Engel-5 Lie algebras

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1 Introduction

A Lie algebra L is called an Engel Lie algebra if for each ordered pair (x, y) there is an integer $n(x, y)$ such that

$$(((y\underbrace{x)x)\cdots)x}_{n(x,y)} = 0. \tag{1}$$

One of the basic classical results for Engel Lie algebras is Engel's Theorem. It states that every finite dimensional Engel Lie algebra over a field is nilpotent. So for finite dimensional Lie algebras the Engel condition is equivalent to nilpotency. This is however not true in general.

Now suppose $n = n(x, y)$ in (1) can be chosen independently of x and y. We then say that L is an Engel-n Lie algebra. A different way of stating this is to say that $\text{ad}(x)^n = 0$ for all $x \in L$. Here $\text{ad}(x)$ is the multiplication from right. We have the following two results of E. I. Zel'manov.

Theorem Z1[10] Every Engel-n Lie algebra over a field k with $\text{char}\, k = 0$ is nilpotent.

Theorem Z2[11,12] An Engel-n Lie algebra over an arbitrary field is locally nilpotent.

The natural question that now arises is what can be said about the nilpotency classes. How does the nilpotency class depend on n and the number of generators r? In [9] E. I. Zel'manov and M. Vaughan-Lee give upper bounds. Before we state their results we introduce some notation. Define a function $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by induction in the following way: $T(m, 1) = m$, $T(m, s + 1) = m^{T(m,s)}$. Let L be an Engel-n Lie algebra generated by r elements. It follows from the work of Zel'manov and Vaughan-Lee that L is nilpotent of class at most $T(r, n^{n^n})$. When the characteristic of the field is greater than n they get smaller bounds. So if $25 \leq n < p$ then L is nilpotent of class at most $T(r, 2^n)$ and when $26 > n < p$ we have that L is nilpotent of class at most $T(r, 3^n)$. The authors nevertheless believe that these bounds are too high and make the conjecture that the class can always be bounded by a function which is polynomial in r.

There is still not much evidence that this conjecture is true. But there are some supporting facts. From Theorem Z1 it follows that for each n there is a constant n_0 such that every Engel-n Lie algebra over a field k with $\operatorname{char} k > n_0$ or $\operatorname{char} k = 0$ is nilpotent. Here the nilpotency class does not depend on r, so we have a constant upper bound. This means that for each n the conjecture is true for almost all characteristics. We also have some detailed information for small values of n. For $n \leq 3$ the conjecture is known to be true. It is well known that Lie algebras satisfying the Engel-2 identity are nilpotent of class at most 3. In [5] it is shown that Engel-3 Lie algebras with $\operatorname{char} k \neq 2, 5$ are nilpotent of class at most 4 and that when the characteristic is 5 we have that the class is at most $2r$. In [6] it is shown that the class is at most $2(r + 1)^6$ when $\operatorname{char} k = 2$. For $n = 4$ the conjecture is also known to be true in most cases. For characteristics not equal to one of $2, 3$ or 5 we have that the class c is at most 7 [1, 5]. For $\operatorname{char} k = 3$ we have that $c \leq 3r$ [5] and $c \leq 6r$ when $\operatorname{char} k = 5$ [2]. In [6] a polynomial upper bound is also given for c when $\operatorname{char} k = 2$ and $|k| \neq 2$. Before we turn to Engel-5 Lie algebras we also mention that Vaughan-Lee [8] has recently shown that Engel-6 Lie algebras over field with $\operatorname{char} k = 7$ have nilpotency class at most $51r^8$.

In this article we will be looking at Engel-5 Lie algebras. It is not difficult to show that if L is an Engel-p Lie algebra over a field of characteristic p, then ab^{p-1} is central in L for all $a, b \in L$. It follows that the class of an r-generator Engel-5 Lie algebra over a field of characteristic 5 is at most $6r+1$. But it seems unlikely that one has a linear upper bound for characteristic 2 and 3 since in that case it is known that the ideal generated by an element need not be nilpotent. In this article we will get linear upper bounds for the class c when $\operatorname{char} k > 5$. Our main theorem is the following.

Theorem 1 *Let L be an Engel-5 Lie algebra with r generators. If $\operatorname{char} k \neq 2, 3, 5, 7$ then the nilpotency class c is at most $59r$. If $\operatorname{char} k = 7$ then $c \leq 80r$.*

Finally it should be mentioned that Engel Lie algebras played an important role in the solution of the "restricted Burnside problem". For a detailed discussion of the Burnside problem we refer to [4, 7].

2 An outline of the approach

Let F be a free Lie algebra freely generated by $z, z_1, z_2, \ldots$ over a field k where $\operatorname{char} k > 5$. We let

$$L = F/J$$

where J is the ideal in F generated by $\{uv^5 | \, u, v \in F\}$. That is L is a relatively free Engel-5 Lie algebra over k freely generated by $y = z + J, y_1 = z_1 + J, y_2 = z_2 + J, \ldots$. Since $\operatorname{char} k > 5$ we have that

$$J = \langle \{ \sum_{\sigma} uv_{\sigma(1)} \cdots v_{\sigma(5)} | \, u, v_1, \ldots, v_5 \in F \} \rangle.$$

Therefore J is a multigraded ideal and thus L is a multigraded Lie algebra.

Now let $I_1 = \operatorname{Id}\langle \{ y_i y_j | \, i, j \in \mathbb{N} \} \rangle$ and $L_1 = L/I_1$. Then L_1 is generated by $x = y + I_1, a_1 = y_1 + I_1, a_2 = y_2 + I_1, \ldots$ and the a's commute in L_1. We want to study the nilpotency of $I_y < L$ but it follows from the following proposition, which is due to G. Higman [3], that it is sufficient to consider the ideal $I_x < L_1$.

Proposition 1 *If $I_x^c = \{0\}$ in L_1 then $I_y^c = \{0\}$ in L.*

Proof We assume that $I_x^c = \{0\}$ in L_1 and prove by induction on m that

every product of multiweight $(c, \underbrace{1, 1, \cdots, 1}_{m})$ in $y, y_1, y_2, \ldots, y_m$ is zero in L. (*)

From property (*) it follows that in every Engel-5 Lie algebra $\tilde{L}$ we have that all products of multiweight $(c, \underbrace{1, 1, \cdots, 1}_{m})$ in $u, u_1, \ldots, u_m$ are zero for arbitrary $u, u_1, \ldots, u_m \in \tilde{L}$.

Basis of induction: $m = 1$. Let $L_{(c,1)}$ be the subspace of L consisting of elements of multiweight $(c, 1)$ in y, y_1.Now $L_{(c,1)} \cap I_1 = \{0\}$ (L is multigraded). But since $I_x^c = \{0\}$ in L_1 we have that $L_{(c,1)} \leq I_1$ and thus $L_{(c,1)} = \{0\}$.

