A note on supersoluble Fitting classes

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Abstract

In this paper we give an elementary construction of a non-nilpotent supersoluble Fitting class in which every group is an extension of a p-group, where p is an arbitrary prime greater than or equal to 5, by a 2-group.

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1 Introduction

A class $\mathfrak{F}$ of groups is a Fitting class if it has the following two properties:

1. If $G \in \mathfrak{F}$ and $H \trianglelefteq G$ then $H \in \mathfrak{F}$.
2. If $H, K \in \mathfrak{F}$, $H, K \trianglelefteq G$ and $G = HK$ then $G \in \mathfrak{F}$.

It is not difficult to see that Fitting classes are closed with respect to forming subnormal products. For a given group G we will denote by $\mathfrak{F}(G)$, the Fitting class generated by G. That is, the smallest Fitting class that contains G. It is easy to determine $\mathfrak{F}(G)$ in the case when G is either nilpotent or simple, but in other cases the problem seems to be quite difficult. While Fitting classes of nilpotent groups are fully understood the same is not true for metanilpotent groups and even the problem of determining $\mathfrak{F}(S_3)$ still remains unsolved. In recent years there has been much work in this area (see [1], [2]-[6] for example).

In this paper we will give an elementary construction of a supersoluble Fitting class $\mathfrak{F}(G)$. Since supersoluble groups are metanilpotent our example is therefore an example of a metanilpotent Fitting class. Notice however that the class of all supersoluble groups is not a Fitting class. Because of this and since we want to be able to compute the class $\mathfrak{F}(G)$ explicitly, we have to be careful about the choice of the generating group G.

In [6] Menth constructed a family of examples of supersoluble groups in which every group is an extension of a p-group by a 3-group where p is a prime different from 3. His construction can be generalized to include examples of supersoluble Fitting classes in which every group is an extension of a p-group by a q-group for other odd primes q. This leaves out however the case when $q = 2$. In this note we will deal with this exceptional case. For each prime p greater than or equal to 5, we will construct an example of a supersoluble Fitting class, in which every group is an extension of a p-group by a 2-group. Like in Menth's examples our construction can be described in terms of a more general pattern, the Fitting classes of Dark type (see [2]).

2 The Fitting class

Let p be a prime number such that $p \geq 5$. We define groups T and E as follows

$$\begin{aligned} T &= \langle a, b, c : a^p = b^p = c^p = [a,b,b] = [b,a,a] = [b,c,c] = [c,b,b] = \\ &\quad [a,c,c] = [c,a,a] = 1, [w_1, \ldots, w_5] = 1 \text{ when } w_1, \ldots, w_5 \in \{a,b,c\}\rangle; \end{aligned}$$

$$E = \langle T, x : x^2 = 1, a^x = a^{-1}, b^x = b^{-1}, c^x = c^{-1}\rangle.$$

It is clear that E is supersoluble with $Z(E) = Z(T) = \gamma_4(T)$. Since the nilpotency class of T is less than p, we also have that T has exponent p. Furthermore we have that the order of T is p^{11}. We want to determine the Fitting class genertated by E. We will first determine all the p-perfect groups. Before we descripe the class of all the p-perfect groups, we will derive some helpful properties of the group T.

Definition 1 *We say that $\{u, v, w\} \subseteq T$ is a good set of generators, if u, v, w generate T and every commutator of length 3 in u, v, w with an element repeated is the identity..*

Lemma 1 *(a) If $u, v \in T$ and $[u, v] \in \gamma_3(T)$, then u and v are dependent modulo $\gamma_2(T)$.*

(b) Suppose that $\{u, v, w\}$ is a good set of generators for T. Then $\{\langle u\rangle T', \langle v\rangle T', \langle w\rangle T'\} = \{\langle a\rangle T', \langle b\rangle T', \langle c\rangle T'\}$.

