Contests with a Jury*

Javier Rivas
University of Bath†

November 11, 2024

Abstract

We study contests with a jury, i.e. a set of agents each of whom selects a player, and the winner of the contest is the player selected by the most jurors. Contrary to contests with multiple battlefields, where the problem involves resource allocation, we find that the presence of a jury creates a novel pivotality effect, marginal effort is only positive in scenarios where the jury is split among candidates, i.e. there is a pivotal juror. Increasing jury size decreases the probability that there is a pivotal juror and therefore decreases effort. However, there is also an uncertainty effect: increasing jury size may increase the probability that the player who exerted the most effort wins, which can lead to players exerting more effort. We explore optimal jury size and how a jury affects contests.

JEL Classification: C72, D72, D74, D82.

Keywords: Contest Theory; Juries; Rent Dissipation; Tullock Contest; Voting.

---

*I would like to thank Carmen Beviá, Christian Ewerhart, Elnaz Bajoori, Francesco Giovannoni, Jaideep Roy, Jörg Franke, Luis Corchón, Matthias Dahm, Santiago Oliveros, Subhasish Chowdhury, and Zenan Wu.

†Department of Economics, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom. j.rivas@bath.ac.uk, http://people.bath.ac.uk/fjrr20/.

1 Introduction

In competitive environments, whether in political contests, sports events, funding competitions, or procurement contracts, the structure of the process plays a crucial role in determining the effort exerted by participants. Central to this structure is the process by which a winner is chosen, typically through a panel of jurors, such as voters, academic referees, or government officials. Previous research on contests has primarily focused on single-juror decision-making through a contest success function (see the seminal work of Tullock (1980) and, more recently, Corchón and Serena (2018) and references therein). This paper aims to address this gap by examining how the presence of a jury affects participants' behaviour in terms of effort provision and the overall characteristics of contest outcomes.

A jury is a set of agents that, based on players' effort levels, each select a player. The winner of the contest is then the player who is selected by the majority of jurors. There are several features that arise when the winner of a contest is decided by jury instead of a single juror, and which are also absent in contests with multiple battlefields. Most notably, it is the fact that a jury gives rise to a novel pivotality effect that cannot be replicated by a traditional contest success function. To understand this effect, consider a contest between two players. Competition happens through effort choice, and once both players have chosen their effort levels, a jury of three jurors chooses the winner by simple majority, where each juror is represented by an independent contest success function. When choosing optimal effort levels, each player considers the marginal gain of increasing (or decreasing) effort. Consider the four possible states of the world depending on the various vote splits that may happen: (v_1, v_2) with $v_1 + v_2 = 3$ and $v_1, v_2 \geq 0$, where v_i represents the votes received by player $i \in \{1, 2\}$.

Conditional on a state such that $v_1 \geq 2$, player 1 does not increase their profit by increasing their effort marginally as they have already won the contest. If $v_1 = 0$, then marginal increases in effort have negligible effects on the probability of winning, as the marginal increase in effort has to "convince" two jurors to change their vote.

It is only in the state where $v_1 = 1$ where a marginal increase in effort is useful for player 1. The same logic applies from player 2's point of view. The result of this is that marginal effort is only useful in situations where one juror is voting for player 1, one juror is voting for player 2, and the third juror "swings", i.e. there is a pivotal juror.

The presence of a jury induces another effect, which we call the uncertainty effect. Increasing the jury size has the result that, ceteris paribus, it is more likely that the player who exerted the most effort wins. This is the Condorcet Jury Theorem at play. Assume that there is a pool of voters choosing between two options, one of which gives a better payoff to all, but there is uncertainty about which of the two options is the better one, where each voter is more likely to vote for the correct option than to the other one. The Condorcet Jury Theorem states that the higher the number of voters, the more likely it is that simple majority selects the right alternative. In our setting, increasing the jury size thus reduces uncertainty about the contest, which, in contests where there is not too much player heterogeneity, increases returns to effort.