Induction step: Suppose property (*) is true for some $m = r \geq 1$ we prove it is true for $m = r + 1$. Let u be some product of multiweight $(c, \underbrace{1, 1, \cdots, 1}_{r+1})$ in $y, y_1, \ldots, y_{r+1}$. Since $I_x^c = \{0\}$ in L_1 we have that $u \in I_1$. Since I_1 is multigraded we have that u is a linear combination of elements of the form

$$(y_i y_j) v_1 \cdots v_{r+c-1}$$

where $v_1, \ldots, v_{r+c-1}$ is some permutation of $\underbrace{y, \ldots, y}_{c}, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{r+1}$. But such a product is a product of multiweight $(c, \underbrace{1, 1, \cdots, 1}_{r})$ in $y, u_1, u_2, \cdots, u_r$, where $u_1 = y_i y_j$ and $\{u_2, \ldots, u_r\} = \{y_l |\ l \notin \{i, j\}\ \}$, and is therefore zero by the induction hypothesis. □

We will therefore be working in L_1 for the rest of the section. Let $A = \{a_1, a_2, \ldots\}$ and $B = A \cup \{x a_i a_j a_r a_s | a_i, a_j, a_r, a_s \in A\}$. We have that L_1 is generated by x and A. Now we want to study the nilpotency of I_x in L_1. We shall now reduce this problem in a few steps as follows.

Step 1. We let $I_2 = \mathrm{Id}\langle\{x a_i a_j a_k a_l | i, j, k, l \in \mathbb{N}\}\rangle$ and $L_2 = L_1/I_2$. In Section 3 we shall show that $(x a_i a_j a_k)(x a_r a_s a_t a_l) = 0$ for all $i, j, k, r, s, t, l \in \mathbb{N}$. This implies the following.

Proposition 2 *If every product in L_1 with d elements of A lies in I_2 then every product with $4d$ elements of A is zero. If $u_1, u_2, u_3, u_4 \in B$ and one of the u_i's is in $B \setminus A$ then $x u_1 u_2 u_3 u_4 = 0$.*

Proof Let $\tilde{L}$ be any Engel-5 Lie algebra generated by $u, b_1, b_2, \ldots$,where the b's commute. Then $\tilde{L}$ is a homomorphic image of L_1 under the map which takes x to u and each a_i to b_i. It follows that $(ub_ib_jb_p)(ub_rb_sb_tb_l) = 0$ in $\tilde{L}$ for all $i, j, p, r, s, t, l \in \mathbb{N}$.

Since $(xa_ia_ja_ra_s)a_l = 0$ by the Engel identity and

$$\begin{aligned}(xa_ia_ja_ra_s)(xa_ta_la_ma_p) &= xa_ia_ja_r(xa_ta_la_ma_p)a_s \\ &= 0,\end{aligned}$$

we have that the elements in B commute. We have therefore

$$(xb_1b_2b_3)(xb_4b_5b_6b_7) = 0 \qquad (*)$$

for all $b_1, b_2, \cdots, b_7 \in B$. So we have proved the second assumption in the proposition.

Suppose u is a product in L_1 with $t \geq d$ elements from A. Then $u \in I_2$ and u is therefore a linear combination of elements of the form

$$xa_ia_ja_ra_su_1u_2\cdots u_m$$

where each u_l is either x or lies in A. Since I_2 is multigraded we can assume that each summand has the same multiweight as u.

Then assume u is a product with $t \geq d$ elements from B. Then it follows from the fact above that u is a linear combination of elements of the form

$$xb_1b_2b_3b_4u_1\cdots u_m$$

where each u_i is either x or in B. But since $(xc_1c_2c_3)(xc_4c_5c_6c_7) = 0$ for all $c_1, c_2, \ldots, c_7 \in B$ it follows that $xb_1b_2b_3b_4 = 0$ unless b_1, b_2, b_3, b_4 all lie in A. So we can assume that in each summand $b_1, b_2, b_3, b_4 \in A$. But then the summand is a product with $t-3$ elements of B, $xb_1b_2b_3b_4$ and the b's among $u_1, u_2, \ldots, u_m$.

Now let u be a product in L_1 including $4d$ elements from A. By using the argument above repeatedly, we see that we can write u as a linear combination of products with $4d - 3d = d$ elements of B. But then for all the summands we must have that the d elements from B are all of the form $xa_ia_ja_ra_s$.

Now we apply the argument once more to each of the summands and we see that u is a linear combination of products of the form

$$xb_1b_2b_3b_4u_1\cdots u_m$$

where b_1, b_2, b_3, b_4 are all of the form $xa_ia_ja_ra_s$. But then each summand is zero by remark (*) above. Hence u is zero. $\square$

We have thus reduced the problem to working in L_2. Now let $C = A \cup \{xa_ia_ja_r | a_i, a_ja_r \in A\}$.

<u>Step 2</u>. We let $I_3 = \mathrm{Id}\langle\{xa_ia_ja_r|\ i,j,r \in \mathbb{N}\}\rangle$. Then $I_2 \leq I_3 \leq L_1$. In Section 3 we shall prove that $(xa_ia_j)(xa_ra_sa_t) \in I_2$. This implies the following.

Proposition 3 *If every product with d elements of A lies in I_3 then every product with $3d$ elements of A lies in I_2. If $u_1, u_2, u_3 \in C$ and one of the u_i's is in $C \setminus A$ then $xu_1u_2u_3 \in I_2$.*

Proof We have that every product of the form $xa_ia_ja_r$ commutes with all the a's modulo I_2 and we also have that two such elements commute together modulo I_2. The rest of the proof is similar to the proof of Proposition 2. $\square$

This implies that we have reduced the problem to working in $L_3 = L_1/I_3$. Define a sequence of sets $\{A_i\}$ by induction as follows,

$$A_1 = A, \quad A_{i+1} = A_i \cup \{xb_1b_2|\ b_1, b_2 \in A_i\}.$$

Then we let $A_\infty = \bigcup A_i$.

<u>Step 3</u>. We let $I_4 = \mathrm{Id}\langle\{xa_ia_j|\ i,j \in \mathbb{N}\}\rangle$. Then $I_2 \leq I_3 \leq I_4 \leq L_1$. In Section 4 we shall prove that all products of weight 9 involving $x, x, x, x, a_i, a_j, a_r, a_s, a_t$ are in I_3. We shall also see that every product of weight 6 involving x, x, a_i, a_j, a_r, a_s is in I_3. This implies the following.

