On the Hirsch-Plotkin radical of stability groups

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Abstract

We study the stability group of subspace series of infinite dimensional vector spaces. In [1], the authors proved that when the vector space has countable dimension then the Hirsch-Plotkin radical of the stability group coincides with the set of all space automorphisms that fix a finite subseries and they conjectured that this would hold in all dimensions. We give a counter example of dimension $2^{\aleph_0}$. We however show that in general the result remains true if the Hirsch-Plotkin radical is replaced by the Fitting group, the product of all the normal nilpotent subgroups of the stability group. We also show that the Hirsch-Plotkin radical has a certain strong local nilpotence property.

0 Introduction

We start with a precise definition, taken from [1], of what we will mean generally by series of subspaces in a given vector space V.

Definition. Let V be a vector space over a field $\mathbb{F}$. A set $\mathcal{L}$, consisting of subspaces of V, is said to be a *series* in V if

(1) Both 0 and V belong to $\mathcal{L}$.

(2) The set $\mathcal{L}$ is linearly ordered with respect to inclusion.
(3) For every $\mathcal{F} \subseteq \mathcal{L}$ both $\bigcap\{W : W \in \mathcal{F}\}$ and $\bigcup\{W : W \in \mathcal{F}\}$ are in $\mathcal{L}$.

The definition thus follows the Kurosh notion of a series rather than the version of P. Hall (see for example pages 9-10 in [6] for a discussion of these). For our purposes it doesn't really matter which definition one uses and the main results of the paper are true for both versions.

A *subseries* of $\mathcal{L}$ is any subset of $\mathcal{L}$ which is a series. In particular every finite subset of $\mathcal{L}$ containing 0 and V is a subseries.

Definition. Let $\mathcal{L}$ be a series in a vector space V. A *jump* of $\mathcal{L}$ is an ordered pair (B, T) of elements $B, T \in \mathcal{L}$ such that $B < T$ and there is no $U \in \mathcal{L}$ where $B < U < T$.

Now let $0 \neq v \in V$ and let

$$T = \bigcap\{U \in \mathcal{L} : v \in U\}, \quad B = \bigcup\{U \in \mathcal{L} : v \notin U\}.$$

Notice that (B, T) is a jump and $v \in T \setminus B$. Now let $\mathcal{J}$ be the collection of all the jumps of $\mathcal{L}$. It follows from the discussion above that

$$V \setminus 0 = \bigcup_{(B,T)\in\mathcal{J}} T \setminus B$$

where the union is a disjoint union.

Definition. Let $\mathcal{L}$ be a series in a vector space V. We say that an element $g \in \mathrm{GL}(V)$ stabilises $\mathcal{L}$ if $T(g-1) \leq B$ for all jumps (B, T) of $\mathcal{L}$.

To a given series $\mathcal{L}$, there are certain associated subgroups of $\mathrm{GL}(V)$ that will play a role later on. We let $S(\mathcal{L})$ be the subgroup consisting of all the elements in $\mathrm{GL}(V)$ that stabilise $\mathcal{L}$. We let $\mathrm{HP}(\mathcal{L})$ be the Hirsch-Plotkin radical of $S(\mathcal{L})$ and $\mathrm{Fit}(\mathcal{L})$ be the Fitting subgroup of $S(\mathcal{L})$, that is the product of all the normal nilpotent subgroups of $S(\mathcal{L})$. Finally we let $F(\mathcal{L})$ be the normal subgroup of $S(\mathcal{L})$ consisting of the elements that stabilise some finite subseries of $\mathcal{L}$.

It is well known, and not difficult to see, that any subgroup G of the group of linear automorphisms GL(V) that stabilises a finite series is nilpotent. Here the elements of G act unipotently on V, that is $(g-1)$ is a nilpotent linear operator for all $g \in G$. For finite dimensional vector spaces the converse is true (see for example 8.1.10 in [5]) and thus any subgroup of GL(V), consisting of elements that act unipotently on V, stabilises a finite series of subspaces. The situation is very different for infinite dimensional vector spaces and is studied in [1]. To start with it is easy to come up with examples of nilpotent subgroups of GL(V), where V has countably infinite dimension, that stabilise some series of V but stabilise no finite series of subpaces.