Thus, a jury creates two effects that can go in opposite directions. The uncertainty effect can be replicated in the traditional contest by a single contest success function. However, the pivotality effect cannot be replicated, as with only one juror, this juror is always pivotal. On top of that, in simultaneous or sequential contests with multiple battlefields (discussed in more depth later on), the problem involves resource allocation: how much effort to allocate to each battlefield. In our paper, as the number of jurors increases, the probability that there is a pivotal juror decreases, and thus the incentives for players to exert effort change. That is, jury size has a crucial effect on equilibrium efforts. The pivotality effect and how it affects contests are our main contribution to the literature.

In our results, we find that there is an effort-maximising jury size. The effort-maximising jury size balances the pivotality effect with the uncertainty effect. We find that in most cases the relationship between jury size and the sum of efforts of players is concave, meaning that for small juries, increasing jury size increases the sum of efforts while for larger jury sizes, increases in the jury size decrease the sum of efforts.

Furthermore, we explore how the presence of a jury affects discrimination. Consider a two-player contest where all jurors discriminate against the player who happens to be better, in that they have a lower cost of effort than the other player. We find that increasing the jury size, even if the new jurors also discriminate in the same way as existing jurors, can dampen and even eliminate discrimination. The mechanism is as follows: increasing the jury size can lead to an increase in equilibrium efforts, and since the player facing discrimination has a lower cost of effort, they increase their effort more than the other player. The overall effect is then ambiguous, and there are situations where the extra increase in effort not only offsets the fact that there are now more jurors discriminating against this player, but it can also turn the contest around in favour of the player being discriminated against.

On top of this, we also find that the presence of a jury may crowd out players from the contest, even those players who have a lower cost of effort. Consider a situation with three players, one of whom has a lower cost function than the other two. In this case, there can be equilibria where the two higher-cost players exert positive effort and the low-cost player does not exert any effort. Due to the pivotality effect, marginal increases in effort starting from zero effort provide no benefit, because the chances that the marginal effort leads enough jurors to change their vote are negligible. This does not happen in traditional contests. Moreover, even when considering non-negligible increases in effort, the cost of effort needed to have a significant probability of winning given the effort level of the two other contestants may be too high, even for the low-cost player. Thus, the low-cost player finds it optimal to not participate in the contest, i.e., to exert zero effort.

The rest of the paper is organized as follows. In the remainder of this section, we present a literature review. In Section 2, we present the main model. In Section 3, we present our results. In Section 4, we test the robustness of our model by introducing alternative assumptions and extensions. Finally, Section 5 concludes.

1.1 Literature

Contest theory can be traced back to the work of Tullock (1980), which introduced the concept of rent-seeking contests and the idea that individuals expend resources to secure a valuable prize (see also the early work by Hirshleifer (1989), Skaperdas (1996), Clark and Riis (1998), and more recently Moldovanu and Sela (2001) and Konrad (2009)).

Most closely related to our paper are simultaneous contests with multiple battlefields, or multibattle contests. These are contests that are won if a majority of battlefields (i.e., single contests) are won. Snyder (1989), from a political economy angle, compares two-player multibattle contests under two different objective functions: win the most battles versus win the majority of battles. The setting we consider corresponds to the one where the objective is to win a majority of the battles. However, our work is different from Snyder (1989), and also expands on it. The fact that rational contestants should consider the probability with which half the battlers are awarded to either candidate was noted in Snyder (1989) (right after comment 4.2 on page 648) but its implications were overlooked and not explored neither by Snyder nor in any of the subsequent literature. We, on the contrary, put this issue in focus and explore its consequences for effort choice and contest design. On top of this, we expand on Snyder (1989) by considering non-concave payoff functions. This, as we show in the paper, has a qualitative impact on equilibrium efforts. We also expand on Snyder (1989) by providing tight conditions for the existence of equilibria in pure strategies.