Proposition 4 *If every product with d elements of A lies in I_4 then every product with $4d$ elements of A lies in I_3. If an element $u \in A_\infty$ includes either 5 elements from A or 4 occurrences of x then $u \in I_3$.*

Proof We have that every product of the form xa_ia_j commutes with all the a's modulo I_3 and we also have that two such elements commute together modulo I_3. The difference between this situation and steps 1 and 2 is that we do not have $(xa_i)(xa_ja_r) \in I_3$. We therefore have to change the argument slightly.

We first show that the elements in A_∞ commute modulo I_3 and that $xb_1b_2b_3 \in I_3$ for all $b_1, b_2, b_3 \in A_\infty$. We prove by induction on n the following hypothesis:

If $b_1, b_2, b_3 \in A_n$ then $\underline{(1)\ b_1b_2 \in I_3}$ and $\underline{(2)\ xb_1b_2b_3 \in I_3}$.

This is obviously true for $n = 1$. To prove the induction step it is clearly sufficient to show that if B is a subset of A_∞ and if (1) and (2) are satisfied for all $b_1, b_2, b_3 \in B$ then (1) and (2) are satisfied for $b_2, b_3 \in B$ and $b_1 = xb_4b_5$ with $b_4, b_5 \in B$. But then

$$b_1b_2 = xb_4b_5b_2 \in I_3 \quad \text{(by induction)}$$

and

$$xb_1b_2b_3 = xb_2b_3(xb_4b_5)$$

modulo I_3. And since $(xa_1a_2)(xa_3a_4) \in I_3$ we have that

$$xb_2b_3(xb_4b_5) \in \mathrm{Id}\langle b_ib_j, xb_ib_jb_k \mid 2 \leq i, j, k \leq 5\rangle \leq I_3$$

by induction. We have therefore proved the hypothesis.

It follows from this that for all $b_1, b_2, \ldots, b_5 \in A_\infty$ we have that every product of weight 6 involving x, x, b_1, b_2, b_3, b_4 and of weight 9 involving $x, x, x, x, b_1, b_2, b_3, b_4, b_5$ is in I_3.

We next prove that if $u \in A_\infty$ includes at least 5 elements from A then it must be in I_3. So suppose that u is in A_∞ and that u contains at least 5 elements from A. Then we have that there are two possibilities for the form of u. We have that

$$u = x(x(xb_1b_2)b_3)b_4$$

or

$$u = x(xb_1b_2)(xb_3b_4),$$

where $b_1, b_2, b_3, b_4 \in A_\infty$ and one of them is of the form xc_1c_2 with $c_1, c_2 \in A_\infty$. But then u is a product including 4 occurrences of x and 5 elements from A_∞ and is therefore in I_3. It now follows easily that the second assumption in the proposition holds.

Now if we have a product u including $e \geq d$ elements $b_1, b_2, \ldots, b_e \in A_\infty$, then u can be written as a linear combination of elements of the form

$$xb_ib_ju_1u_2\cdots u_r,$$

where $u_1, u_2, \ldots, u_r$ are each either x or b_l for $l \in \{1, 2, \ldots, e\} \setminus \{i, j\}$. But then each summand is a product including $e-1$ elements from A_∞.

Now let u be a product including $4d$ elements from A. By using the argument above repeatedly, we get a linear combination of products each with $d-1$ elements from A_∞. But $4(d-1) < 4d$ so one of these elements must contain at least 5 elements from A. As we showed above, any element of A_∞ containing at least 5 elements from A is in I_3, and so $u \in I_3$. $\square$

In the next 3 sections we shall show that the assumptions we have made above are true.

3 Reduction steps 1 and 2

In this section we shall assume that $\mathrm{char}\, k \geq 7$ or $\mathrm{char}\, k = 0$. Since L_1 is an Engel-5 Lie-algebra and $\mathrm{char}\, k > 5$ then linearization of the Engel identity gives us that

$$0 = uc^4v + uc^3vc + uc^2vc^2 + ucvc^3 + uvc^4$$

for all $u, v, c \in L_1$. Then also

$$\begin{aligned} 0 &= -vc^4u - vc^3uc - vc^2uc^2 - vcuc^3 - vuc^4 \\ &= u(vc^4) + u(vc^3)c + u(vc^2)c^2 + u(vc)c^3 + uvc^4 \\ &= uc^4v - 5uc^3vc + 10uc^2vc^2 - 10ucvc^3 + 5uvc^4. \end{aligned}$$

We will call this latter identity the skew-Engel identity.

We will first prove the following proposition.

Proposition 5 *If $a, b \in A$ then $(xb^3)(xa^4) = 0$ in L_1.*

Then linearization gives us that

$$(xa_ra_sa_t)(xa_ia_ja_la_m) = 0$$

in L_1. This was the assumption we made in reduction step 1 in last section.

Proof of Proposition 5 The Engel identity and the skew-Engel identity give that

$$0 = xa^4x + xa^3xa + xa^2xa^2 + xaxa^3$$

and

$$0 = xa^4x - 5xa^3xa + 10xa^2xa^2 - 10xaxa^3.$$

If we now postmultiply by b three times, we get

$$0 = xa^4xb^3 + xa^3xab^3$$

and

$$0 = xa^4xb^3 - 5xa^3xab^3.$$

It follows that $xa^4xb^3 = 0$ and hence $(xa^4)(xb^3) = 0$. (This uses the fact that $xa^rxa^sb^3 = 0$ if $s \geq 2$. That is clearly true since L_1 is an Engel-5 algebra.) $\square$

Recall from last section that $I_2 = Id\langle\{xa_ia_ja_ka_l | i, j, k, l \in \mathbb{N}\}\rangle$. From now on we will be calculating modulo I_2 so $u = v$ will mean $u = v$ modulo I_2.

Proposition 6 *If $a, b \in A$ then every product of x, x, a, a, a, b, b is zero.*

It follows that $(xb^2)(xa^3) = 0$. In particular linearization gives

$$(xa_ia_j)(xa_ra_sa_l) \in I_2$$

for all $i, j, r, s, t \in \mathbb{N}$. But this was the assumption we made in reduction step 2 in Section 1.

Proof of Proposition 6 We first show that every product of x, x, a, a, a, a, b is zero. From the Engel and skew-Engel identity we have

$$0 = xbxa^4 + xbaxa^3 + xba^2xa^2$$

and

$$0 = xbxa^4 - 2xbaxa^3 + 2xba^2xa^2.$$

But we also have

$$0 = xb(xa^4) = xbxa^4 - 4xbaxa^3 + 6xba^2xa^2.$$

Solving these equation together gives $xbxa^4 = xbaxa^3 = xba^2xa^2 = 0$. In particular we have $xaxa^4 = xa^2xa^3 = xa^3xa^2 = 0$. It follows that

$$-4xaxa^3b = xbxa^4, \quad -3xa^2xa^2b = 2xabxa^3, \quad 2xa^3xab = -3xa^2bxa^2$$

so

$$xaxa^3b = xa^2xa^2b = xa^3xab = 0.$$

It is now clear that all products of x, x, a, a, a, a, b are zero.

