dim_F(\tilde{\mathrm{H}}^1(h_0, h_1) + \tilde{\mathrm{H}}^1(h_0, h_2)) \\ &\leq \dim_F \mathrm{H}^1(H, V) = r. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we deduce

$$r + s_V(h_1, h_2) - s_V(h_0, h_1, h_2) \geq n.$$

Since $r \leq n/2$, it follows

$$s_V(h_1, h_2) - s_V(h_0, h_1, h_2) \geq \frac{n}{2}.$$

This is only possible if $s_V(h_0, h_1, h_2) = 0$, $s_V(h_1, h_2) = n/2 = r$ and consequently $\tilde{\mathrm{H}}^1(h_1, h_2) = \mathrm{H}^1(H, V)$. With the same argument we can also deduce $\tilde{\mathrm{H}}^1(h_0, h_1) = \tilde{\mathrm{H}}^1(h_0, h_2) = \mathrm{H}^1(H, V)$. This implies $\tilde{\mathrm{H}}^1(h_0, h_1, h_2) = \mathrm{H}^1(H, V)$, contradicting $r = n/2 > s_V(h_0, h_1, h_2) = 0$. □

When $d(H) = 2$, the proof of Theorem 7 shows that, for every generating set (h_1, h_2) of H either (h_1, h_2), or $(h_1, h_1 h_2)$, or $(h_2, h_1 h_2)$ does not satisfy the strong complement property.

4. The case where N is non-abelian

The case where N is a complemented non-abelian minimal normal subgroup of $G = NH$ is quite more complicate. In this section we will work under the following assumptions:

- $G = NH$ is a finite group with a unique minimal normal subgroup, say N, which is complemented by the subgroup H.
- $N \cong S^n$, where $n \in \mathbb{N}$ and S is a finite non abelian simple group.
- κ is the number of complements of N in G.

Let $H = \langle h_1, \dots, h_d \rangle$ and assume that $(N, (h_1, \dots, h_d))$ satisfies the strong complement property, i.e. $\langle h_1^{n_1}, \dots, h_d^{n_d} \rangle$ is a complement of N in G for every choice of $n_1, \dots, n_d \in N$. Let

$$\Omega = \{\langle h_1^{n_1}, \dots, h_d^{n_d} \rangle \mid (n_1, \dots, n_d) \in N^d\}.$$

As κ is the number of all complements of N in G, we must have $\kappa \geq |\Omega|$. Note that

$$|\Omega| = \prod_{1 \leq i \leq d} \frac{|N|}{|C_N(h_i)|}.$$

We may identify G with a subgroup of the wreath product $\mathrm{Aut}\, S \wr \mathrm{Sym}(n)$. In particular any $h \in H$ can be written in the form $(\alpha_1, \dots, \alpha_n)\sigma_h$, with $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}\, S$ and $\sigma_h \in \mathrm{Sym}(n)$. Define $v(h) = n - r(h)$ where $r(h)$ is the number of orbits of $\langle \sigma_h \rangle$ on $\{1, \dots, n\}$. It can be easily seen that $|C_N(h)| \leq |S|^{r(h)}$ and $|N : C_H(h)| \geq |S|^{v(h)}$, hence

$$|\Omega| \geq |S|^{\sum_{1 \leq i \leq d} v(h_i)}.$$

Therefore we obtain the following bound:

$$\sum_{1 \leq i \leq d} v(h_i) \leq \log_{|S|} \kappa. \tag{4.1}$$

Proposition 8. *Let $G = NH$ be a finite group with a unique minimal normal subgroup N, where N is non abelian and complemented by H. If (N, H) satisfied the strong complement property, then*

$$d(2n - 2) \leq (d + 1) \log_{|S|} \kappa,$$

where $d = d(H)$ and κ is the number of complements of N in G.