Proof (a) Suppose $uT' = a^i b^j c^k T'$ and $vT' = a^r b^s c^t T'$. Modulo $\gamma_3(T)$, we have

$$1 = [u, v] = [a, b]^{is-jr}[b, c]^{jt-ks}[c, a]^{kr-it}.$$

Since $\gamma_2(T)/\gamma_3(T)$ is a vectorspace with basis $[a, b]\gamma_3(T)$, $[b, c]\gamma_3(T)$ and $[c, a]\gamma_3(T)$, we must have that either $(r, s, t) = (0, 0, 0)$ or (i, j, k) is a multiple of (r, s, t).

(b) Suppose $uT' = a^i b^j c^k T'$, $vT' = a^r b^s c^t T'$ and $wT' = a^\alpha b^\beta c^\gamma T'$. We show that each of the triples (i, j, k), (r, s, t), (α, β, γ) has two entries that are zero. We do this by showing that $rs = st = tr = ij = jk = ki = \alpha\beta = \beta\gamma = \gamma\alpha = 0$. Suppose one of these were nonzero. Without loss of generality, we can assume that $rs \neq 0$. We will show that this leads to the contradiction that u,v and w are dependent modulo T'. We calculate modulo $\gamma_4(T)$. Using $[c, a, b] = [a, b, c]^{-1}[b, c, a]^{-1}$ we get that

$$1 = [u, v, v] = [a, b, c]^{2ist-jrt-krs}[b, c, a]^{jtr+its-2ksr}.$$

Since $[a, b, c]$ and $[b, c, a]$ are independent modulo $\gamma_4(T)$, it follows that

$$ist = jrt = krs.$$

If $t \neq 0$ we get $i/r = j/s = k/t$ and u is a power of v modulo T'. So we can assume that $t = 0$. We then have $k = 0$ and $\{u, v\} \subseteq \langle a, b\rangle T'$. Similarly $\{w, v\} \subseteq \langle a, b\rangle T'$ and therefore u, v, w are dependent modulo T'. □

Lemma 2 *Suppose $A \leq Aut(T)$ is a 2-group such that each $y \in A$ either inverts or centralizes T/T'. Then there is a good set of generators $\{a', b', c'\}$ for T such that each $y \in A$ either inverts or centralizes (a', b', c').*

Proof We let $S = \{(au, bv, cw) : u, v, w \in T'$ and $\{au, bv, cw\}$ is a good set of generators for $T\}$. We first show that the order of S is a power of p. Suppose

$(au, bv, cw) \in S$ and that modulo $\gamma_3(T)$

$$\begin{array}{rcl} au &=& a[a,b]^{r_1}[b,c]^{s_1}[c,a]^{t_1}; \\ bv &=& b[b,c]^{s_2}[c,a]^{t_2}[a,b]^{r_2}; \\ cw &=& c[c,a]^{t_3}[a,b]^{r_3}[b,c]^{s_3}. \end{array}$$

We have

$$1 = [au, bv, bv] = [b, c, a, b]^{-2t_1-2s_2}[a, b, c, a]^{t_2}.$$

By symmetry we have $t_1 + s_2 = t_2 = r_2 + t_3 = r_3 = s_3 + r_1 = s_1 = 0$ and $S \subseteq \{(a[a,b]^r[c,a]^{-s}\alpha,\ b[b,c]^s[a,b]^{-t}\beta,\ c[c,a]^t[b,c]^{-r}\gamma) : \alpha, \beta, \gamma \in \gamma_3(T)$ and $r, s, t \in \mathbb{Z}_p\}$.

But it is easy to check that the latter set is contained in S and therefore $|S|$ is a power of p. We let $\tilde{S} = \{\ \{x, x^{-1}\} : x \in S\}$. A acts on $\tilde{S}$ and since A is a 2-group, every A orbit has an order which is a power of 2. Since $|\tilde{S}|$ is odd, one orbit must have one element only. □

Lemma 3 *Suppose $s \in Aut(T)$ is a 2-element such that $a^s \in b^nT'$, $b^s \in a^mT'$ and $c^s \in c^\epsilon T'$ where $mn = 1(\mod p)$ and $\epsilon \in \{-1, 1\}$. Then there is a good set of generators $\{a_1, b_1, c_1\}$ for T such that $(a_1, b_1, c_1)^s = (b_1^n, a_1^m, c_1^\epsilon)$.*

