Proposition 7.3 *If $\alpha : V \to V$ is self-adjoint, then*

1) Every eigenvalue of α is real.
2) Eigenvectors with distinct eigenvalues are orthogonal.

Proof 1) Suppose that $\alpha(v) = \lambda v$ with $v \neq 0$. Then

$$\begin{array}{rcl} \langle \alpha(v), v \rangle & = & \langle v, \alpha(v) \rangle \\ & \Downarrow & \\ \langle \lambda v, v \rangle & = & \langle v, \lambda v \rangle \\ & \Downarrow & \\ \overline{\lambda}\langle v, v \rangle & = & \lambda \langle v, v \rangle. \end{array}$$

As $\langle v, v \rangle \neq 0$ it follows that $\overline{\lambda} = \lambda$ and thus $\lambda \in \mathbb{R}$.

2) If $\alpha(v_i) = \lambda_i v_i$ where $\lambda_1 \neq \lambda_2$ then

$$\lambda_1 \langle v_1, v_2 \rangle = \langle \alpha(v_1), v_2 \rangle = \langle v_1, \alpha(v_2) \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

(notice that we have used the fact that $\lambda_1 \in \mathbb{R}$). It follows that $\langle v_1, v_2 \rangle = 0$ as $\lambda_1 \neq \lambda_2$. $\Box$

Corollary 7.4 *If A is a hermitian $n \times n$ matrix then*

$$\Delta_A(t) = (\lambda_1 - t) \cdots (\lambda_n - t)$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Proof The corresponding linear map $\alpha : \mathbb{C}^n \to \mathbb{C}^n$, given by $\alpha(v) = Av$, is self-adjoint and it follows from the last proposition that all the eigenvalues are real. $\Box$.

Remark. (Version for self-adjoint maps). If we state last corollary in terms of linear maps, we get the following: if $\alpha : V \to$ is self-adjoint then

$$\Delta_\alpha(t) = (\lambda_1 - t) \cdots (\lambda_n - t)$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

If V is an inner product space, then any subspace W is also an inner product space with respect to the same inner product. Furthermore, if $\alpha : V \to V$ is self-adjoint and W is α-invariant, then $\alpha|_W$ is self-adjoint as well.

Lemma 7.5 *If $\alpha : V \to V$ is self-adjoint and W is α-invariant, then*

$$W^\perp = \{v \in V : \langle v, w \rangle = 0 \quad \forall w \in W\}$$

is also α-invariant.

Proof Let $v \in W^\perp$. We want to show that $\alpha(v) \in W^\perp$ but for any $w \in W$ we have

$$\langle \alpha(v), w \rangle = \langle v, \alpha(w) \rangle$$

which is zero as $\alpha(w) \in W$ and $v \in W^\perp$. This shows that $\alpha(v) \in W^\perp$ as required. $\Box$.

The next result is the main result of this chapter and tells us that a self-adjoint operator $\alpha : V \to V$ on a finite dimensional inner product space is always diagonalisable. In fact more is true as we will see that one can moreover choose the basis of eigenvectors to be orthonormal.

Theorem 7.6 *(Spectral Theorem) Let $\alpha : V \to V$ be a self-adjoint linear operator on an n-dimensional inner product space. Then there is an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of α.*

Proof (Not examinable). We first prove by induction on n that we can find a orthonormal basis of eigenvectors. For the case $n = 1$ pick a orthonormal vector (which is then ofcourse an eigenvector).

Now suppose that $n > 1$ and the result is true for all V with $\dim V < n$. We know from Corollary 7.4 that $\Delta_\alpha(t)$ has a real eigenvalue λ. Let $W_1 = E_\alpha(\lambda)$ be the corresponding eigenspace and choose any orthonormal basis for W_1. By Lemma 7.5 we know that $W_2 = W_1^\perp$ is α-invariant. The restriction $\alpha|_{W_2}$ is then self-adjoint and by induction hypothesis, we can choose a orthonormal basis of or W_2 consisting of eigenvectors of $\alpha|_{W_2}$, i.e. of α. Finally, $V = W_1 \oplus W_2$ is an orthogonal direct sum decomposition, so we can combine the two bases above to obtain an orthonormal basis of V. □

Corollary 7.7 *If $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues with respect to the ON basis $v_1, \ldots, v_n$, then α can be written explicitly by the formula*

$$\alpha(w) = \sum_{k=1}^{n} \lambda_k \langle v_k, w \rangle v_k.$$

Proof The explicit formula for α follows from the fact that

$$w = \sum_{k=1}^{n} \langle v_k, w \rangle \, v_k.$$

It then follows that

$$\begin{aligned} \alpha(w) &= \sum_{k=1}^{n} \langle v_k, w \rangle \, \alpha(v_k) \\ &= \sum_{k=1}^{n} \langle v_k, w \rangle \, \lambda \, \alpha(v_k). \end{aligned}$$

Theorem 7.8 *(The matrix version of the Spectral Theorem)*

(1) If A is a symmetric $n \times n$ matrix over $\mathbb{R}$ then there is an orthogonal invertible real matrix P (i.e. $P^{-1} = P^t$) such that $P^{-1}AP$ is diagonal.

(2) If A is a hermitian $n \times n$ matrix over $\mathbb{C}$ then there is a unitary complex matrix P (i.e. $P^{-1} = \overline{P}^t$) such that $P^{-1}AP$ is diagonal.

Proof (Not examinable) In both cases the linear map $\alpha : \mathbb{F}^n \to \mathbb{F}^n$ mapping v to Av is self-adjoint. By the Spectral Theorem we have an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors. Let P be the base change matrix from the orthonormal basis to the standard basis $e_1, \ldots, e_n$. The columns of P are the coordinates of the orthonormal vectors $v_1, \ldots, v_n$ expressed in $e_1, \ldots, e_n$. This translates into $P^tP = I$ when P is real and $\overline{P}^tP = I$ when P is complex. □

Example. Consider the hermitian matrix

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

We have $\Delta_A(t) = (t-1)^2 - 1 = t(t-2)$ and the eigenvalues are 0 and 2. We next determine the eigenspaces. Firstly we solve $Av = 0$. That is

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that has the solution $\mathbb{C}\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Then we solve $(A - 2I)v = 0$ or

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that has the solution $\mathbb{C}\begin{pmatrix} i \\ 1 \end{pmatrix}$.

In order to get an orthonormal basis we need to normalise these eigenvectors. This gives thus basis vectors

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

So if we let

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix},$$

then $P^{-1}AP = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.