We have seen that $xbxa^4 = 0$. Linearization gives that $xbxa_1a_2a_3a_4 = 0$, for all $b, a_1, a_2, a_3, a_4 \in A$. Linearization of $xabxa^3 = 0$ and $xa^2xa^2b = 0$ gives

$$xb^2xa^3 = -3xabxa^2b, \quad xa^2xab^2 = -xabxa^2b.$$

Then similarly since $xa^3xab = xa^2bxa^2 = 0$ we have

$$xa^3xb^2 = -3xa^2bxab, \quad xab^2xa^2 = -xa^2bxab.$$

Therefore all products of x, x, a, a, a, b, b are in the linear span of $xabxa^2b$ and xa^2bxab. From the Jacobi identity we have

$$\begin{aligned} 0 &= (xa^3)(xb)b + (xb)b(xa^3) + b(xa^3)(xb) \\ &= -3xabxa^2b \end{aligned}$$

and then from the Engel identity

$$\begin{aligned} 0 &= axa(xa)b^2 + 2axb(xa)ab + axb^2(xa)a + 2axab(xa)b \\ &= -2xa^2bxab. \end{aligned}$$

So we have proved the proposition. $\square$

4 Reduction step 3

Recall that $I_3 = Id\langle\{xa_ia_ja_r | i,j,r \in \mathbb{N}\}\rangle$. We also had defined A_∞ as follows,

$$A_1 = A, \quad A_{i+1} = A_i \cup \{xb_1b_2 | b_1, b_2 \in A_i\}$$

and $A_\infty = \bigcup A_i$.

In this section $u = v$ will mean that $u = v$ modulo I_3. We will prove the following two propositions.

Proposition 7 *Every product of x, x, a, b, c, d is zero if $a, b, c, d \in A_\infty$.*

Proposition 8 *Every product of $x, x, x, x, a, b, c, d, e$ is zero if $a, b, c, d, e \in A$.*

This is what was needed to go through the reduction step 3 in Section 2.

Proof of Proposition 7. We first prove that every product of x, x, a, a, a, b is zero if $a, b \in A_\infty$. Using the Engel and Jacobi identities we have

$$\begin{aligned} 0 &= bx^2a^3 + bxaxa^2 + bxa^2xa + bxa^3x \\ &= -xbaxa^2 - xbxa^3 \end{aligned}$$

and

$$\begin{aligned} 0 &= xb(xa^3) \\ &= xbxa^3 - 3xbaxa^2. \end{aligned}$$

Solving these two equations together gives $xbaxa^2 = 0$ and $xbxa^3 = 0$. In particular we have $xa^2xa^2 = xaxa^3 = 0$ and linearization gives $xa^2xab = xbaxa^2 = 0$ and $xaxa^2b = -1/3 \cdot xbxa^3 = 0$. It is therefore clear that all products of x, x, a, a, a, b are zero.

Let us now turn to the general case. Since $xbxa^3 = 0$ we have that linearization gives that $xbxacd = 0$. Also linearizations of $xabxa^2 = 0$ and $xa^2xab = 0$ give

$$xb^2xa^2 = -2xabxab, \quad xa^2xb^2 = -2xabxab.$$

It follows that all products of x, x, a, a, b, b are multiples of $xabxab$. But

$$0 = xab(xab) = xabxab.$$

We have therefore showed that all products of x, x, a, a, b, b are zero.

Linearization of $xa^2xb^2 = 0$ gives then $xabxcd = 0$ for all $a, b, c, d \in A_\infty$. $\square$

Now we turn to the proof of Proposition 8. In the following argument we need to assume that $\text{char}\, k \neq 7, 17$. Using a computer program the nilpotent quotient algorithm [see 2] was applied for these exceptional characteristics and it was verified that Proposition 8 also holds in these cases. In the rest of the section we will therefore assume that $\text{char}\, k \neq 7, 17$.

Lemma 1 *If $a \in A$ then every product of x, x, x, x, a, a, a, a is zero.*

Proof First of all

$$0 = xa(xa) = xaxa - xa^2x. \tag{2}$$

It follows that $xaxaxa = xa^2x^2a$ and $xaxa^2x = xa^2xax$. It is then clear that every product of x, x, x, a, a, a is in the linear span of xa^2x^2a, xa^2xax and xax^2a^2. From the Jacobi identity we have using (2)

$$\begin{aligned} 0 &= (xa^2)(xa)x + (xa)x(xa^2) + x(xa^2)(xa) \\ &= 3xa^2xax + xax^2a^2 - 3xa^2x^2a \end{aligned} \tag{3}$$

and from the Engel-identity

$$\begin{aligned} 0 &= xax^2a^2 + xaxaxa + xaxa^2x + xa^2x^2a + xa^2xax \\ &= 2xa^2xax + xax^2a^2 + 2xa^2x^2a. \end{aligned} \tag{4}$$

From (2)-(4) it follows that

$$\begin{aligned} xa^2xax &= xaxa^2x = 5xa^2x^2a \\ xax^2a^2 &= -12xa^2x^2a \\ xaxaxa &= xa^2x^2a. \end{aligned} \tag{5}$$

Every product of x, x, x, x, a, a lies in the span of xa^2x^3, xax^2ax and xax^3a. From the Engel identity we have

$$\begin{aligned} 0 &= -ax^4a - ax^3ax - ax^2ax^2 - axax^3 \\ &= xax^3a + xax^2ax + 2xaxax^2 \end{aligned}$$

and we also have

$$\begin{aligned} 0 &= (ax)x(ax^2) \\ &= -xaxax^2 + 2xax^2ax - xax^3a. \end{aligned}$$

It follows that

$$\begin{aligned} xa^2x^3 &= xaxax^2 = -3xax^2ax \\ xax^3a &= 5xax^2ax. \end{aligned} \tag{6}$$

We can now easily complete the prove of the lemma. By Proposition 8 every product of x, x, a, a, a, a is zero. We can therefore assume that a product of x, x, x, x, a, a, a, a ends in xa, ax or aa. By (5) and (6) it is then in the linear span of

$$U_1 = xa^2x^2axa, \quad U_2 = xa^2x^2a^2x, \quad U_3 = xax^2axa^2.$$

Using (5) and (6) we have

$$\begin{aligned} 0 &= xa^2x(xa^2x) \\ &= 2U_1 - 10U_2 + 3U_3 \end{aligned}$$

and using the Engel identity

$$\begin{aligned} 0 &= xa^2x^3a^2 + xa^2x^2axa + xa^2x^2a^2x \\ &\quad + xa^2xa^2x^2 + xa^2xaxax + xa^2xax^2a \\ &= 6U_1 + 6U_2 - 3U_3. \end{aligned}$$