Proof We let $S = \{(au, bv, cw) : u, v, w \in T'$ and $\{au, bv, cw\}$ is a good set of generators for $T\}$. In the proof of Lemma 2 we saw that $|S|$ is a power of p. For each $(a_1, b_1, c_1) \in S$ we define an element $\alpha(a_1, b_1, c_1) = (b_1^n, a_1^m, c_1^\epsilon)$. We let $\tilde{S} = \{\ \{x, \alpha(x)\} : x \in S\}$. We have that s acts on $\tilde{S}$ and since s is a 2-element, every s-orbit has order which is a power of 2. Since $|\tilde{S}|$ is odd, one orbit must have one element only. □

Definition 2 *We define a class $\mathfrak{F}_0$ of finite groups as follows. $G \in \mathfrak{F}_0$ if it is an extension of a p-group X by a 2-group Y and*

(i) X is a central product of groups $T_1, \ldots, T_m$ isomorphic to T and $T_i \trianglelefteq G$ for $i = 1, \ldots, m$;

(ii) for all $i \in \{1, \ldots, m\}$ we have $Y/C_Y(T_i/T_i') \cong \mathbb{Z}_2$ and the generator acts on T_i/T_i' as the inverse automorphism.

We will see later that $\mathfrak{F}_0$ is the subclass of all the p-perfect groups in $\mathfrak{F}(E)$.

Lemma 4 *Let $G = XY \in \mathfrak{F}_0$ and $C = C_Y(X)$.*
(a) G is p-perfect; (b) G is supersoluble; (c) $Fit(G) = X \times C$ and $G/Fit(G) \cong Y/C$ is an elementary abelian 2-group.

Proof (a) By definition of $\mathfrak{F}_0$, we can find $y_i \in Y$ which inverts T_i/T_i'. By Lemma 2 we can find generators a_i, b_i and c_i for T_i such that $(a_i^{y_i}, b_i^{y_i}, c_i^{y_i}) = (a_i^{-1}, b_i^{-1}, c_i^{-1})$. Then $a_i = y_i^{-1} \cdot y_i a_i$, $b_i = y_i^{-1} \cdot y_i b_i$ and $c_i = y_i^{-1} \cdot y_i c_i$, so $\langle T_i, y_i \rangle$ is generated by 2-elements. For each i we can find such y_i and G is therefore generated by 2-elements.

(b) Each $y \in Y$ either inverts or centralizes T_i/T_i', $T_i'/\gamma_3(T_i)$, $\gamma_3(T_i)/Z(T_i)$ and $Z(T_i)$ for $i = 1, \ldots, m$.

(c) Everything is clear except that Y/C is an elementary abelian 2-group. Since $Y/C_Y(T_i/T_i')$ is of order 2, we have that y^2 centralizes T_i/T_i' for all $i \in \{1, \ldots, m\}$. It follows from Lemma 2 that y^2 centralizes $T_1 \cdots T_m = X$. $\square$

We want to prove that $\mathfrak{F}_0$ is closed with respect to forming normal products. The following lemma will be useful.

Lemma 5 *Suppose $G = XY \in \mathfrak{F}_0$ and that X can be written in two ways as a central product $X = T_1 \cdots T_m = U_1 \cdots U_l$, where the T_i and U_j satisfy the conditions of the definition for $\mathfrak{F}_0$. Then $m = l$ and one can reindex the U_j such that $U_i = T_i$ for $i = 1, \ldots, m$.*

Proof We have that $X/Z(X) = T_1Z(X)/Z(X) \times \cdots \times T_mZ(X)/Z(X) = U_1Z(X)/Z(X) \times \cdots \times U_lZ(X)/Z(X)$. By considering orders we clearly have $m = l$. Furthermore, since $T/Z(T)$ is indecomposible, we have from the Krull-Remak-Schmidt theorem, that we can reindex the U_j such that

$$\begin{aligned} X/Z(X) &= T_1Z(X)/Z(X) \times \cdots \times T_{i-1}Z(X)/Z(X) \times U_iZ(X)/Z(X) \times \\ &\quad T_{i+1}Z(X)/Z(X) \times \cdots \times T_mZ(X)/Z(X) \end{aligned}$$

for all $i \in \{1, \ldots, m\}$. Since $[U_i, T_j] \leq Z(X)$ for $j \neq i$, we can for every i find $x_i, y_i, z_i \in T_i'Z_2(X)$ such that $T_i = \langle a_i, b_i, c_i \rangle$ and $U_i = \langle a_ix_i, b_iy_i, c_iz_i \rangle$.