Other papers that study multibattle contests include Szentes and Rosenthal (2003a and 2003b), who study situations where players aim to win a majority of objects in simultaneous all-pay auctions. However, their papers consider a limit case of our model, where the probability of winning an auction is deterministic. This equates to a situation where jurors' behaviour is deterministic. We instead assume that jurors' behaviour is smooth based on players' effort levels. Their assumption translates into different incentives and a lack of the pivotality effect, as in their work there is a degenerate probability of any juror being pivotal. Klumpp and Polborn (2006) compare simultaneous multicontest battles to US primaries, but focus on symmetric candidates and symmetric equilibria, whereas we allow for asymmetric players, jurors, and equilibria. As we shall discuss in more depth in Section 3.3, the particular symmetric case considered in Klumpp and Polborn (2006) is the only case where the pivotality effect is not present. Moreover, considering asymmetric situations allows us to study issues such as discrimination and crowding out of more efficient players. Feng et al. (2024) study team multibattle contests where two teams face off in a number of pairwise battles. A different strand of literature that is related to multibattle contests includes Rai and Sarin (2009) and Arbatskaya and Mialon (2010), who study contest where effort is multidimensional and aggregated into a single variable that determines the chances of winning the contest.

In terms of sequential multi-battle contests, where participants compete in a sequential series of interrelated battles, Alcalde and Dahm (2007) propose a serial contest that integrates relative efforts akin to Tullock's contest, and absolute effort differences from difference-form contests. Altman et al. (2012) investigate the effects of carry-over benefits, where performance in one battle affects the probability of success in subsequent battles. Hortala-Vallve and Llorente-Saguer (2012) examine a Colonel Blotto game where opposing parties have varying relative intensities and characterize the conditions under which pure strategy equilibria exist. Fu et al. (2015) examine multi-stage R&D contests where contest entry is an endogenous choice. Morgan et al. (2018) explore self-selection dynamics where entry into larger or more meritocratic contests varies non-monotonically with ability. Other relevant papers are those of Harris and Vickers (1987), Malueg and Yates (2010), Kovenock and Roberson (2010), and Fu et al. (2015). We study sequential contests with a jury as an extension to our main model in Section 4.6 and show the differences with the simultaneous case.

In terms of experimental work, Knight and Schiff's (2010) applied paper on dynamic electoral competition highlights the role of cumulative advantages and strategic momentum in political campaigns. Chowdhury et al. (2013) explore Colonel Blotto games, focusing on the differences in strategies used by players across different allocation treatments. Mago and Sheremeta (2017) experimentally study subjects in simultaneous and sequential multi-battle contests and find incomplete bidding in simultaneous contests and overbidding in sequential contests. Chowdhury et al. (2021) investigate the impact of battlefield salience in Colonel Blotto games and find that subjects deviate from Nash equilibrium predictions when battlefields have salient targets but do not otherwise. Stephenson (2024) examines contests with complementary prizes where agents allocate fixed budgets across multiple battlefields and shows the existence of a unique pure strategy Nash equilibrium under highly sensitive battlefield success functions. Hortala-Vallve and Llorente-Saguer (2015) experimentally test a Colonel Blotto game with heterogeneous and asymmetric battlefield valuations and find that learning improves aggregate welfare despite initial deviations from theoretical predictions. Deck et al. (2017) show experimentally that subjects prioritize specific winning combinations rather than all battles.

From a technical point of view, our paper also contributes to the contest literature by being the first to employ a result from the statistics literature (Wadsworth (1960) and Abramowitz and Stegun (1965)) that relates the Binomial cumulative distribution function (which is used to calculate the probability of receiving the support of the majority of jurors/winning the majority of the battlefields) with the Incomplete Beta Function. As we show in the paper, this equality significantly simplifies our calculations and is at the core of the pivotality effect.

