I. Inner product spaces

We start by recalling what you have already seen in the course Algebra 1.

Definition. An inner product space is a vector space \( V \) over \( \mathbb{F} \) (\( \mathbb{R} \) or \( \mathbb{C} \)) with an inner product \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{F} \) satisfying:

1. (linearity) \( \langle u, \lambda_1 v_1 + \lambda_2 v_2 \rangle = \lambda_1 \langle u, v_1 \rangle + \lambda_2 \langle u, v_2 \rangle \).
2. \( \langle u, v \rangle = \overline{\langle v, u \rangle} \).
3. (positive definite) \( \|v\|^2 = \langle v, v \rangle > 0 \) for \( v \neq 0 \).

Remark. Notice that \( \langle \lambda u, v \rangle = \overline{\langle v, \lambda u \rangle} = \overline{\lambda \langle v, u \rangle} = \lambda \langle v, u \rangle \).

Matrix version. Consider a fixed basis \( v_1, \ldots, v_n \) for \( V \). If \( u = \sum_i x_i v_i \) and \( v = \sum_i y_i v_i \) then

\[
\langle u, v \rangle = \langle x_1 v_1 + \cdots + x_n v_n, y_1 v_1 + \cdots + y_n v_n \rangle = \sum_{i,j} \bar{x}_i y_j \langle v_i, v_j \rangle.
\]

Or in other words

\[
\langle u, v \rangle = \bar{x}^t B y
\]

where

\[
x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},
\]

and \( B = (b_{ij}) \) where \( b_{ij} = \langle v_i, v_j \rangle \). The matrix \( B \) has the following properties:

1. \( B^t = B \) (as \( \langle v_j, v_i \rangle = \langle v_i, v_j \rangle \)).
2. \( B \) is positive definitive (that is \( \bar{x}^t B x > 0 \) if \( x \neq 0 \)).

For the given basis \( v_1, \ldots, v_n \) there is a 1-1 correspondence between inner products on \( V \) and matrices \( B \) satisfying (I) and (II). A matrix satisfying (I) is said to be symmetric if we are working over \( \mathbb{R} \) and hermitian if we are working over \( \mathbb{C} \).

Recall that by Gram-Schmidt, every finite dimensional inner product space has an orthonormal basis \( v_1, v_2, \ldots, v_n \), that is a basis satisfying

\[
\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Or equivalently the representing matrix \( A \) is the identity matrix \( I \). Then we get the familiar simple formula for the inner product

\[
\langle u, v \rangle = \bar{X}^t Y.
\]

II. The adjoint

Definition. An adjoint of a linear map \( \alpha : W \to V \) is a linear map \( \alpha^* : V \to W \) such that

\[
\langle v, \alpha(w) \rangle = \langle \alpha^*(v), w \rangle.
\]
Let \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) be fixed orthonormal bases for \( V \) and \( W \) respectively.

**Lemma 7.1** Supposing that \( \alpha^* \) exists. If \( A = (a_{ij}) \) is the matrix representing \( \alpha \) with respect to the bases above. Then \( A^t \) is the matrix representing \( \alpha^* \).

**Proof** Suppose that \( B = (b_{ij}) \) is the matrix representing \( \alpha^* \). Then

\[
\alpha^*(v_j) = b_{1j}w_1 + \cdots + b_{ij}w_i + \cdots + b_{nj}w_n.
\]

It follows that

\[
b_{ij} = \langle w_i, \alpha(v_j) \rangle = \langle \alpha^*(w_i), v_j \rangle = \langle v_j, \alpha^*(w_i) \rangle = a_{ji}.
\]

This finishes the proof. \( \square \).

**Lemma 7.2** \( \alpha^* \) always exists and is unique.

**Proof** (Uniqueness). If \( \alpha^* \) exists then it is unique. This is because if \( \langle \alpha^* v, w \rangle = \langle v, \alpha w \rangle = \langle \alpha' v, w \rangle \) for all \( w \in V \) then \( \langle \alpha^* v - \alpha' v, w \rangle = 0 \) for all \( w \in V \) and therefore in particular \( \| \alpha^* v - \alpha' v \|^2 = \langle \alpha^* v - \alpha' v, \alpha^* v - \alpha' v \rangle = 0 \Rightarrow \alpha^* v = \alpha' v \).

(Existence). Suppose \( V, W \) are finite dimensional with orthonormal bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) respectively. Suppose that \( A \) is the matrix representing \( \alpha \) with respect to these bases. Let \( \alpha^*: V \rightarrow W \) be the linear map defined by the matrix \( A^t \). Let us see that \( \alpha^* \) is the adjoint of \( \alpha \). Suppose that

\[
v = \sum_i x_i v_i, \quad w = \sum_i y_i w_i,
\]

then (for \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) and \( y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \))

\[
\langle \alpha^* v, w \rangle = \langle A^t x, y \rangle = \langle x, Ay \rangle = \langle v, \alpha w \rangle.
\]

**III. Self-adjoint operators**

**Definition.** An operator \( \alpha : V \rightarrow V \) on an inner product space is **self-adjoint** if \( \alpha^* = \alpha \). In other words if

\[
\langle \alpha(v), w \rangle = \langle v, \alpha(w) \rangle
\]

for all \( v, w \in V \).

**Remark.** If we choose orthonormal basis for \( V \) then we have seen that if \( A \) is the matrix for \( \alpha \) with respect to this basis then \( A^t \) is the matrix for \( \alpha^* \). So the assumption that \( \alpha \) is self-adjoint implies that \( A^t = A \). So \( A \) is symmetric if we are working over \( \mathbb{R} \) and \( A \) is hermitian if we are working over \( \mathbb{C} \).