It follows that $U_2 = 2U_1$ and $U_3 = 6U_1$. From the skew-Engel identity we have

$$0 = -10ya^2yay^3 + 10ya^2y^2ay^2 - 5ya^2y^3ay + ya^2y^4a$$

for all y. If we substitute $x + a$ for y we get

$$\begin{aligned} 0 &= -4xa^2x^3a^2 + 11xa^2x^2axa + 5xa^2x^2a^2x - 9xa^2xax^2a - 15xa^2xaxax \\ &= -34U_1 - 70U_2 + 12U_3 = -102U_1 = -6 \cdot 17U_1. \end{aligned}$$

So $U_1 = 0$ since $\operatorname{char} k \neq 17$. □

Lemma 2 *If $a, b \in A$ then every product of x, x, x, x, a, b is in the linear span of $xaxbx^2$ and $xabx^3$. In particular*

$$\begin{aligned} 6xax^2bx &= 9xaxbx^2 - 11xabx^3; \\ 6xax^3b &= -15xaxbx^2 + 5xabx^3; \\ xbxax^2 &= -xaxbx^2 + 2xabx^3; \\ 6xbx^2ax &= -9xaxbx^2 + 7xabx^3; \\ 6xbx^3a &= 15xaxbx^2 - 25xabx^3. \end{aligned}$$

Proof From the Engel and skew-Engel identity we have

$$0 = axbx^3 + ax^2bx^2 + ax^3bx + ax^4b$$

and

$$0 = -10axbx^3 + 10ax^2bx^2 - 5ax^3bx + ax^4b.$$

Solving these together gives the first two identities in the lemma. The rest is easy and is left to the reader. □

Lemma 3 *Every product of x, x, x, a, b, b is in the linear span of $xabxbx$, $xaxb^2x$ and $xaxbxb$. In particular*

$$\begin{aligned} 5xabx^2b &= -11xabxbx + 9xaxb^2x + 15xaxbxb; \\ 5xax^2b^2 &= 6xabxbx - 14xaxb^2x - 20xaxbxb; \\ xb^2xax &= 2xabxbx - xaxb^2x; \\ xbxabx &= 2xabxbx - xaxb^2x; \\ 5xb^2x^2a &= 26xabxbx - 19xaxb^2x - 30xaxbxb; \\ 5xbxaxb &= -22xabxbx + 18xaxb^2x + 25xaxbxb; \\ 5xbxbxa &= 26xabxbx - 19xaxb^2x - 30xaxbxb; \\ 5xbx^2ab &= -27xabxbx + 13xaxb^2x + 10xaxbxb. \end{aligned}$$

Proof From the Engel identity we have

$$0 = axbxbx + axbx^2b + ax^2b^2x + ax^2bxb + ax^3b^2.$$

Then from the skew-Engel identity we have

$$0 = -10ayby^3 + 10ay^2by^2 - 5ay^3by + ay^4b$$

for all y, which implies that

$$0 = -15axbxbx - 9axbx^2b + 5ax^2b^2x + 11ax^2bxb - 4ax^3b^2.$$

Solving these two equations together gives the first two identities. The rest follows easily. □

Lemma 4 *If $a, b \in A$ then every product of x, x, x, x, a, b, b, b is zero.*

Proof We first show that every such product ending in x is zero. Let $V_1 = xabxbxbx$, $V_2 = xaxb^2xbx$ and $V_3 = xaxbxb^2x$. Now every product of x, x, a, b, b, b is zero by Proposition 7. It is then clear by Lemma 3 that all products ending in x are in the linear span of V_1, V_2 and V_3. Now from Lemma 1 we have $5xa^2x^2a^2x = 0$ and linearization gives

$$5xabx^2b^2x = -5xb^2x^2abx.$$

Using Lemma 3 we then get $-11V_1 + 9V_2 + 15V_3 = -26V_1 + 19V_2 + 30V_3$ which implies that

$$0 = 3V_1 - 2V_2 - 3V_3.$$

Since $5xax^2a^3x = 0$ we get by similar reasoning

$$0 = 15V_1 - 5V_2 - 2V_3.$$

If we solve these equation together we have $5V_2 = -13V_3$ and $15V_1 = -11V_3$. Therefore we only have to show that $V_3 = 0$. But

$$\begin{aligned} 0 &= 5xax(xb^3)x \\ &= 6V_1 + V_2 - 35V_3. \end{aligned}$$

If we multiply this by 5 we get $0 = -2 \cdot 3 \cdot 5 \cdot 7 \cdot V_3$. So $V_3 = 0$ since $\operatorname{char} k \notin \{2, 3, 5, 7\}$.

Let $W = xabx^3b^2$. From Lemma 6 we have $xax^3a^3 = 0$ and it follows that

$$6xax^3b^3 + 18xbx^3ab^2 = 0.$$

We now use Lemma 2 and get

$$0 = 30xaxbx^2b^2 - 70W \tag{7}$$

From previous work we have that all products ending in x are zero and that all products are in the linear span of W, U_1, U_2 and U_3 where

$$U_1 = xabxbx^2b, \quad U_2 = xaxb^2x^2b, \quad U_3 = xaxbxbxb, \quad W = xabx^3b^2.$$

Now from Lemma 3 ((xab) commutes with b) we have

$$5x(xab)x^2b^2 = 6x(xab)bxbx - 14x(xab)xb^2x - 20x(xab)xbxb$$

which implies that $5xabx^3b^2 = -20xabx^2bxb$ and then Lemma 3 gives

$$5W = 44U_1 - 36U_2 - 60U_3. \tag{8}$$

From the Engel-identity we then have

$$0 = 5xabxx^2bb + 5xabxxbxb + 5xabxbx^2b$$

which with help of Lemma 3 gives

$$5W = 6U_1 - 9U_2 - 15U_3. \tag{9}$$

Then we also have

$$\begin{aligned} 0 &= 30xax^2(xb^3) \\ &= -300W + 108U_1 - 252U_2 - 360U_3 \end{aligned}$$

which gives

$$50W = 18U_1 - 42U_2 - 60U_3. \tag{10}$$

Now solve equations (8)-(10) together and we have $3U_2 = 17U_1$ and $9U_3 = -23U_1$. We then only have to show that $U_1 = 0$. Now use the Engel identity. We leave the routine calculations to the reader

$$\begin{aligned} 0 &= 30[ax(xb^2)x^2b + ax^2(xb^2)xb + 2ax^3(xb^2)b + axbx^2(xb^2) + ax^2bx(xb^2)] \\ &= 400W - 180U_1 + 300U_2 + 660U_3 \\ &= -4 \cdot 25 \cdot 7 \cdot U_1. \end{aligned}$$

and $U_1 = 0$ since char $k \notin \{2, 5, 7\}$. □

Lemma 5 *If $a, b, c, d \in A$ and then every product of x, x, x, x, a, b, c, d, with last two elements from A, is zero.*