This paper is also connected to the literature on political economy and voting. In particular with the strand of the literature that began with the seminal work of Palfrey and Rosenthal (1983), who were the first to highlight the implications of the so-called pivotal logic on elections. According to the pivotal logic, since a voter's action only affects their payoff when that voter is pivotal, voters should choose their payoff-maximising action assuming they are pivotal. Given equilibrium strategies, being pivotal confers information about the behaviours and types of other voters, which should then be internalized by each player when choosing an action. Since Palfrey and Rosenthal's seminal work, a significant number of papers have made further advancements into how pivotality and the pivotal logic affect incentives in elections. For more on this literature see, for instance, Austen-Smith and Banks (1996) and Mengel and Rivas (2017).

2 The Model

Assume 2 players compete to win a prize worth $v > 0$. To compete for the prize, each player $i \in \{1, 2\}$ simultaneously chooses how much effort to exert, denoted by $e_i \geq 0$. Effort has a cost given by the function

$$c_i(e_i) \quad = \quad \alpha_i e_i,$$

with $\alpha_i > 0$ for all $i \in \{1, 2\}$. Once all players have chosen their effort levels, a jury decides who is the winner of the prize.

There are $N \geq 1$ members of the jury (jurors henceforth), where N is an odd number. Each juror simultaneously chooses a player and the winner of the prize is the player who is chosen by most jurors, i.e. a simple majority voting rule.

Jurors' choices of players are independent and identically distributed, where the probability that a juror chooses player i is denoted by the function p_i and given by the Tullock Contest Success Function (TCSF):

$$p_i(e_i, e_j) = \begin{cases} \frac{n_i e_i^m}{n_i e_i^m + n_j e_j^m} & \text{if } e_i + e_j \neq 0, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

with $n_i > 0$ and $m \in [0, 2]$ for all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus i$. Whenever it is convenient, we do not include efforts as an argument in p_i. Note that we do not assume $m \leq 1$. That is, the impact functions may be non-concave. For more information on the TCSF employed in this paper see Dahm and Corchón (2010) and references therein.

In Section 4, we relax a number of the assumptions made in this section and show that our qualitative results are robust to these modifications. In Section 4.1, we allow jurors to be strategic instead of their behaviour being abstracted by a TCSF.

In Section 4.2, we consider supermajority voting rules and also allow for an even number of jurors. In Section 4.4, we explore the case where there are more than two contestants.

Note that any contest with homogeneous valuations, heterogeneous marginal costs, and biased CSF, such as the one we consider in this paper, is strategically equivalent to a contest with heterogeneous valuations, homogeneous marginal costs and non-biased CSF. That is, our paper deals with a contest that is strategically equivalent to one where each players values the prize $v_i > 0$ for $i \in \{1, 2\}$, $\alpha_1 = \alpha_2$ and $n_1 = n_2$. The equivalence is found by setting $v_i = v\frac{n_i^{\frac{1}{m}}}{\alpha_i}$, and the effort in this modified model, $\hat{e}_i$, is given by $\hat{e}_i = n_i^{\frac{1}{m}} e_i$.

We assume that effort choice is single-dimensional. This is in contrast with multi-battle contests where a potentially different effort level is chosen for each battlefield (each jury in our setting). The reason why we look at the case where there is a single effort choice is because it better represents the real world situations that we want to model: a single contest decided by multiple jurors (such as grant competitions where a single application is presented by each contestant and assessed by multiple jurors, or sports competitions like gymnastics or diving where a single performance is assessed by multiple jurors).

Snyder (1989) and Klumpp and Polborn (2006) show that in the setting considered in this paper, if $m \leq 1$, optimal effort choices are such that the same effort level is chosen for all battlefields. Therefore, single dimensional effort choice is equivalent to multidimensional effort choice if $m \leq 1$. It is not currently known what the equilibrium looks like in a model where $m > 1$ and multidimensional effort choice is allowed.