We will show that $x_i, y_i, z_i \in T_i' Z(X)$. Suppose

$$\begin{array}{ll} a_1x_1 = a_1[a_2,b_2,c_2]^{r_2}[b_2,c_2,a_2]^{s_2}\alpha_1 & a_2x_2 = a_2[a_1,b_1,c_1]^{r_1}[b_1,c_1,a_1]^{s_1}\alpha_2 \\ b_1y_1 = b_1[a_2,b_2,c_2]^{t_2}[b_2,c_2,a_2]^{u_2}\beta_1 & b_2y_2 = b_2[a_1,b_1,c_1]^{t_1}[b_1,c_1,a_1]^{u_1}\beta_2 \\ c_1z_1 = c_1[a_2,b_2,c_2]^{v_2}[b_2,c_2,a_2]^{w_2}\gamma_1 & c_2z_2 = c_2[a_1,b_1,c_1]^{v_1}[b_1,c_1,a_1]^{w_1}\gamma_2 \end{array}$$

where $\alpha_i, \beta_i, \gamma_i \in T_i' \prod_{j\geq 3} \gamma_3(T_i)$. The commutivity of the generators of U_1 with those of U_2 gives

$$\begin{array}{lcl} [a_2,b_2,c_2,a_2]^{r_2-2s_2} & = & [a_1,b_1,c_1,a_1]^{r_1-2s_1}; \\ [b_2,c_2,a_2,b_2]^{r_2+s_2} & = & [a_1,b_1,c_1,a_1]^{t_1-2u_1}; \\ [c_2,a_2,b_2,c_2]^{-2r_2+s_2} & = & [a_1,b_1,c_1,a_1]^{v_1-2w_1}; \\ [a_2,b_2,c_2,a_2]^{t_2-2u_2} & = & [b_1,c_1,a_1,b_1]^{r_1+s_1}; \\ [b_2,c_2,a_2,b_2]^{t_2+u_2} & = & [b_1,c_1,a_1,b_1]^{t_1+u_1}; \\ [c_2,a_2,b_2,c_2]^{-2t_2+u_2} & = & [b_1,c_1,a_1,b_1]^{v_1+w_1}; \\ [a_2,b_2,c_2,a_2]^{v_2-2w_2} & = & [c_1,a_1,b_1,c_1]^{-2r_1+s_1}; \\ [b_2,c_2,a_2,b_2]^{v_2+w_2} & = & [c_1,a_1,b_1,c_1]^{-2t_1+u_1}; \\ [c_2,a_2,b_2,c_2]^{-2v_2+w_2} & = & [c_1,a_1,b_1,c_1]^{-2v_1+w_1}. \end{array}$$

Now $(r_2-2s_2)+(r_2+s_2)+(-2r_2+s_2) = 0$. If not both r_2 and s_2 are 0, then two of r_2-2s_2, r_2+s_2 and $-2r_2+s_2$ must be nonzero. This would imply that two of $[a_2,b_2,c_2,a_2]$, $[b_2,c_2,a_2,b_2]$ and $[c_2,a_2,b_2,c_2]$ would be in $\langle [a_1,b_1,c_1,a_1] \rangle$ and thus dependent which is a contradiction. Therefore $r_2 = s_2 = 0$. By symmetry $r_1 = s_1 = t_1 = u_1 = v_1 = w_1 = r_2 = s_2 = t_2 = u_2 = v_2 = w_2 = 0$. Applying this argument two every pair of indices, we see that $x_i, y_i, z_i \in T_i'Z(X)$ for all $i \in \{1, \ldots, m\}$. Notice that it follows that $U_i' = T_i'$. Let K_i be a complement of $Z(T_i)$ in $Z(X)$ for $i = 1, \ldots, m$. We then have $a_ix_i = a_i\alpha_ik_i$ for some $\alpha_i \in T_i'$ and $k_i \in K_i$. By definition of $\mathfrak{F}_0$ there is an element $s \in Y$ which inverts T_i/T_i'. Then s must also invert U_i/U_i' (since it does not centralize it). It is also clear that s centralizes $Z(X) = \gamma_4(X)$. Now we calculate modulo $T_i' = U_i'$.