Example. Let V be a vector space with basis v_i, $i \in \mathbb{Z}$ and let $V_i = \langle v_j : j \leq i\rangle$. Let

$$\mathcal{L} = \{V_i : i \in \mathbb{Z}\} \cup \{0, V\}.$$

The linear automorphism g that maps v_i to $v_i + v_{i-1}$ for $i \in \mathbb{Z}$ is in $S(\mathcal{L})$ but as it is not unipotent, it can't stabilise any finite series in V. Thus we have a cyclic subgroup, $\langle g\rangle$, of a series stabiliser that stabilises no finite series.

Remark. In [1] the authors however show that all the elements of HP($\mathcal{L}$) are unipotent and thus stabilise a finite subseries. The proof of this is quite tricky and delicate. Using this result and a classic result of Gruenberg [3], that tells us that any finitely generated Engel group is nilpotent, it is not difficult to obtain the following generalisation: if H is a finitely generated subgroup of HP($\mathcal{L}$), then there exists a positive integer m such that $[V,_m H] = 0$. In particular H stabilises the finite series $\mathcal{S} = \{[V,_k H] :\ 0 \leq k \leq m\}$. We will do this in Section 2.

In [1] there is also an example given of a group acting unipotently on a vector space that doesn't stabilise any series of subspaces. Let G be a Tarski p-group and let $\mathbb{F}$ be any field of characteristic p. Then G acts unipotently on $\mathbb{F}G$. It cannot however stabilise a series in V. To see this one argues by contradiction and assumes that G stabilises some series in V. According to a result of [4], G itself then has a series with abelian factors. This is however clearly not the case.

In fact the structure of the group $S(\mathcal{L})$ is in general not much restricted and the authors of [1] demonstrate that every countable free group faithfully stabilises an ascending series in a suitable vector space.

In view of the remark above however, $\mathrm{HP}(\mathcal{L})$ has a restricted structure as we have mentioned that its elements act unipotently on V. As a result of their study, the authors of [1] came up with the conjecture that $F(\mathcal{L}) = \mathrm{HP}(\mathcal{L})$ in general. They proved this conjecture for any vector space of countable dimension as well as for few other special situations. Quite surprisingly, it turns out that the conjecture is however not true in general and in Section 2 we show that there exists a vector space V of dimension $2^{\aleph_0}$ with a series $\mathcal{L}$ such that $F(\mathcal{L}) < \mathrm{HP}(\mathcal{L})$. We will however show in Section 1 that in general the weaker equality $F(\mathcal{L}) = \mathrm{Fit}(\mathcal{L})$ holds.

1 A characterisation of elements that stabilise a finite subseries

We start this section by introducing an associated 'quotient series' to a given pair$(\mathcal{L}, U)$, where $\mathcal{L}$ is a series of a vector space V and $U \in \mathcal{L}$. There is a natural associated series of subspaces in V/U, namely

$$\mathcal{L}/U = \{W + U/U : W \in \mathcal{L}\}.$$

There is a another associated series that can be defined with respect to any element U of $\mathcal{L}$. This is the series

$$\mathcal{L} \cap U : \{W \in \mathcal{L} : W \leq U\}$$

that is a series in U. We will need both these terms later on.