Proof From Lemmas 2 and 4 we have

$$0 = 6xax^3bcd = -15xaxbx^2cd + 5xabx^3cd$$

and

$$0 = 6xbx^3acd = 15xaxbx^2cd - 25xabx^3cd.$$

Therefore $xaxbx^2cd = xabx^3cd = 0$. □

Let $a, b, c \in A$ and let

$$\begin{array}{lll} V_1 = xabxbxcx, & V_2 = xaxb^2xcx, & V_3 = xaxbxbcx, \\ U_1 = xabxbxxc, & U_2 = xaxb^2xxc, & U_3 = xaxbxbxc. \end{array}$$

It follows from previous lemmas that all products ending in xc or cx are in the linear span of these products.

Lemma 6 *We have* $U_2 = 4U_1$, $U_3 = -U_1$, $V_2 = 4V_1$ *and* $2V_3 = -5V_1$

Proof From Lemma 4 we have

$$(xa)(xb^2)xcx = -xb^2(xa)xcx = 2xab(xb)xcx$$

and therefore

$$xaxb^2xcx - 2xabxbxcx = 2xabxbxcx$$

which implies that $V_2 = 4V_1$. Similarly we get $U_2 = 4U_1$. Then from Lemmas 3 and 4 we have

$$\begin{aligned} 0 &= 5xax^2b^2cx \\ &= 6V_1 - 14V_2 - 20V_3 \\ &= -50V_1 - 20V_3 \end{aligned}$$

which implies that $2V_3 = -5V_1$. Finally from the Jacoby identity and Lemmas 3 and 5 we have

$$\begin{aligned} 0 &= 10 \cdot [(ax^2)(bx^2)bc + (bx^2)b(ax^2)c + b(ax^2)(bx^2)c] \\ &= 10[(bx^2)b(ax^2)c - (ax^2)b(bx^2)c] \\ &= -140U_1 - 140U_3. \quad \square \end{aligned}$$

Lemma 7 *If $a, b \in A$ then all products of x, x, x, x, a, a, b, b are zero.*

Proof We let U_i' and V_i' be defined as U_i and V_i with c replaced by a. It follows from last lemma that it is sufficient to show that the products U_1' and V_1' are zero. From the Engel identity and Lemmas 3 and 5 we have

$$\begin{aligned} 0 &= 5xb^2x^2axa + 5xb^2x^2a^2x + 5xb^2xax^2a + 5xb^2xaxax \\ &= 15V_1' - 30U_1' \end{aligned} \tag{11}$$

and from the skew-Engel identity (see proof of Lemma 3) (11) and previous lemmas we have

$$\begin{aligned} 0 &= 5 \cdot [11xb^2ax^4 + 5xb^2x^2a^2x - 9xb^2xax^2a - 15xb^2xaxax] \\ &= 420U_1' \end{aligned}$$

and thus $U_1' = 0$ since $\mathrm{char}\, k \notin \{2, 3, 5, 7\}$. Hence also $V_1' = 2U_1' = 0$. □

Lemma 8 *If $a, b, c \in A$ then all products of x, x, x, x, a, b, b, c are zero.*

Proof From Lemma 5 we have that all products with last two elements from A are zero. From Lemma 3 we have

$$5xabx^2bcx = -11V_1 + 9V_2 + 15V_3.$$

Lemma 7 therefore implies that

$$0 = -22V_1 + 18V_2 + 30V_3 = -25V_1$$

so all products ending in cx are zero. By symmetry all products ending in ax are also zero. By Lemma 7 we have $xacxbxbx = 0$ and $xbxacxbx = 0$. Also $2xabxcxbx = -xb^2xcxax = 0$, where the first identity comes form Lemma 4. Similarly $2xaxbcxbx = -xaxb^2xcx = 0$. Since all products of x, x, a, b, b, c are zero we have that all products ending in x are 0.

Now from the Engel-identity we have

$$\begin{aligned} 0 = \; & ax(xb^2)x^2c + ax^2(xb^2)xc + ax^3(xb^2)c \\ & +axcx^2(cxb^2) + ax^2cx(xb^2) + ax^3c(xb^2). \end{aligned} \tag{12}$$

Then we have from the skew-Engel identity

$$0 = -10au(xb^2)u^3 + 10au^2(xb^2)u^2 - 5au^3(xb^2)u + au^4(xb^2).$$

for all $u \in L_1$. That implies that

$$\begin{aligned} 0 &= -10ax(xb^2)x^2c + 10ax^2(xb^2)xc - 5ax^3(xb^2)c \\ &\quad +axcx^2(xb^2) + ax^2cx(xb^2) + ax^3c(xb^2). \end{aligned} \tag{13}$$

Now (12) and (13) together give

$$\begin{aligned} 0 &= 5 \cdot (11ax(xb^2)x^2c - 9ax^2(xb^2)xc + 6ax^3(xb^2)c) \\ &= 128U_1 - 52U_2 - 150U_3 \\ &= 70U_1 \end{aligned}$$

so $U_1 = 0$ since char $k \notin \{2, 5, 7\}$. All products ending in xa are also zero by symmetry. The only products ending in xb that are not obviously zero are $xaxbxcxb$ and $xaxcxbxb$. But

$$0 = -(xb^2)(xaxcx) = xaxcx(xb^2) = -2xaxcxbxb = 2xaxbxcxb + 2xaxbxbxc.$$

So we have proved that every product of x, x, x, x, a, b, b, c is zero for all $a, b \in A$. □

Proof of Proposition 8 It follows from last lemma that if we interchange two of a, b, c, d in a product of x, x, x, x, a, b, c, d then the sign changes. This implies that if two of a, b, c, d occur in a row in such a product, then it must be zero. Therefore every product of x, x, x, x, a, b, c, d is a multiple of $xaxbxcxd$. But from the Jacobi identity we have

$$\begin{aligned} 0 &= (xaxb)xcxd + xc(xaxb)xd + c(xaxb)xxd \\ &= xaxbxcxd + xc(xax)bxd \\ &= xaxbxcxd + 2xcxaxbxd \\ &= 3xaxbxcxd. \end{aligned}$$

To complete the proof it is sufficient to show that for all $a, b, c, d, e \in A$ we have that every product of x, x, x, a, b, c, d, e is zero.