$$a_i^{-1}k_i = a_i^s k_i^s = (a_ix_i)^s = x_i^{-1}a_i^{-1} = a_i^{-1}k_i^{-1}$$

and $k_i^2 \in T_i' \cap K_i = \{1\}$. Thus $a_ix_i = a_i\alpha_i \in T_i$. Similarly $b_iy_i, c_iz_i \in T_i$ and $U_i = T_i$. $\square$

Proposition 1 $\mathfrak{F}_0 \subseteq \mathfrak{F}(E)$

Proof Since 2 divides the order of E we have that $\mathfrak{F}(E)$ must contain all 2-groups. Suppose now that $G = XY \in \mathfrak{F}_0$ with $X \neq 1$. Then G is generated by the subnormal subgroups $\langle X, s\rangle$, $s \in Y$. It is thus sufficient to show that $\langle X, s\rangle \in \mathfrak{F}(E)$ for all $s \in Y$. Now $\langle X, s\rangle$ is a normal subgroup of $\langle X, \sigma\rangle \times \langle s\rangle$, where σ is the automorphism on X induced by s. It is therefore enough to show that $\langle X, \sigma\rangle$ is in $\mathfrak{F}(E)$. Let σ_i be the automorphism on $X = T_1 \cdots T_m$, which acts on T_i like σ but centralizes T_j for $j \neq i$. Then $\sigma = \sigma_1 \cdots \sigma_m$ and $\langle X, \sigma\rangle$ is a normal subgroup of the normal product $\langle X, \sigma_1\rangle \cdots \langle X, \sigma_m\rangle$. Each $\langle X, \sigma_i\rangle$ is a normal product $T_1 \cdots T_{i-1}\langle T_i, \sigma_i\rangle T_{i+1} \cdots T_m$ and we have only left to show that $\langle T_i, \sigma_i\rangle \in \mathfrak{F}(E)$. If σ_i centralizes T_i, this is clear. If not, then σ_i inverts T_i/T_i' and by Lemma 2 we have that $\langle T_i, \sigma_i\rangle$ is isomorphic to E. $\square$

Proposition 2 *Let $G_1 = X_1Y_1$ and $G_2 = X_2Y_2$ be in $\mathfrak{F}_0$ and suppose that G_1G_2 is a normal product of G_1 and G_2. Then $G_1G_2 \in \mathfrak{F}_0$.*

Proof If both X_1 and X_2 are trivial this is clear. Suppose then that $X_1 \neq 1$ but $X_2 = 1$. Let $S = Y_1G_2$, then S is a Sylow 2-subgroup of G_1G_2. We also have $X_1 = O_p(G_1G_2)$. Suppose X_1 is a central product of $T_1, \dots, T_m$, where $T_1, \dots, T_m$ are described as in Definition 2. For each $i \in \{1, \dots, m\}$ we have $[T_i, G_2] = 1$ and thus $T_i \trianglelefteq G_1G_2$. We also get

$$\begin{aligned} S/C_S(T_i/T_i') &= Y_1G_2/C_{Y_1}(T_i/T_i')G_2 \\ &\cong Y_1/Y_1 \cap C_{Y_1}(T_i/T_i')G_2 \\ &= Y_1/C_{Y_1}(T_i/T_i') \\ &\cong \mathbb{Z}_2 \end{aligned}$$

and since $X_1Y_1 \in \mathfrak{F}_0$, there is an element in Y_1 which acts on T_i/T_i' as the inverse automorphism. It follows that $G_1G_2 \in \mathfrak{F}_0$.