Proof. By hypothesis, $(N, (h_1, \dots, h_d))$ satisfies the strong complement property for all the generating sequence of H of cardinality $d = d(H)$. If $d = 1$, then the bound follows from (4.1). So assume $d \geq 2$ and choose a generating sequence $(h_1, \dots, h_d)$. Let

$$x_1 = h_1 h_2 \cdots h_{d-1} h_d, \quad x_2 = h_2 h_3 \cdots h_d h_1, \quad \dots, \quad x_d = h_d h_1 \cdots h_{d-2} h_{d-1}.$$

Since $x_1, \dots, x_d$ are conjugate in H, we have $v(x_1) = \cdots = v(x_d)$. Set

$$\gamma_0 = v(x_1), \gamma_1 = v(h_1), \dots, \gamma_d = v(h_d)$$

and, for $i \in \{0, \dots, d\}$,

$$\kappa_i = \sum_{0 \leq j \leq d, j \neq i} \gamma_j.$$

Notice that

$$\sum_{0 \leq i \leq d} \kappa_i = d(\gamma_0 + \gamma_1 + \cdots + \gamma_d).$$

By applying (4.1) to the generating sequence $(h_1, \dots, h_d)$ we get $\kappa_0 \leq \log_{|S|} \kappa$; similarly considering the generating sequence $(h_1, \dots, h_{i-1}, x_i, h_{i+1}, \dots, h_d)$ we obtain $\kappa_i \leq \log_{|S|} \kappa$ for every $i \neq 0$. This implies

$$d(\gamma_0 + \gamma_1 + \dots + \gamma_d) = \sum_{0 \leq i \leq d} \kappa_i \leq (d+1) \log_{|S|} \kappa. \tag{4.2}$$

On the other hand, by a theorem of Ree [12, Theorem 1],

$$\gamma_0 + \gamma_1 + \dots + \gamma_d \geq 2n - 2. \tag{4.3}$$

Comparing (4.2) and (4.3) we deduce

$$d(2n-2) \leq (d+1) \log_{|S|} \kappa,$$

as desired. □

It is quite hard to produce a good bound for the number of complements of $N = S^n$ in G. An easy case occurs when $|N|$ and $|G/N|$ are coprime, indeed by the Schur-Zassenhaus theorem N is complemented and all the complements are conjugate in G, hence $\kappa \leq |N|$.

The case $n = 1$ of the forthcoming Theorem 10 follows from the following lemma.

Lemma 9. *Let G be a finite almost simple group and $S = \operatorname{soc} G$. If $(|G/S|, |S|)$ are coprime, then G/S is cyclic.*

Proof. We have to prove that if a subgroup X of $\operatorname{Out} S$ has order coprime with the order of S, then X is cyclic. If S is an alternating or a sporadic simple group, then $|\operatorname{Out} S|$ divides 4 so, since $|S|$ is even, we are done. Thus we may assume that S is a simple group of Lie type. In this case $\operatorname{Out} S$ is a semidirect product (in this order) of groups of order d (diagonal automorphisms), f (field automorphisms), and g (graph automorphisms modulo field automorphisms), except that for $B_2(2^f), G_3(3^f), F_4(2^f)$ the graph automorphism square to the generating field automorphism. The groups of order d, f, g are cyclic except in the case $D_4(q)$ (where the graph automorphisms generates a group isomorphic to $\operatorname{Sym}(3)$). On the other hand it follows from [4, Table 5 and Table 6] that d and g divides $|S|$ so X must be a subgroup of a cyclic group of order dividing f. □

In the following we will need Gaschütz's Lemma [5], which says that if K is a normal subgroup of a finite group G and $\langle g_1, \dots, g_d, K \rangle = G$ with $d \geq d(G)$, then we can find $k_1, \dots, k_d \in K$ such that $\langle g_1 k_1, \dots, g_d k_d \rangle = G$.

Theorem 10. *Let G be a finite group with a unique minimal normal subgroup N, where N is non abelian and $(|G : N|, |N|) = 1$. Let H be a complement of N in G. If (N, H) satisfies the strong complement property, then H is cyclic.*

Proof. Let $N \cong S^n$, where $n \in \mathbb{N}$ and S is a finite non abelian simple group. By the Schur-Zassenhaus theorem, $\kappa \leq |N| = |S|^n$ so by Proposition 8

$$d(2n-2) \leq (d+1)n, \tag{4.4}$$

that is $(n-2)d \leq n$. Hence for $n > 5$, we must have $d \leq 1$ and H cyclic.