It is clear that if g stabilises a finite subseries of $\mathcal{L}$ then $\langle g \rangle^{S(\mathcal{L})}$ is nilpotent. We will now see that the converse holds. In other words we will show that $\mathrm{F}(\mathcal{L}) = \mathrm{Fit}(\mathcal{L})$. Let $E = S(\mathcal{L})$ and let g be an element in E where $\langle g \rangle^E$ is nilpotent. We will argue by contradiction and and assume that g stabilises no finite subseries and show that $\langle g \rangle^E$ is not nilpotent. Notice that if $U \in \mathcal{L}$ then $S(\mathcal{L}/U) = E/N$ where N is the normal subgroup consisting of all the elements h in E where $V(h-1) \leq U$. If $\langle gN \rangle^{E/N}$ is not nilpotent then clearly the same is true for $\langle g \rangle^E$. Also if $\mathcal{S}$ is some subseries of $\mathcal{L}$ that is stabilised by g, then $S(\mathcal{S}) \trianglelefteq E$ and again we have that non-nilpotence of $\langle g \rangle^{S(\mathcal{S})}$ implies non-nilpotence of $\langle g \rangle^E$. In view of this we will first reduce our problem to a specific series where we can more easily obtain a contradiction. We need the following lemma whose proof is derived from arguments uses in the proofs of Theorem 3.9 and Theorem 3.10 from [1].

Lemma 1.1 *Let $\mathcal{L}$ be a series in a vector space $V \neq 0$ and let $g \in HP(\mathcal{L})$. Then there exists $V \neq W \in \mathcal{L}$ such that $[V, g] \leq W$.*

Proof We argue by contradiction and suppose that there is no such $W \in \mathcal{L}$. This is saying that there is no jump of the form (W, V). As $g \in S(\mathcal{L})$ this implies that

$$V/U = \bigcup_{U \leq W < V} W/U$$

for all $U \in \mathcal{L}$ such that $U < V$. In particular, taking $U = W_0 = 0$, we see that there is $W_0 < W_1 < W$ such that $[W_1, g] \neq 0$. Repeating this argument we get $W_1 < W_2 < V$ such that $[W_2, g] \not\leq W_1$ and continuing in this manner we get a strictly ascending sequence of subspaces

$$0 = W_0 < W_1 < W_2 < \dots,$$

in $\mathcal{L}$ such that $[W_{i+1}, g] \not\leq W_i$ for $i \geq 0$. Notice that this implies that g does not stabilise any subseries of $\mathcal{L} \cap W_n$ of length less than or equal to n. The same argument works for the series $\mathcal{L}/U$ for any $U \in \mathcal{L}$ such that $U < V$. We can thus come of with a strictly ascending chain of subspaces

$$0 = U_0 < U_1 < \dots,$$

in $\mathcal{L}$ such that g stabilises no subseries of $\mathcal{L} \cap U_n/U_{n-1}$ that is of length less than or equal to n. Let M be a complement to $\bigcup_i U_i$. Let A_i be a complement to U_{i-1} in U_i for $i \geq 1$ and for each i fix an isomorphism $\sigma_i : U_i/U_{i-1} \to A_i$. These isomorphisms can be used to define an action of g on each A_i. Moreover we can use the isomorphism σ_i to transfer the series $\mathcal{L} \cap U_i/U_{i-1}$ into an equivalent series $\mathcal{S}_i$ in A_i. As $g \in \mathrm{HP}(\mathcal{L})$ we know from the introduction that g acts unipotently on V. Suppose $(g-1)^k = 0$. By Lemma 3.8 from [1], there exists for each $i > k+2$, a $h_i \in S(\mathcal{S}_i)$ such that $(gg^{h_i} - 1)^{[(i-2)/k]-1} \neq 0$ on A_i. We define an automorphism h on $V = M \oplus \bigoplus_{i>0} A_i$ by setting $h = 1$ on $M \oplus \bigoplus_{i \leq k+2} A_i$ and $h = h_i$ on A_i when $i > k+2$. Now for each positive integer m there is an i such that $(gg^h - 1)^m$ acts non-trivially on A_i. Hence it follows that gg^h does not act unipotently on V. But this contradicts the fact that all elements in $\mathrm{HP}(\mathcal{L})$ act unipotently on V. $\square$

For an element $g \in \mathrm{Fit}\,(\mathcal{L})$ we say that g has unipotence degree n if $[v,_n g] = 0$ for all $v \in V$. Suppose that we have an element $g \in \mathrm{Fit}\,(\mathcal{L})$ that does not stabilise any finite subseries of $\mathcal{L}$ and pick g, V and $\mathcal{L}$ such that the unipotence degree n of g is the smallest possible. Notice that $n \geq 2$.