3 Results

3.1 Equilibrium

We focus first on equilibria in pure strategies and study mixed strategy equilibria later on. The payoff of player i as a function of effort levels is given by π_i, where

$$\pi_i \;=\; v \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} - c_i(e_i).$$

At the core of the pivotality effect is an equality that relates the Binomial cumulative distribution function with the Incomplete Beta Function:

$$\sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} \;=\; \frac{N!}{\frac{N-1}{2}!2} \int_0^{p_i} x^{\frac{N-1}{2}} (1-x)^{\frac{N-1}{2}} \mathrm{d}x. \tag{1}$$

This result is featured in Wadsworth (1960) and Abramowitz and Stegun (1965), among others. For completeness, we present the proof for a general voting rule q in the appendix (simple majority is given by $q = \frac{1}{2}$). The proof involves iteratively integrating by parts on the right-hand side of the equation above.

The convenience of (1) is that it allows us to easily derive the best responses of players without having to deal with a sum where the number of terms depends on the jury size. The best response of player i, given the effort level of the other player, is found by setting $\frac{\partial \pi_i}{\partial e_i} = 0$ and is implicitly given by

$$v \frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}} \frac{\partial p_i}{\partial e_i} \;=\; c_i'(e_i). \tag{2}$$

In the standard contest with only one juror, we have that $v\frac{\partial p_i}{\partial e_i} = c_i'(e_i)$. Thus, the addition of a jury introduces the term $\frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}}$, which is equal to the probability that the jury is exactly one vote away from awarding the prize to either player. That is, $\frac{N-1}{2}$ jurors are voting for player i and $\frac{N-1}{2}$ jurors are voting for player j, while the remaining jury member is still undecided. Using the terminology from the voting literature, $\frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}}$ is the probability that there is one pivotal juror. This is the pivotality effect.

Given equation (2), we have that the best response effort levels are related via the equation

$$\frac{c_1'(e_1)}{c_2'(e_2)} = \frac{\frac{\partial p_1}{\partial e_1}}{\frac{\partial p_2}{\partial e_2}}. \tag{3}$$

Note that in equation (3), the jury size N is not present. While N affects the level of effort exerted by both players, it does not affect how the best responses are related.

Substituting the functions c_i and p_i, we obtain that the best response effort levels are related via the equation

$$\frac{e_2}{e_1} = \frac{\alpha_1}{\alpha_2}. \tag{4}$$

Define $\theta = \frac{n_2}{n_1}\left(\frac{\alpha_1}{\alpha_2}\right)^m$. The equation above, together with equation (2), leads to

$$\begin{aligned}
\alpha_1 &= v\frac{N!}{\frac{N-1}{2}!2}\left(\frac{e_1^m}{e_1^m+\theta e_1^m}\right)^{\frac{N-1}{2}}\left(\frac{\theta e_1^m}{e_1^m+\theta e_1^m}\right)^{\frac{N-1}{2}} \\
&\quad \times \frac{m e_1^m \theta e_1^m}{(e_1^m+\theta e_1^m)^2}\frac{1}{e_1}, \\
\alpha_1 e_1 &= v\frac{N!}{\frac{N-1}{2}!2}\left(\frac{\theta}{(1+\theta)^2}\right)^{\frac{N-1}{2}}\frac{m\theta}{(1+\theta)^2}, \\
e_1 &= v\frac{N!}{\frac{N-1}{2}!2}\left(\frac{\theta}{(1+\theta)^2}\right)^{\frac{N+1}{2}}\frac{m}{\alpha_1},
\end{aligned}$$

Since $e_2 = \frac{\alpha_1}{\alpha_2}e_1$, we have that

$$e_i = v\frac{N!}{\frac{N-1}{2}!2}\left(\frac{\theta}{(1+\theta)^2}\right)^{\frac{N+1}{2}}\frac{m}{\alpha_i} \tag{5}$$

for $i \in \{1,2\}$.

Note that if an equilibrium in pure strategies exists, it is unique. This is because we have just proven by construction that there is a unique value for the pair (e_1, e_2) at which both players play a best response.

Finally, for the pair of efforts in (5) to be an equilibrium, it must be that players' profits are positive, and that the second-order conditions are met. We develop this analysis in the Appendix and summarize its findings in our next result.