Let us analyze the cases where $n \leq 4$. Notice that H acts transitively on the n factors of N: so if $n = 2$ or $n = 4$ then 2 divides $|H|$, but then $|H|, |N|$ are not coprime, against our assumption. If $n = 1$, it follows from Lemma 9 that H is cyclic. So the only case that is left is $n = 3$. In this case, by the above bound,

$d \leq 3$. First assume $d = 3$. In this case, by Gaschütz's Lemma [5], we can choose the generators h_1, h_2, h_3 so that $\sigma_{h_1} = \sigma_{h_2} = (1, 2, 3), \sigma_{h_3} = (1, 3, 2)$ but then $v(h_1) = v(h_2) = v(h_3) = 3 - 1 = 2$ so $v(h_1) + v(h_2) + v(h_3) = 6 > 3$ against (4.1). Similarly if $d = 2$ we can choose choose h_1, h_2 so that $\sigma_{h_1} = \sigma_{h_2} = (1, 2, 3)$ but then $v(h_1) = v(h_2) = 2$ so $v(h_1) + v(h_2) = 4$ and again (4.1) fails. So H is cyclic in all the cases. □

5. The coprime case

Proof of Theorem 1. Let $G = H \ltimes N$ be a semidirect product of finite groups N and H where H is not cyclic and where H and N have coprime orders. We want to prove that if (N, H) satisfies the strong complement property, then H acts trivially on N.

We argue by contradiction and take a counter example where the order of N is smallest possible. Let U be a minimal normal subgroup of G contained in N. It is easy to see that both (U, H) and $(N/U, H)$ satisfy the strong complement property. If U is a non-trivial proper subgroup of N, then, by minimality, H acts trivially on N/U and U. Since the action is coprime, we deduce that H acts trivially on N, against our assumption. Thus N is a minimal normal subgroup of G.

Let $C = C_H(N)$ and $d = d(H)$. For short, we will use the bar notation to denote the image of the elements and subgroups in H/C. We want to prove that for every d-generating sequence $(\bar{h}_1, \ldots, \bar{h}_d)$ of $\bar{H}$, $(\bar{N}, (\bar{h}_1, \ldots, \bar{h}_d))$ satisfies the strong complement property. So, given $n_1, \ldots n_d \in N$ we need to prove that $\langle \bar{h}_1^{\bar{n}_1}, \ldots, \bar{h}_d^{\bar{n}_d} \rangle$ intersects trivially $\bar{N}$ in $\bar{G}$. By Gaschütz's Lemma, there exist elements $c_1, \ldots, c_d \in C$ such that $\langle h_1 c_1, \ldots, h_d c_d \rangle = H$. As (N, H) satisfies the strong complement property, the subgroup $\langle (h_1 c_1)^{n_1}, \ldots, (h_d c_d)^{n_d} \rangle$ is a complement of N, hence, by the Schur-Zassenhaus theorem, it is a conjugate H^n of H for some $n \in N$. Note that $C \leq H^n$. Therefore

$$\langle h_1^{n_1} C, \ldots, h_d^{n_d} C \rangle \leq \langle (h_1 c_1)^{n_1}, \ldots, (h_d c_d)^{n_d} \rangle = H^n.$$

and we conclude that $\langle \bar{h}_1^{\bar{n}_1}, \ldots, \bar{h}_d^{\bar{n}_d} \rangle$ is a complement of $\bar{N}$. This proves that $(\bar{N}, (\bar{h}_1, \ldots, \bar{h}_d))$ satisfies the strong complement property for any generating sequence $\bar{h}_1, \ldots, \bar{h}_d$ of $\bar{H}$ of length d. In particular, if $d > d(\bar{H})$, then by Remark 5 the action results to be trivial and thus $H = C$ against our assumption. We deduce that $d(\bar{H}) = d = d(H)$ and $(\bar{N}, \bar{H})$ satisfies the strong complement property.

Note that $\bar{N}$ is the unique minimal normal subgroup of $\bar{G}$. If N is non-abelian, then by Theorem 10 we get that $\bar{H}$ is cyclic. If N is abelian, then by the Schur-Zassenhaus theorem all complements are conjugate, hence $\mathrm{H}^1(\bar{H}, \bar{N})$ is trivial. Therefore from Lemma 6 it follows that $\bar{H}$ is cyclic. In both cases, since $d(\bar{H}) = d(H)$, it follows that H is cyclic, against our assumption. □

In the case where H is of odd order and nilpotent of class at most 2 acting coprimely on a faithful irreducible H-module N, the strong complement property is quite limiting: namely it suffices that $(N, (h_1, \ldots, h_d))$ satisfies the strong complement property for one generating sequence $h_1, \ldots, h_d$ of H to deduce that H is cyclic.