We let $V_0 = V$ and let

$$V_1 = \bigcap \{U \in \mathcal{L} : \, [V_0, g] \leq U\}.$$

By Lemma 1.1 we know that $V_1 < V_0$. As g stabilises no finite subseries, we also have that $V_1 > 0$. Let g_1 be the restriction of g on V_1. Then it is not difficult to see that g_1 is in $\mathrm{HP}(\mathcal{L} \cap V_1)$ and clearly g_1 cannot stabilise any finite subseries of $\mathcal{L} \cap V_1$. We can thus apply Lemma 1.1 again. In this manner we obtain recursively a strictly descending sequence of subspaces such that

$$V_{i+1} = \bigcap \{U \in \mathcal{L} : \, [V_i, g] \leq U\}.$$

Let $U = \bigcap_{n \in \mathbb{N}} V_n$ and consider the subseries

$$\{V_i/U : \, i \in \mathbb{N}\} \cup \{0\}$$

of $\mathcal{L}/U$. If $S(\mathcal{L}/U) = E/N$, then gN stabilises the new series.

Theorem 1.2 *Let $\mathcal{L}$ be any series. We then have that $F(\mathcal{L}) = Fit(\mathcal{L})$.*

Proof In view of what we said at the beginning of this section, we can assume without loss of generality that

$$\mathcal{L} = \{V_i : i \in \mathbb{N}\} \cup \{0\}$$

where $V_0 > V_1 > \ldots$ and $\bigcap_{i \in \mathbb{N}} V_i = 0$. Notice that we can't have $[V_i,_{n-1} g] = 0$ for any $i \in \mathbb{N}$ since otherwise, by our choice of n, there would be a finite subseries in V_i of $\{V_j : \, j \geq i\} \cup \{0\}$ that is stabilised by g but this is not the case. We now construct a sequence of elements $v_1, v_2, \ldots$ and a descending chain $V = A_0 > A_1 > A_2 > \ldots$, with $A_1, A_2, \ldots \in \mathcal{L}$, recursively such that

$$[v_i,_{n-2} g] \in A_{2i-1} \setminus A_{2i}, \;\; [v_i,_{n-1} g] \in A_{2i} \setminus A_{2i+1}$$

for $i = 1, 2, \ldots$. To start with we let $v_1 \in V$ such that $[v_1,_{n-1} g] \neq 0$. Let

$$A_1 = \bigcap \{V_i \in \mathcal{L} : \, [v_1,_{n-2} g] \in V_i\}$$

and then

$$A_2 = \bigcap\{V_i \in \mathcal{L} : [v_{1,n-1}\, g] \in V_i\}.$$

Next pick $v_2 \in A_2$ where $[v_{2,n-1}\, g] \neq 0$ and repeat the process by letting

$$A_3 = \bigcap\{V_i \in \mathcal{L} : [v_{2,n-2}\, g] \in V_i\}$$

and then

$$A_4 = \bigcap\{V_i \in \mathcal{L} : [v_{2,n-1}\, g] \in V_i\}.$$

Continuing like this gives us the sequences of elements and subspaces we wanted.

Pick now an integer $k = 1 + 2^m$ where m is arbitrarily large. Let

$$W = W_0 \oplus W_1 \oplus \cdots \oplus W_k$$

be a complement to A_{k+1} such that $A_i = W_i \oplus \cdots \oplus W_k \oplus A_{k+1}$. We pick a basis β_i for W_i such that $[v_{i,n-2}\, g] \in \beta_{2i-1}$ and $[v_{i,n-1}\, g] \in \beta_{2i}$. We let $z : V \to V$ be the endomorphism that maps A_{k+1} to 0 and all the basis elements in $\beta_0 \cup \ldots \cup \beta_k$ also to 0 apart from