So we can assume that $X_1 = T_1 \cdots T_m$ and $X_2 = U_1 \cdots U_l$ with m and l nonzero. Let s be a 2-element of G_2. We have that s induces an automorphism σ on X_1. We also have that s^2 centralizes X_2 and since s^2 is a 2-element and $(2, p) = 1$ we get $[X_1, \langle s^2\rangle] = [X_1, \langle s^2\rangle, \langle s^2\rangle]$ (s^2 fixes every coset of $[X_1, \langle s^2\rangle]$ in X_1 and every coset of $[X_1, \langle s^2\rangle, \langle s^2\rangle]$ in $[X_1, \langle s^2\rangle]$). Therefore $[X_1, \langle s^2\rangle] \leq [X_2, \langle s^2\rangle] = 1$ and $\sigma^2 = 1$. The following lemma will be useful for the completion of the proof.

Lemma 6
(a) $[T_i, X_2] \leq T_i'$ and $[U_j, X_1] \leq U_j'$ for $1 \leq i \leq m$ and $1 \leq j \leq l$.

(b) $[\gamma_i(X_1), \gamma_j(X_2)] \leq \gamma_{i+j}(X_1) \cap \gamma_{i+j}(X_2)$ *for all integers* $i, j \geq 1$.
(c) $T_i, U_j \trianglelefteq G_1G_2$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq l$.

Proof of Lemma 6 We first show that $T_i \trianglelefteq X_1X_2$. Suppose this is not the case. Then there is $u \in X_2$ such that $T_i^u \neq T_i$. By Lemma 5 we have that u permutes the set $\{T_1, \ldots, T_m\}$ by conjugation and we have that $T_i, T_i^u, \ldots, T_i^{u^4}$ commute elementwise. Let a, b and c be good generators for T_i. Now $[[a, u], [b, u], [c, u], u, u] \in \gamma_5(X_2) = \{1\}$. Thus we have

$$\begin{aligned} [a, b, c]^{u^3-3u^2+3u-1} &= [[a, b, c], u, u, u] \\ &= [[a, u], [b, u], [c, u], u, u] \\ &= 1 \end{aligned}$$

and since $p > 3$ this implies that $[a, b, c, a] = 1$ which is a contradiction. Therefore we must have $T_i \trianglelefteq X_1X_2$.

We now prove (a). By symmetry it is sufficient to show that $[T_i, X_2] \leq T_i'$. Let $u \in X_2$. We have seen that $T_i^u \leq T_i$. By lemma 1 we have that u permutes the set $\{\langle a\rangle T_i', \langle b\rangle T_i', \langle c\rangle T_i'\}$. But since p does not divide $|\mathrm{Sym}(3)|$, we have that u fixes this set. If $a^u = a^r$ modulo T_i' then $a = a^{u^p} = a^{r^p} = a^r$ and so u centralizes T_i/T_i' and we have proved (a). It follows from (a) that $[X_1, X_2] \leq X_1'$ and therefore it follows by induction on i (using the three subgroups Lemma) that $[\gamma_i(X_1), X_2] \leq \gamma_{i+1}(X_1)$. Then on has by induction on j that $[\gamma_i(X_1), \gamma_j(X_2)] \leq \gamma_{i+j}(X_1)$. By symmetry $[\gamma_j(X_2), \gamma_i(X_1)] \leq \gamma_{i+j}(X_2)$ and so (b) is true. We have now only left to prove (c). Suppose T_i is not normal in G_1G_2. By (a) there is a 2-element $s \in G_2$ such that $T_i^s \neq T_i$. By lemma 5 we have that $[T_i, T_i^s] = 1$. Let a, b and c be good generators for T_i. Now $aa^{-s} \in X_2$ and $b \in X_1$ and therefore it follows from (b) that $[a, b] = [aa^{-s}, b] \in [X_1, X_2] \leq \gamma_2(X_2)$. Then $[[a, b], s] \in \gamma_3(X_2)$, since it is inverted by s. It follows that

$$[a, b, c, a]^{-1} = [[a, b], s], c, a] \in [\gamma_3(X_2), X_1, X_1] \leq \gamma_5(X_2).$$

Where in the last inclusion we are using (b). So we have the contradiction that $[a, b, c, a] = 1$ which finishes the proof of the Lemma.