By Proposition 7 we have that every product with two occurrences of x and four elements of A_∞ is zero. Therefore we have

$$xa^2xcxd^2 = xa^2xc(xd^2) = 0 \tag{14}$$

for all $a, c, d \in A$ and then

$$\begin{aligned} 0 &= xa^2x(xc^3) \\ &= xa^2x^2c^3 - 3xa^2xcxc^2 \\ &= xa^2x^2c^3 \text{ (by (14))}. \end{aligned}$$

From this we have

$$xabxcxde = xabx^2cde = 0$$

so every product starting in xab is zero. Then

$$xaxbcxd^2 = xa(xbc)xd^2 = 0$$

and therefore

$$0 = xaxb(xc^3) = xaxbxc^3.$$

Therefore $xaxbcxde = xaxbxcde = 0$. □

5 The Proof

Assume in the following that char $k > 7$ or char $k = 0$. We shall consider the case char $k = 7$ in the end of the section. Before we prove the main theorem, we need to carry out reduction step 4 by proving the following proposition. In the proof of this proposition $u = v$ will mean that $u = v$ modulo I_4

Proposition 9 *If U is a product in L which includes either 5 occurrences of x or 3 elements from A, then $U = 0$.*

Lemma 9 *If $a, b, c \in A$ then every product of x, x, x, a, b, c is zero.*

Proof We first show that every product of x, x, x, a, b, b is zero. We can assume that the product starts in xa. The next letter must be x. Because

$$0 = xa(xb^2) = xaxb^2$$

all such products are in the span of xax^2b^2 and $xaxbxb$. From the Engel and Jacobi identities we have

$$\begin{aligned} 0 &= (xax)(xb)b + (xb)b(xax) + b(xax)(xb) \\ &= xax^2b^2 - 2xaxbxb \end{aligned}$$

and

$$0 = xax^2b^2 + xaxbxb.$$

This implies that $xax^2b^2 = xaxbxb = 0$ and we have thus proved that every product of x, x, x, a, b, b is zero.

Now consider a product of x, x, x, a, b, c. It follows from the previous work that if we interchange two of a, b, c the sign changes. If we have two of a, b, c in a row, we therefore get a zero. Therefore every product of x, x, x, a, b, c is a multiple of $xaxbxc$. But from the Jacobi identity we have

$$\begin{aligned} 0 &= (xaxb)xc + xc(xaxb) + c(xaxb)x \\ &= xaxbxc + xcxaxb + xcxaxb \\ &= 3xaxbxc. \end{aligned}$$

□

Lemma 10 *If $a, b, c \in A$ then every product of x, x, x, x, a, b, c is zero.*

Proof The proof follows similar pattern as the proof of last lemma . We first show that every product of x, x, x, x, a, a, b is 0. By Lemma 9 they clearly lie in the span of

$$U_1 = bx^2ax^2a, \quad U_2 = bx^3axa, \quad U_3 = bx^4a^2.$$

From the Engel identity we have

$$0 = bx^2ax^2a + bx^3axa + bx^4a^2$$

and we also have

$$0 = bx^3(xa^2) = bx^4a^2 - 2bx^3axa.$$

This implies that $U_3 = 2U_2$ and $U_1 = -3U_2$. Now we use the Engel identity again. We have

$$\begin{aligned} 0 &= 2(xa)x^3ab + (xa)x^2axb + (xa)x^2bxa + (xa)xax^2b + (xa)xbx^2a \\ &= 2b(ax^4)a + b(ax^3ax) + b(ax^3)xa + b(ax^2)x^2a \\ &= 19U_1 - 12U_2 + 3U_3 \\ &= -7 \cdot 3^2U_2. \end{aligned}$$

Since $\operatorname{char} k \notin \{3, 7\}$ we have $U_2 = 0$ and we have thus shown that every product of x, x, x, x, a, a, b is zero.

Now consider a product of x, x, x, x, a, b, c. By the preceeding work it follows that if we interchange two of a, b or c then the sign changes. By the same argument as was used in last lemma, we have that the products are in the linear span of $U_1 = ax^2bx^2c$ and $U_2 = ax^3bxc$. (Note that the product can not end in an x by last lemma). From the Jacobi and Engel identities we then have

$$\begin{aligned} 0 &= (ax^2)(bx^2)c + (bx^2)c(ax^2) + c(ax^2)(bx^2) \\ &= 3U_1 - 2U_2 \end{aligned}$$

and

$$\begin{aligned} 0 &= ax(xc)x^2b + ax^2(xc)xb + ax^3(xc)b + ax^2bx(xc) + ax^3b(xc) \\ &= U_1 + U_2. \end{aligned}$$

It follows that $U_1 = U_2 = 0$. □

Lemma 11 *If $a, b, c \in A$ then every product of x, x, x, x, x, a, b, c is zero.*

Proof As in previous two lemmas we first prove that every product of x, x, x, x, x, a, a, b is zero. From Lemma 10 it follows that they are in the span of

$$U_1 = bx^2ax^3a, \quad U_2 = bx^3ax^2a, \quad U_3 = bx^4axa.$$

We have

$$0 = bx^4(xa^2) = -2bx^4axa = U_3.$$

Then the Engel and skew-Engel identities give us

$$\begin{aligned} 0 &= (bx^2ax^3 + bx^3ax^2)a \\ &= U_1 + U_2 \end{aligned}$$

and

$$\begin{aligned} 0 &= (10bx^2ax^3 - 5bx^3ax^2 + bx^4ax)a \\ &= 5(2U_1 - U_2). \end{aligned}$$

It then follows that $U_1 = U_2 = U_3 = 0$ and every product of x, x, x, x, x, a, a, b is zero.

Now consider a product of x, x, x, x, x, a, b, c. It now follows that if we interchange two of a, b or c then the sign changes. Then the product is in the linear span of

$$U_1 = ax^2bx^3c, \quad U_2 = ax^3bx^2c, \quad U_3 = ax^4bxc$$

and as above we have from the Engel and skew-Engel identity

$$U_1 + U_2 + U_3 = 0$$

and

$$10U_1 - 5U_2 + U_3 = 0.$$

It follows that $2U_2 = 3U_1$ and $2U_3 = -5U_1$. From the Jacobi identity we have

$$\begin{aligned} 0 &= (ax^3)(bx^2)c + (bx^2)c(ax^3) + c(ax^3)(bx^2) \\ &= -U_1 + 2U_2 - 2U_3 \\ &= 7U_1. \quad \square \end{aligned}$$

Proof of Proposition 9 We first prove that all products including $a, b, c \in A$ are zero. We will use induction on the number of occurrences of x in the product. This holds obviously when this number is 0 or 1. Because $ax^2bc = ax(xbc) = 0$, this is also true when the number of x's is 2. Lemmas 9, 10 and 11 show that this is also true if the number of occurrences of x is

3, 4 or 5. So suppose the number of x's, say $i+6$, is greater than 5 and the statement holds if the number is less than $i+6$. Let

$$U_1 = ax^2bx^ix^4c, \quad U_2 = ax^3bx^3x^ic, \quad U_3 = ax^4bx^2x^ic.$$

As before we get from the Engel identity and skew-Engel identity that $2U_2 = 3U_1$ and $2U_3 = -5U_1$. But from the Engel identity we have also

$$\begin{aligned} 0 &= ax^2bx^icx^4 + ax^2bx^ixcx^3 + \cdots + ax^2bx^ix^4c \\ &= ax^2bx^ix^4c. \end{aligned}$$

So $U_1 = 0$ and the induction statement holds.