$$[v_{i,n-1}\, g]z = [v_{i+1,n-2}\, g]$$

for $i = 1, \ldots, k$. Notice that $V_j z \leq V_{j+1}$ for all $j \in \mathbb{N}$ and thus $\phi = \phi(m) = 1 + z$ stabilises $\mathcal{L}$. Let $x = g - 1$ and let

$$W = \langle [v_{i,n-2}\, g],\ [v_{i,n-1}\, g] :\ i = 1, 2, \ldots \rangle$$

Notice that W is z and x invariant and that the restrictions of z and x to W satisfy $x^2 = z^2 = 0$ (we will be using the same notation for these as we will be from now on working in W only). Let $\phi_r = [\phi,_r\, g]$. To finish the proof of the theorem we first prove a lemma.

Lemma 1.3 *We have*

$$x\phi_r^{-1}x = -x\phi_r x$$

and

$$x\phi_{r+1}x = x\phi_r x\phi_r x$$

for all r.

Proof Notice first that

$$x\phi_0^{-1}x = x(1-z)x = -xzx = -x(1+z)x = -x\phi_0 x$$

For $r \geq 0$ we then have

$$\begin{aligned} x\phi_{r+1}x &= x[\phi_r, g]x \\ &= x\phi_r^{-1}(1-x)\phi_r(1+x)x \\ &= -x\phi_r^{-1}x\phi_r x \\ &= -x(1-x)\phi_r^{-1}(1+x)\phi_r x \\ &= -x[g, \phi_r]x \\ &= -x\phi_{r+1}^{-1}x. \end{aligned}$$

This proves the first claim. We use this to prove the second claim. We saw from the calculations above that

$$x\phi_{r+1}x = -x\phi_r^{-1}x\phi_r x$$

and by the first claim this is equal to $x\phi_r x\phi_r x$. □

We can now finish the proof of Theorem 1.2. It follows from the previous lemma by induction that

$$x\phi_r x = (x\phi_0)^{2^r} x = (x(1+z))^{2^r} x = (xz)^{2^r} x$$

for all r. Now

$$[v_{1\,,n-2}\, g](xz)^{2^m} x = [v_{1+2^m\,,n-1}\, g] \neq 0.$$

This shows that

$$0 \neq x\phi_m x = x[\phi_{,m}\, g]x$$

and hence $[\phi_{,m}\, g] \neq 1$. For a given $k = 1 + 2^m$ we can thus find $\phi(m) \in E$ such that $\langle g, g^{\phi(m)}\rangle$ is nilpotent of class at least $m-1$. Thus

$$\langle g, g^{\phi(1)}, g^{\phi(2)}, \ldots \rangle$$

is not nilpotent that gives the contradiction that $\langle g\rangle^E$ is not nilpotent. This finishes the proof of Theorem 1.1. □

2 Counterexamples for spaces of uncountable dimension

When V is a vector space of countable dimension the authors in [1] proved the stronger result that $\mathrm{F}(\mathcal{L}) = \mathrm{HP}(\mathcal{L})$. They then made the conjecture that this result holds for all vector spaces. In this section we give a general example that shows that the conjecture does not hold in any vector space of dimension greater than or equal to $2^{\aleph_0}$. Let F be any field. Let $U = \bigoplus_{i=0}^{\infty} Fu_i$ be a direct sum of countably many 1-dimensional vector spaces over F and let $W = \prod_{i=0}^{\infty} Fw_j$ be a cartesian sum of countably many 1-dimensional vector spaces over F. Then let

$$V = U \oplus W.$$

Remark. The dimension of V is the cardinality of W (for a proof see Lemma 1 of [2]). In particular if F is one of the prime fields $\mathbb{Z}_p$ or $\mathbb{Q}$ then the dimension of V is $2^{\aleph_0}$.