Continuation of the proof of Proposition 2 Let s be a 2-element of G_2 and a, b and c be good generators for T_i. We will prove that T_i/T_i' is either inverted or centralized by s. From Lemma 6 we have that the set $\{a^s, b^s, c^s\}$ is also a good set of generators for T_i. It follows from Lemma 1(b) that

$$a^s, b^s, c^s \in \langle a\rangle T_i' \cup \langle b\rangle T_i' \cup \langle c\rangle T_i'.$$

We first show that $a^s \in \langle a\rangle T_i'$, $b^s \in \langle b\rangle T_i'$ and $c^s \in \langle c\rangle T_i'$. If that is not the case, then one of $\langle a\rangle T_i'$, $\langle b\rangle T_i'$ and $\langle c\rangle T_i'$ is fixed by s but the other two interchanged. Without loss of generality we can suppose that the first two are interchanged and the last fixed. By Lemma 3 we can (by taking a new set of good generators) find integers m and n which satisfy $mn = 1 (\mathrm{mod}\ p)$ such that

$$a^s = b^n,\ b^s = a^m \text{ and } c^s = c^\epsilon,$$

where $\epsilon \in \{-1, 1\}$ We then have $[a, b] = [b^{-n}a, b] = [a^{-s}a, b] \in [X_2, X_1] \leq \gamma_2(X_2)$ by Lemma 6. But $[a, b]^s = [b^n, a^m] = [b, a]^{mn} = [a, b]^{-1}$ and so $[a, b]$ must be in $\gamma_3(X_2)$ (since all elements in $\gamma_2(X_2)$ are fixed by s modulo $\gamma_3(X_2)$). But then $[a, b, c, a] \in \gamma_5(X_2)$ by Lemma 6. That is $[a, b, c, a] = 1$ which is a contradiction. We therefore have that $\langle a\rangle T_i'$, $\langle b\rangle T_i'$ and $\langle c\rangle T_i'$ are all fixed by s. We can then (by Lemma 2) choose the good generators a, b and c such that each generator is either fixed or inverted by s. We want to show that either all are fixed or all inverted. Suppose this is not the case and without loss of generality we can assume that $a^s = a^{-1}$ and $b^s = b$. But then we have that $a^{-2} = a^{-1}a^s = [a, s] \in [X_1, X_2]$ and thus we have by Lemma 6 that $a \in \gamma_2(X_2)$ and as before we get the contradiction that $[a, b, c, a] = 1$. We have thus shown that T_i/T_i' is either inverted or fixed by s. Similarly one has that for every 2-element s in G_1 either U_j/U_j' is centralized or inverted by s.

Suppose we have reindexed the U_j such that U_i/U_i' is centralized by G_1 when $1 \leq i \leq k$ but that U_i/U_i' inverted by some 2-element of G_1 when $k + 1 \leq i \leq l$. By Lemma 2 we have, that when $k + 1 \leq i \leq l$, some 2-element in G_1 inverts some three generators of U_i. Then $U_{k+1} \cdots U_l \leq X_1$ and

$$X_1X_2 = T_1 \cdots T_mU_1 \cdots U_k.$$

Since G_1 is generated by 2-elements it also follows from Lemma 2 that $U_1, \ldots U_k$ are centralized by G_1 and it is thus clear that the product above is a central product. In Lemma 6 we proved that each of the factors is normal in G_1G_2. Now let Y be a Sylow 2-subgroup of G_1G_2. We let $Y_1 = Y \cap G_1$

and $Y_2 = Y \cap G_2$, so Y_i is a Sylow 2-subgroup of G_i. We have seen, that each element in Y_2 inverts or centralizes each T_i/T_i'. Since $G_1 \in \mathfrak{F}_0$, we have that this is true for all elements in Y and that furthermore $Y/C_Y(T_i/T_i') \cong \mathbb{Z}_2$. Same is true for each qoutient U_j/U_j', $j = 1, \ldots, k$. It is therefore clear that $G_1G_2 \in \mathfrak{F}_0$ and the proof of Proposition 2 is completed. $\square$

Proposition 3 *Let $G = XY \in \mathfrak{F}_0$ and $N \trianglelefteq G$. Then $N = O_p(N)O^p(N)$ with $O^p(N) \in \mathfrak{F}_0$.*