We will need the following consequence of Corollary 3.

Lemma 11. *Suppose that N is a faithful irreducible H-module and that $\mathrm{H}^1(H, N) = 0$. If $(N, (h_1, \ldots, h_d))$ satisfies the strong complement property and $d \geq 2$, then $C_N(h_i) \neq 0$ for every $1 \leq i \leq d$.*

Proof. Assume by contradiction that $C_N(h_i) = 0$ for some index i. Let $F = \mathrm{End}_H(N)$ and let $n = \dim_F N$. As $\mathrm{H}^1(H, N) = 0$, by Corollary 3 we have that

$$\sum_{1 \le j \le d} \dim_F C_N(h_j) = n(d-1).$$

Since $\dim_F C_N(h_i) = 0$ and $\dim_F C_N(h_j) \le n$ for $j \ne i$, we deduce that $\dim_F C_N(h_j) = n$ for all $j \ne i$. Hence the action is not faithful, a contradiction. □

Proposition 12. *Suppose that N is a faithful irreducible H-module, where $H = \langle h_1, h_2, \dots, h_d\rangle$ is nilpotent of class at most 2, $|H|$ is odd and $(|N|, |H|) = 1$. If $(N, (h_1, \dots, h_d))$ satisfies the strong complement property, then H is cyclic.*

Proof. By standard arguments we only need to consider the case when H is a p-group where p is an odd prime. Let $F = \mathrm{End}_H(N)$ and let $n = \dim_F N$. We may identify H with a subgroup of $\mathrm{GL}(n, F)$. Moreover for every $z \in Z = Z(H)$, there exists $\lambda \in F$ such that z acts on N as the scalar multiplication by λ. In particular Z is isomorphic to a subgroup of F^* and it is cyclic. So we may assume that H is non-abelian, in particular there exists $i \ne j$ such that, setting $a = h_i$ and $b = h_j$, we have $[b, a] \ne 1$. Hence $[b, a]$ acts as the scalar multiplication by an element λ of F whose order is a non-trivial p-power. By Lemma 11, that there exists an $0 \ne x \in N$ that commutes with a. Notice that

$$x^{b^j a} = x^{ab^j[b,a]^j} = \lambda^j x^{b^j}.$$

Thus x^{b^j} is an eigenvector for a with eigenvalue λ^j. Since $(N, (h_1, \dots, h_d))$ satisfies the strong complement property, $\langle a^{x^b}, b\rangle$ is contained in a complement of N, and therefore we should have that $[a^{x^b}, b, b]$ is trivial. However calculations show that

$$[a^{x^b}, b, b] = (\lambda^{-1}(1-\lambda)x + \lambda^{-1}(\lambda^2 - 1)x^b + (1-\lambda)x^{b^2})^b.$$

Notice that $\lambda^2 \ne 1$ as $p \ge 3$. In particular x, x^b, x^{b^2} are eigenvectors for the different eigenvalues $1, \lambda, \lambda^2$ so they are linearly independent. But then $(\lambda^{-1}(1-\lambda)x + \lambda^{-1}(\lambda^2-1)x^b + (1-\lambda)x^{b^2})^b \ne 0$, a contradiction. [Notice that for $p = 2$ and $\lambda = -1$ this expression becomes $(-2x + 0 + 2x)^b = 0$]. □

In the next example we show that the assumption that $|H|$ is odd in Proposition 12 is essential.

Example. Let $N = \langle x\rangle \times \langle y\rangle$ be an elementary abelian p group where p is an odd prime. Let $P = \langle a, b\rangle$ be the subgroup of $\mathrm{Aut}(N)$ where the action of P on N is given by

$$x^a = x,\ y^a = y^{-1},\ x^b = y,\ y^b = x.$$

Notice that P is the dihedral group of order 8 with $x^{[a,b]} = x^{-1}$ and $y^{[a,b]} = y^{-1}$. Thus in particular P is nilpotent of class 2. The action of P on N is faithful. Let us see also that $(N, (a, b))$ satisfies the strong complement property. This follows from the fact that a commutes with x and b with xy. Thus, if $w = x^\alpha (xy)^\beta$ and $z = x^\gamma (xy)^\delta$ are any two elements of N, then

$$\langle a^w, b^z\rangle = \langle a^{wx^{\gamma-\alpha}}, b^{z(xy)^{\beta-\delta}}\rangle = \langle a, b\rangle^{x^\gamma (xy)^\beta}$$

and so $\langle a^w, b^z\rangle$ is a complement of N.