Now let $V_n = \bigoplus_{i=n}^{\infty} Fu_i + \prod_{j=n}^{\infty} Fw_j$. We consider then the series

$$\mathcal{L} = \{V_i : i \in \mathbb{N}\} \cup \{0\}.$$

Let R be the subring of $\mathrm{End}(V)$ consisting of those endomorphisms that map V_i into V_{i+1} for all $i \in \mathbb{N}$. Then $G = 1 + R$ is the stabiliser of $\mathcal{L}$.

Let $x : V \to V$ be the endomorphism that maps u_i to w_{i+1} for $i \in \mathbb{N}$ and maps W to zero. Notice that $g = 1 + x$ is in G and that g stabilises no finite subseries of $\mathcal{L}$. It remains to see that $g \in \mathrm{HP}(G)$. The following lemma will be a key result in establishing this.

Lemma 2.1 *Let $y \in R$. There exists a positive integer m such that*

$$w_j y \in W$$

for all $j \geq m$.

Proof We argue by contradiction and suppose that there is no such bound m. Pick some element w_{k_1} such that $w_{k_1} y \notin W$, say

$$w_{k_1} y = \alpha_{k_1+1}^{1} u_{k_1+1} + \cdots + \alpha_{k_1+m_1}^{m_1} u_{k_1+m_1} + w_1$$

where $\alpha^{m_1}_{k_1+m_1} \neq 0$ and $w_1 \in W \cap V_{k_1+1}$. Now pick $k_2 > k_1 + m_1$ such that $w_{k_2} y \notin W$, say

$$w_{k_2} y = \alpha^1_{k_2+1} u_{k_2+1} + \cdots + \alpha^{m_2}_{k_2+m_2} u_{k_2+m_2} + w_2$$

where $\alpha^{m_2}_{k_2+m_2} \neq 0$ and $w_2 \in W \cap V_{k_2+1}$. Continuing in this manner we get a strictly ascending sequence of positive integers

$$k_1 < k_1 + m_1 < k_2 < k_2 + m_2 < \ldots < k_j < k_j + m_j < \ldots$$

and a sequence of elements $w_1, w_2, \ldots$ with $w_j \in W \cap V_{k_j+1}$ and where

$$w_{k_j} y = \alpha^1_{k_j+1} u_{k_j+1} + \ldots + \alpha^{m_j}_{k_j+m_j} u_{k_j+m_j} + w_j$$

with $\alpha^{m_j}_{k_j+m_j} \neq 0$. Now let $u \in U$ and $w \in W$ be such that

$$(w_{k_1} + w_{k_2} + \cdots) y = u + w.$$

Suppose that $u \in Fu_1 + \cdots + Fu_{k_j+m_j-1}$. By our construction above we must have that, modulo $\bigoplus_{i=k_j+m_j+1}^{\infty} Fu_i$,

$$\alpha^1_{k_1+1} u_{k_1+1} + \cdots + \alpha^{m_1}_{k_1+m_1} u_{k_1+m_1} + \cdots + \alpha^1_{k_j+1} u_{k_j+1} \cdots + \alpha^{m_j}_{k_j+m_j} u_{k_j+m_2} = u.$$

But this is absurd as $\alpha^{m_j}_{k_j+m_j} \neq 0$. $\square$

Corollary 2.2 *Let $y \in R$. There exists a positive integer m (such that*

$$wy \in W$$

for all $w \in W \cap V_m$.

Proof Let m be as in Lemma 2.1. We argue by contradiction and suppose that we have (for some $\alpha_j \in F$, $j = m, \ldots$)

$$(\alpha_m w_m + \alpha_{m+1} w_{m+1} + \cdots) y = u + w$$

for some $u \in U$ and $w \in W$ where $u \neq 0$. Let $k > m$ be a large enough positive integer such that $u \notin V_k$. Calculating modulo V_k we get

$$(\alpha_m w_m + \cdots + \alpha_{k-1} w_{k-1} + V_k) y = u + w + V_k$$

where $u \notin W + V_k$. It follows that $(\alpha_m w_m + \cdots + \alpha_{k-1} w_{k-1})y \notin W$ that contradicts Lemma 2.1. $\square$