Proof Let k be such that (after reindexing) for $i = 1, 2, \ldots, k$ there is some 2-element $y \in N$ that inverts T_i/T_i' but that every 2-element in N centralizes T_i/T_i' when $k + 1 \leq i \leq m$. By Lemma 2 we have that $T_1, T_2, \ldots T_k$ have generators that are inverted by some 2-element from N and $T_{k+1}, \ldots, T_m$ are centralized by all 2-elements in N. Let $X_1 = T_1 \cdots T_k$ and $M = T_{k+1} \cdots T_m$. Then $X_1 \leq N$ and M is centralized by all 2-elements in N. We now consider two cases.

Case 1.$k = 0$. Then each 2-element of N centralizes X and N is nilpotent. We now have that $N = O_p(N)O^p(N)$ and $O^p(N) = Syl_2(N)$ is in $\mathfrak{F}_0$.

Case 2. $k \geq 1$. We have $O_p(N) = X \cap N = X_1(M \cap N)$. Let $R = N \cap Y$. Then R is a Sylow 2-subgroup of N. Let $G_1 = X_1R$. Because $[M \cap N, R] = 1$ we have that $G_1 \trianglelefteq N$. Also $M \cap N \trianglelefteq N$ and N is a normal product of G_1 and $X_1(M \cap N)$. Then for $1 \leq i \leq k$ we have

$$
\begin{aligned}
R/C_R(T_i/T_i') &= R/C_Y(T_i/T_i') \cap R \\
&\cong RC_Y(T_i/T_i')/C_Y(T_i/T_i') \\
&\leq Y/C_Y(T_i/T_i').
\end{aligned}
$$

So $G_1 \in \mathfrak{F}_0$ if $R/C_R(T_i/T_i')$ is non-trivial. But if every element of R centralizes T_i/T_i' then, since X centralizes T_i/T_i', every 2-element of N would centralize T_i/T_i' which is a contradiction. So $G_1 \in \mathfrak{F}_0$ and since N/G_1 is isomorphic to a quotient of $M \cap N$ we have that $G_1 = O^p(N)$. $\square$

Definition 3 *We define a class $\mathfrak{F}$ of finite groups as follows. $G \in \mathfrak{F}$ if $G = O_p(G)O^p(G)$ and $O^p(G) \in \mathfrak{F}_0$.*

Theorem 1 *$\mathfrak{F}$ is the Fitting class generated by E and $\mathfrak{F}$ is a supersoluble class.*

Proof Let $G \in \mathfrak{F}$. By Lemma 4 we have that $O^p(G)$ is supersoluble. G is then a normal product of a nilpotent group and a supersoluble group, and is therefore supersoluble. We then have only left to show that $\mathfrak{F}$ is a Fitting class since $E \in \mathfrak{F}_0$ and $\mathfrak{F} \leq \mathfrak{F}(E)$.

We first prove that $\mathfrak{F}$ is closed with respect to forming normal products. Let $G_1, G_2 \in \mathfrak{F}$ and G_1G_2 be a normal product of those groups. Now $G_1G_2 = O_p(G_1)O^p(G_1) \cdot O_p(G_2)O^p(G_2) = O_p(G_1G_2)O^p(G_1G_2)$ where $O^p(G_1G_2) = O^p(G_1)O^p(G_2)$ is in $\mathfrak{F}_0$ by Proposition 2.

Now we show that $\mathfrak{F}$ is closed with respect to taking normal subgroups. Let $G \in \mathfrak{F}$ and $N \trianglelefteq G$. Then $O^p(N) \leq O^p(G) \cap N$. Since $H := O^p(G) \cap N$ is a normal subgroup of the $\mathfrak{F}_0$-group $O^p(G)$, we have by Proposition 3 that it is a normal product $O^p(H)O_p(H)$ with $O^p(H) \in \mathfrak{F}_0$. All 2-elements of N lie in H and thus $O^p(H) = O^p(N)$. Since $O_p(N)$ is a Sylow p-subgroup of N we have that $N = O_p(N)O^p(N)$. $\Box$

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