7 Linear operators on inner product spaces

I. Inner product spaces

We start by recalling what you have already seen in the course Algebra 1.

Definition. An *inner product space* is a vector space V over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ satisfying:

(1) (linearity) $\langle u, \lambda_1 v_1 + \lambda_2 v_2\rangle = \lambda_1\langle u, v_1\rangle + \lambda_2\langle u, v_2\rangle$.
(2) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
(3) (positive definite) $\|v\|^2 = \langle v, v\rangle > 0$ for $v \neq 0$.

Remark. Notice that $\langle \lambda u, v\rangle = \overline{\langle v, \lambda u\rangle} = \overline{\lambda\langle v, u\rangle} = \overline{\lambda}\overline{\langle v, u\rangle} = \overline{\lambda}\langle u, v\rangle$.

Matrix version. Consider a fixed basis $v_1, \ldots, v_n$ for V. If $u = \sum_i x_i v_i$ and $v = \sum_i y_i v_i$ then

$$\langle u, v\rangle = \langle x_1 v_1 + \cdots + x_n v_n, y_1 v_1 + \cdots + y_n v_n\rangle = \sum_{i,j} \bar{x}_i y_j \langle v_i, v_j\rangle.$$

Or in other words

$$\langle u, v\rangle = \overline{x}^t B y$$

where

$$x = \begin{pmatrix} x_1 \\ \ddots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \ddots \\ y_n \end{pmatrix},$$

and $B = (b_{ij})$ where $b_{ij} = \langle v_i, v_j\rangle$. The matrix B has the following properties:

(I) $\overline{B}^t = B$ (as $\langle v_j, v_i\rangle = \overline{\langle v_i, v_j\rangle}$).
(II) B is positive definitive (that is $\overline{x}^t B x > 0$ if $x \neq 0$).

For the given basis $v_1, \ldots, v_n$ there is a 1-1 correspondence between inner products on V and matrices B satisfying (I) and (II). A matrix satisfying (I) is said to be *symmetric* if we are working over $\mathbb{R}$ and *hermitian* if we are working over $\mathbb{C}$.

Recall that by Gram-Schmidt, every finite dimensional inner product space has an orthonormal basis $v_1, v_2, \ldots, v_n$, that is a basis satisfying

$$\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Or equivalently the representing matrix A is the identity matrix I. Then we get the familiar simple formula for the inner product

$$\langle u, v\rangle = \bar{X}^t Y.$$

II. The adjoint

Definition. An *adjoint* of a linear map $\alpha : W \to V$ is a linear map $\alpha^* : V \to W$ such that

$$\langle v, \alpha(w)\rangle = \langle \alpha^*(v), w\rangle.$$

Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be fixed orthonormal bases for V and W respectively.

Lemma 7.1 *Supposing that α^* exists. If $A = (a_{ij})$ is the matrix representing α with respect to the bases above. Then $\bar{A}^t$ is the matrix representing α^*.*

Proof Suppose that $B = (b_{ij})$ is the matrix representing α^*. Then

$$\alpha^*(v_j) = b_{1j}w_1 + \cdots + b_{ij}w_i + \cdots + b_{nj}w_n.$$

It follows that

$$b_{ij} = \langle w_i, \alpha(v_j)\rangle = \langle \alpha^*(w_i), v_j\rangle = \overline{\langle v_j, \alpha^*(w_i)\rangle} = \overline{a_{ji}}.$$

This finishes the proof. □.

Lemma 7.2 *α^* always exists and is unique.*

Proof (Uniqueness). If α^* exists then it is unique. This is because if $\langle \alpha^* v, w\rangle = \langle v, \alpha w\rangle = \langle \alpha' v, w\rangle$ for all $w \in V$ then $\langle \alpha^* v - \alpha' v, w\rangle = 0$ for all $w \in V$ and therefore in particular $\|\alpha^* v - \alpha' v\|^2 = \langle \alpha^* v - \alpha' v, \alpha^* v - \alpha' v\rangle = 0 \Rightarrow \alpha^* v = \alpha' v$.

(Existence). Suppose V, W are finite dimensional with orthonormal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ respectively. Suppose that A is the matrix representing α with respect to these bases. Let $\alpha^* : V \to W$ be the linear map defined by the matrix $\overline{A}^t$. Let us see that α^* is the adjoint of α. Suppose that

$$v = \sum_i x_i v_i, \quad w = \sum_i y_i w_i,$$

then (for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$.

$$\begin{aligned} \langle \alpha^* v, w\rangle &= \overline{(\overline{A}^t x)}^t I y \\ &= \overline{x}^t A y \\ &= \langle v, \alpha w\rangle. \end{aligned}$$

III. Self-adjoint operators

Definition. An operator $\alpha : V \to V$ on an inner product space is *self-adjoint* if $\alpha^* = \alpha$. In other words if

$$\langle \alpha(v), w\rangle = \langle v, \alpha(w)\rangle$$

for all $v, w \in V$.

Remark. If we choose orthonormal basis for V then we have seen that if A is the matrix for α with respect to this basis then $\overline{A}^t$ is the matrix for α^*. So the assumption that α is self-adjoint implies that $\overline{A}^t = A$. So A is symmetric if we are working over $\mathbb{R}$ and A is hermitian if we are working over $\mathbb{C}$.