Next we prove that all products with 5 occurrences of x are zero. So suppose we have a product with 5 occurrences of x and r elements from A. If r is 1 then the product is clearly zero by the Engel identity. If $r \geq 3$ then we have just proved that the product is zero. So we can assume that we have a product of x, x, x, x, x, a, b and $a, b \in A$. From the Engel and skew-Engel identity we have

$$0 = ax^2bx^3 + ax^3bx^2 + ax^4bx$$

and

$$0 = 10ax^2bx^3 - 5ax^3bx^2 + ax^4bx.$$

Which implies that $2ax^3bx^2 = 3ax^2bx^3$ and $2ax^4bx = -5ax^2bx^3$. Then

$$\begin{aligned} 0 &= a(bx^5) \\ &= 5(2ax^2bx^3 - 2ax^3bx^2 + ax^4bx) \\ &= 5/2 \cdot (-7)ax^2bx^3. \end{aligned}$$

Since $\operatorname{char} k \neq 7$ we therefore have that all these products are zero and the proposition is therefore proved. $\square$

Next we see what information about the nilpotency class of I_x we can deduce from this. We have seen that every product which includes either 5 occurrences of x or 3 elements of A lies in I_4. By Proposition 4 we have that every product that includes 12 elements of A lies in I_3. But we also have that every product which has 13 occurrences of x lies in I_3. Let us see why this is true.

Recall the definition of A_∞ from the proof of Proposition 4. If we have a product u including $e \geq 5$ occurrences of x and t elements from A_∞ $b_1, b_2, \ldots, b_t$. Then u can be written as a linear combination of elements of the form

$$xb_ib_ju_1u_2\cdots u_{e+t-3}$$

where $u_1, u_2, \ldots, u_{e+t-3}$ is some permutation of $\underbrace{x, \ldots, x}_{e-1}, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots,$ $b_{j-1}, b_{j+1}, \ldots, b_{t-2}$. But then each summand is a product including $e-1$ occurrences of x and $t-1$ elements form A_∞. Namely xb_ib_j and $t-2$ of the elements of A_∞ which we started with. Now let u be a product with 13 occurrences of x and at most 11 elements of A. By using the argument above repeatedly we see that u can be written as a linear combination of products each including 4 x's and at most 2 elements from A_∞. But since 9 x's are involved in these two elements, one of them must include 5 occurrences of x and therefore lie in I_3 by Propostition 4. Hence $u \in I_3$.

So every product with either 13 x's or 12 elements from A is in I_3. By Proposition 3 it follows that every product including 36 elements of A is in I_2. But we also have that every product including

$$13 + [35/3] = 24$$

occurrences of x is in I_2. The reason for this is as follows.

Recall that $C = A \cup \{xa_ia_ja_r |\ a_i, a_j, a_r \in A\}$. We have seen that the elements in C commute modulo I_2. If u is a product including $e \geq 13$ occurrences of x and t elements of C, $c_1, c_2, \ldots, c_t$, then u can be written as a linear combination of elements of the form

$$xc_ic_jc_ru_1u_2\cdots u_{e+t-4}$$

where $u_1, u_2, \ldots, u_{e+t-4}$ is some permutation of $\underbrace{x, \ldots, x}_{e-1}, c_1, \ldots, c_{i-1}, c_{i+1}, \ldots,$ $c_{j-1}, c_{j+1}, \ldots, c_{t-3}$. But then each summand is a product including $e-1$ occurrences of x and $t-2$ elements from C. Now let u then be a product with $13 + [35/3] = 24$ occurrences of x and at most 35 elements of A. Using the above argument repeatedly we see that u can be written as a linear combination of products each involving 13 x's and at most 13 elements of C, with

11 of the elements of C having the form $xa_ia_ja_r$. One further application of the above argument shows that $u \in I_2$.

So every product including either 24 x's or 36 elements of A is in I_2. By Proposition 2 we have that every product with $4 \cdot 36 = 144$ elements of A is 0. But by a similar argument as above we also have that every product including

$$24 + [143/4] = 59$$

occurrences of x is 0. Hence $I_x^{59} = \{0\}$ in L_1 if $\operatorname{char} k \neq 2, 3, 5, 7$.

Using a computer program, the nilpotent quotient algorithm was applied for the case when $\operatorname{char} k = 7$ and it was observed that every product with either 4 elements of A or 7 x's is in I_4. It follows from similar argument as for the other characteristics that every product with either 16 elements of A or $7+11 = 18$ x's is in I_3. (At most 11 x's can be involved in elements of A_∞. If we would have three of the form $(xab)(xc)(xd)$ and one of the form $(xab)(xc)$ then all the 15 elements of A would have been used.) Then it follows from Proposition 3 that every product including $3 \cdot 16 = 48$ elements of A or

$$18 + [47/3] = 33$$

x's. Then finally we have form Proposition 2 that every product which includes either $48 \cdot 4 = 192$ elements of A or

$$33 + [191/4] = 80$$

x's is zero. Hence $I_x^{80} = \{0\}$ if $\operatorname{char} k = 7$. Applying Proposition 1 we then have.

Theorem 1 *Let L be an Engel-5 Lie algebra with r generators. If $\operatorname{char} k \neq 2, 3, 5, 7$ then the nilpotency class c is at most $59r$. If $\operatorname{char} k = 7$ then $c \leq 80r$*

From Theorem Z1 it follows that there exist some number n_0 such that every Engel-5 Lie algebra is nilpotent if the underlying field has characteristic greater than n_0. Applying some representation theory of the symmetric group one can use the theorem above to get some information about the global nilpotency. One can find this method in a more general form than is needed here in the proof of Theorem Z1 (see [4] for an accessible description). Unfortunately it would take too much space for us to go into this here. We therefore only state the corollary.

Corollary 1 *If L is an Engel-5 Lie algebra over a field k with $\mathrm{char}\, k > 195113$ or $\mathrm{char}\, k = 0$ then it is nilpotent of class not more than 975563.*

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