We want to use these results to show that $\langle g \rangle^G$ is locally nilpotent. Let $S = Fx + Rx + xR + RxR$ be the ideal in R generated by x. Notice that

$$\langle g \rangle^G \leq 1 + S.$$

It thus suffices to show that $1 + S$ is locally nilpotent. Take some finitely many elements $x_1, \ldots, x_n$ from $1 + S$, where

$$x_i = 1 + f_i x + r_i x + x s_i + t_i x l_i$$

with $r_i, s_i, t_i, l_i \in R$ and $f_i \in F$. By Corollary 2.2, we know that there exists a positive integer m such that r_i, s_i, t_i, l_i leave $V_m \cap W$ invariant for $i = 1, \ldots, n$. Let T be the subring of R generated by $x, r_i x, x s_i, t_i x l_i$, $i = 1, \ldots, n$. To show that $\langle x_1, \ldots, x_n \rangle$ is nilpotent, it suffices to show that T is nilpotent as a ring. In fact we show that $T^{m+2} = 0$. First notice that $VT^m \leq V_m$. Now x maps V_m into $V_m \cap W$ and, by our choice of m, all the r_i, s_i, t_i, l_i leave $V_m \cap W$ invariant. It follows that $VT^{m+1} \leq V_m \cap W$. Now x maps $V_m \cap W$ to zero and thus, by our choice of m again, we see that $VT^{m+2} = 0$. This shows that T is nilpotent. It follows that $1 + T$ is nilpotent and thus $1 + S$ is locally nilpotent. Hence $\langle g \rangle^G$ is locally nilpotent.

Remark. When the underlying field F is one of the prime fields, we have an example of cardinality $2^{\aleph_0}$. Our example does however not address possible cardinals between $\aleph_0$ and $2^{\aleph_0}$. Thus we do not know what happens in this case in set theories where the continuum hypothesis doesn't hold.

Although we do not have in general that $F(\mathcal{L}) = \mathrm{HP}(\mathcal{L})$, the Hirsch-Plotkin radical of $S(\mathcal{L})$ satisfies a strong local nilpotence property. As we mentioned in the introduction, all the elements of $\mathrm{HP}(\mathcal{L})$ act nilpotently on V through the commutator action. Thus for each $h \in \mathrm{HP}(\mathcal{L})$ there exists a positive integer $n = n(h)$ such that $[v,_n h] = 0$ for all $v \in V$. From this one can easily derive the following generalisation.

Theorem 2.3 *For each finitely generated subgroup $H = \langle h_1, \ldots, h_n \rangle$ of $HP(\mathcal{L})$, there exists a positive integer $d = d(H)$ such that $[V,_d H] = 0$.*

Proof We know from [1] that for each word $h = w(h_1, \dots, h_n)$ in $h_1, \dots, h_n$ there exist a positive integer $n = n(h)$ such that $[v,_n h] = 0$ for all $v \in V$. Suppose that H is nilpotent of class c and let $x \in VH$ and $y = vh$ for some $v \in V$. Then $[x,_c y] \in V$ and thus $[x,_{c+n(h)} y] = 0$. This shows that for any $v \in V$ the subgroup $\langle v, H\rangle$ is a finitely generated Engel group. Furthermore if we let $F = \langle x, x_1, \dots, x_n\rangle$ be the largest group with $\langle x\rangle^F$ abelian, where $F/\langle x\rangle^F$ is isomorphic to H and where F satisfies the extra relations $[x,_{n(w(h_1,\dots,h_n))} y] = 0$ for all words $y = w(x_1, \dots, x_n)$ in $x_1, \dots, x_n$, then F is a finitely generated Engel group and thus nilpotent by a classic result of Gruenberg [3]. Suppose that the class of F is d. Then any $\langle v, H\rangle$ is a homomorphic image of F and thus nilpotent of class at most d for all $v \in V$. Hence $[v,_d H] = 0$ for all $v \in V$. $\Box$

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