Proposition 1. *Define* $p = \frac{1}{1+\theta}$ *and let* e_i^* *for* $i \in \{1,2\}$ *be given by*

$$e_i^* \;=\; v \frac{N!}{\frac{N-1}{2}!^2} \left[p(1-p)\right]^{\frac{N+1}{2}} \frac{m}{\alpha_i}.$$

A pure strategy equilibrium exists, and it is given by (e_1^*, e_2^*)*, if and only if:*

1. *The profit functions are globally concave:* $m\frac{N+1}{2} \leq 1$*, or*

2.a *the profit functions are locally concave at* (e_1^*, e_2^*)*:*

$$\begin{aligned}(1-2p)\, m\frac{N+1}{2} &\leq 1,\\ (2p-1)\, m\frac{N+1}{2} &\leq 1,\end{aligned}$$

2.b *and the profit of both players is positive at* (e_1^*, e_2^*)*:*

$$\begin{aligned}\sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p^k (1-p)^{N-k} - \frac{N!}{\frac{N-1}{2}!^2}\left[p(1-p)\right]^{\frac{N+1}{2}} m &\geq 0,\\ \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} (1-p)^k p^{N-k} - \frac{N!}{\frac{N-1}{2}!^2}\left[p(1-p)\right]^{\frac{N+1}{2}} m &\geq 0.\end{aligned}$$

Furthermore, if a pure strategy equilibrium exists, it is unique.

Corollary 1. *In a fully symmetric contest* ($n_1 = n_2$ *and* $\alpha_1 = \alpha_2$)*, an equilibrium in pure strategies exists, and is unique, if and only if* $m \leq 2^N \frac{\frac{N-1}{2}!^2}{N!}$.

In the case with only one juror ($N = 1$) and a symmetric contest ($\alpha_1 = \alpha_2$ and $n_1 = n_2$), we have that a pure strategy equilibrium exists, and it is unique, if and only if $m \leq 2$ (parts 2.a and 2.b of the proposition). This is a well-known result in the literature (see Tullock (1980) and Baye et al. (1996)).

If players are symmetrical ($\alpha_1 = \alpha_2 = N$) and treated symmetrically by the jurors ($n_1 = n_2$), as in the case considered in Klumpp and Polborn (2006), we obtain the same necessary and sufficient existence conditions as they do (Corollary 1).

If $m \in [1, 2]$, then a pure strategy equilibrium with $N = 1$ always exists (part 1 in the Proposition), whereas a pure strategy equilibrium with $N > 1$ exists only if (part 2.a in the Proposition)

$$\max\left\{\frac{1}{1+\theta}, \frac{\theta}{1+\theta}\right\} \leq \frac{1}{2}\frac{N+3}{N+1}.$$

That is, a necessary condition for the existence of pure strategy equilibria with $m \in [1, 2]$ is that the contest is not too asymmetric given the number of jurors.

Snyder (1989) shows that a sufficient condition for a unique pure strategy equilibrium in our setting is $mN \leq 1$, and acknowledges that this condition is not tight. We provide the tight condition for all n_1, n_2, α_1 and α_2: $m\frac{N+1}{2} \leq 1$ (part 1. of the proposition), and also the necessary and sufficient conditions for the case where $m\frac{N+1}{2} > 1$ (in this case, the conditions depend on the particular values of n_1, n_2, α_1 and α_2, as given in parts 2.a and 2.b of the proposition).

If $m\frac{N+1}{2} > 1$, the profit function is not globally concave. It can be shown (see the Appendix) that the second derivative of the profit function is decreasing in the effort choice of the player. Thus, if the profit function is not globally concave then, given the effort level of the other player, there exists a $\hat{e} > 0$ such that for all $e_i < \hat{e}$ the profit function of i is convex, whereas for all $e_i > \hat{e}$ the profit function is concave. Therefore, if a local maximum is found, this maximum is in fact a global one, and it is unique. In this case, the only requirement left to check for equilibrium is that the player obtains a positive profit.