6. Some examples

Another case in which we can produce suitable bounds for the number of complements of a non-abelian minimal normal subgroup N in G is where $G = S \wr H$ is actually the wreath product of a finite non-abelian simple group S and a d-generated transitive subgroup H of $\mathrm{Sym}(n)$, and $N = S^n$ is the base subgroup.

The so call "McIver-Neumann Half-n Bound" says that if Y is a subgroup of $\mathrm{Sym}(n)$, then $d(Y) \le n/2$ if $n \ne 3$, and $d(Y) \le 2$ if $n = 3$ (see [11, Lemma 5.2] or [3, Section 4]). Moreover, from the main result of [8] it follows that if Y is a p-group, then $d(Y) \le n/p$.

Arguing as in [10, Lemma 2.8], we can derive the following bounds.

Lemma 13. *Let H be a transitive subgroup of $\mathrm{Sym}(n)$, S a finite non-abelian simple group, G the wreath product $S \wr H$ and $N \cong S^n$ the socle of G. For $n \ne 4$, the number κ of complements of N in G is at most $|S|^{(3n-1)/2}$. If H is a p-group, then $\kappa \le |S|^{n+n/p-1}$.*

Proof. We can identify G with a subgroup of the wreath product of S and $\mathrm{Sym}(n)$. So the elements of G are of the kind $g = (x_1, \dots, x_n)\sigma$ with $x_i \in S$ and $\sigma \in \mathrm{Sym}(n)$. For any $1 \le i \le n$, denote by S_i the subset of $N = S^n$ consisting of the elements $x = (x_1, \dots, x_n)$ with $x_j = 1$ for every $j \ne i$. Let $K = S_2 \times \cdots \times S_n$. Note that $N_G(K)$ is isomorphic to a subgroup of $S \times (S \wr \mathrm{Sym}(n-1))$ and $N_G(K)/K$ is isomorphic to a subgroup of $S \times \mathrm{Sym}(n-1)$. Then by [7, Corollary 4.4] or [2, Theorem 2] there is a bijection between the conjugacy classes of complements of N in G and the conjugacy classes of complements of N/K in $N_G(K)/K$.

Let Y be a complement of N/K in $N_G(K)/K$. The number of complements of N/K in $N_G(K)/K$ equals the cardinality of the set $\mathrm{Der}(Y, N/K)$ of derivations from Y to N/K. Since $\delta \in \mathrm{Der}(Y, N/K)$ is uniquely determined by the images $y_1^\delta, \dots, y_m^\delta$ of a set of generators of Y, we derive that $|\mathrm{Der}(Y, N/K)| \le |N/K|^{d(Y)} = |S|^{d(Y)}$. As Y is isomorphic to a subgroup of $\mathrm{Sym}(n-1)$ and $n-1 \ne 3$, we have that $d(Y) \le (n-1)/2$, hence $|\mathrm{Der}(Y, N/K)| \le |S|^{(n-1)/2}$. Finally, as every complement X of N in G has index $|N|$, there are at most $|N|$ conjugates of X in G. Therefore,

$$\kappa \le |S|^{(n-1)/2}|S|^n = |S|^{(3n-1)/2}.$$

Let consider that case where H is a p-group. Then $n = p^t$ and Y is a permutation p-group of degree at most $p^t - 1$. Therefore Y is a permutation group of degree at most $p^t - p$ and so, by [8], $d(Y) \le (p^t - p)/p = p^{t-1} - 1$. Arguing as above we deduce that

$$\kappa \le |S|^{n+n/p-1},$$

as desired. □

Proposition 14. *Let H be a transitive subgroup of $\mathrm{Sym}(n)$, S a finite non-abelian simple group, G the wreath product $S \wr H$ and $N \cong S^n$ the socle of G. If (N, H) satisfies the strong complement property, then $d(H) \le 4$. Moreover, if $n \ge 12$ then $d(H) \le 3$ and if H is a p-group for a prime $p > 3$, then H is cyclic.*

Proof. Let $d = d(H)$. Recall that a permutation group of degree n can be generated by at most $[n/2]$ elements if $n \ne 3$, and at most 2 elements if $n = 3$. So, if $n \le 8$, then $d \le 4$.

If $n \geq 8$, then by Lemma 13 $\kappa \leq |S|^{3n/2-1/2}$. From Proposition 8 it follows that

$$d(2n-2) \leq (d+1)\left(\frac{3n}{2} - \frac{1}{2}\right)$$

i.e.

$$nd \leq 3d + 3n - 1. \tag{6.1}$$

Since $d \leq n/2$, we deduce

$$nd \leq \frac{3n}{2} + 3n - 1 = \frac{9n}{2} - 1$$

an consequently $d \leq 4$.

In particular, if $n \geq 12$ then $8/(n-3) < 1$ and by (6.1) we deduce that

$$d \leq \frac{3n-1}{n-3} = 3 + \frac{8}{n-3} < 4.$$

Suppose now that H is a non cyclic p-group where $p > 3$. In this case $n = p^t$ for some $t > 1$. Then by Lemma 13 $\kappa \leq |S|^{n+n/p-1}$, and by Proposition 8 we deduce

$$d(2n-2) \leq (n + n/p - 1)(d+1)$$

hence

$$\frac{d}{d+1} \leq \frac{p^t + p^{t-1} - 1}{2p^t - 2}.$$

For $p > 3$, we have

$$\frac{p^t + p^{t-1} - 1}{2p^t - 2} < \frac{2}{3}.$$

But $d/(d+1) < 2/3$ would imply $d = 1$, against the assumption that H is not cyclic. □

We now give an example of a semidirect product NH where both N and H are elementary abelian p-groups with H of an arbitrary large rank $r \geq 2$ acting faithfully on N and where (N, H) satisfies the strong complement property. This is in sharp contrast with the coprime case where (N, H) can only satisfy the strong complement property if we either have a direct product or H is cyclic.

Example. Let r be an integer where $r \geq 2$ and p a prime. Let $N = \mathbb{F}v_1 + \cdots + \mathbb{F}v_r + \mathbb{F}w$ be a vector space of dimension $r+1$ over $\mathbb{F} = \mathrm{GF}(p)$. Let $H = \langle a_1, \ldots, a_r \rangle \leq \mathrm{GL}(N)$ where $v_i^{a_i} = v_i + w$ and a_i acts trivially on w and v_j when $j \neq i$. Then H is an elementary abelian p-group acting faithfully on N. We claim that (N, H) satisfies the strong complement property. Let $h_1, \ldots, h_r$ be any generators of H and let $x_1, \ldots, x_r \in N$. Then $K = \langle h_1^{x_1}, \ldots, h_r^{x_r} \rangle \leq H\langle w \rangle$. Thus K is also elementary abelian of rank r. Notice also that $K \cap N \leq H\langle w \rangle \cap N = \langle w \rangle$. Thus if K was not a complement of N, then $K \cap N = \langle w \rangle$ that would imply that $K = K\langle w \rangle = H\langle w \rangle$ giving the contradiction that $|K| = p^{r+1}$. It follows that K is a complement of N and thus (N, H) satisfies the strong complement property.

The next example is again a semidirect product NH where both N and H are p-groups and (N, H) satisfies the strong complement property.

Example. Let $N = \langle a, b \rangle$ be a p-group of rank 2 and class 2 with $[a, b] = c$ and $a^p = b^p = c^p = 1$. Let $P = \langle C, A \rangle$ be the elementary abelian p-group of order p^2 where C acts trivially on N and A is the inner automorphism induced by conjugation by a. We claim that (N,P) satisfies the strong complement property. Let A^rC^s, A^lC^m be any generators of P and let x,y be any elements of N. Notice that the two conjugates $(A^rC^s)^x$ and $(A^lC^m)^y$ are in $\langle A, C, c\rangle$ and thus commute (c and C are in $Z(PN)$). Hence $Q = \langle (A^rC^s)^x, (A^lC^m)^y\rangle$ is also an elementary abelian p-group of order p^2. Notice also that $Q \cap N \leq \langle c\rangle$. Thus if (N,P) does not satisfy the strong complement property we would need $Q \cap N = \langle c\rangle$ for some Q. But then $Q \geq \langle c\rangle$ and then $Q = \langle A, C, c\rangle$ contradicting the previous observation that the order of Q is p^2. Thus (N,P) satisfies the strong complement property. Notice however that P does not act trivially on N as $b^A = bc^{-1}$.

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