Proposition 1 shows that pure strategy equilibria where the profit function is not globally concave are possible. To our knowledge, the case with $N > 1$ where the profit function is not globally concave $\left(m\frac{N+1}{2} > 1\right)$ has not been looked at before in the literature.

A situation where the profit function is not globally concave is depicted in Figure

1. As can be seen, in a neighbourhood around zero effort, the shape of the profit function is fundamentally different between $N = 1$ and $N = 3$. With a single juror ($N = 1$), the profit function is globally concave, as starting from zero effort, a marginal increase in effort increases the player's probability of winning by a proportionately higher amount compared to the increase in effort. On the other hand, with a jury ($N = 3$ and beyond), the profit function is convex near 0. This is because a marginal increase in effort of $\Delta > 0$ increases the probability of winning by a factor of Δ^2 or lower, which is negligible for small enough Δ. That is, at a cost of order Δ and a benefit of order Δ^2, the player goes from zero profit to negative profit. As we shall see later on, this effect has qualitative implications, such as fostering the crowding out of players.

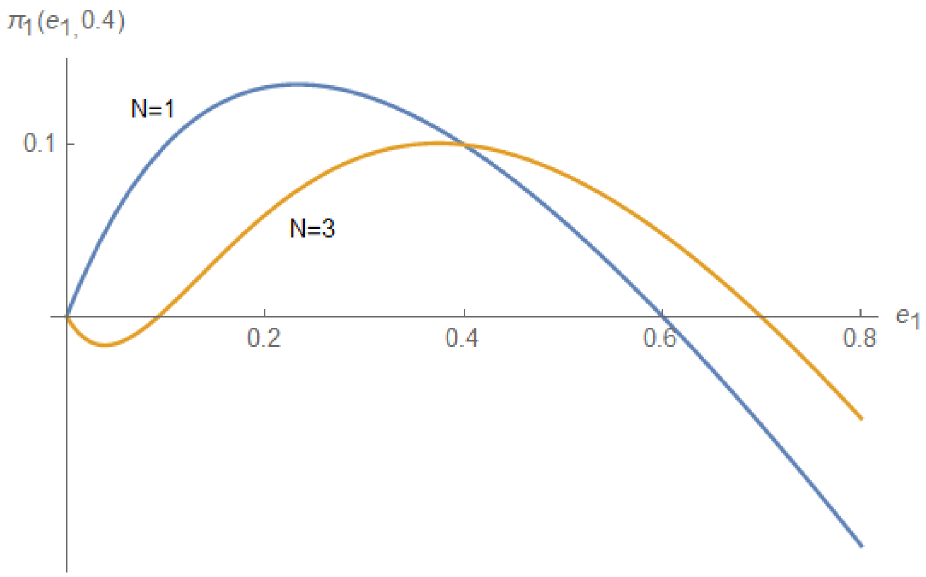

Figure 1: Example of the profit function of player 1 when player 2 exerts effort $e_2 = 0.4$ with $v = \alpha_1 = \alpha_2 = n_1 = n_2 = m = 1$.

3.2 Mixed Strategy Equilibria and Rent Dissipation

Throughout this paper, we focus on pure strategy equilibria. The complexity of mixed strategy equilibria in contests is well-known (see, for instance, Ewerhart (2015) and

Feng and Lu (2017) and references therein). Nevertheless, for symmetric contests we can fully characterize rent dissipation for all jury size N, irrespective of whether a pure strategy equilibrium exists or not.

Proposition 2. *Assume $n_1 = n_2$ and $\alpha_1 = \alpha_2 = \alpha$, and define*

$$\hat{m}(N) \quad = \quad 2^N \frac{\frac{N-1}{2}!^2}{N!}.$$

If $m \leq \hat{m}(N)$ the sum of equilibrium efforts equals $\frac{v}{\alpha}\frac{m}{\hat{m}(N)}$, whereas if $m > \hat{m}(N)$ the sum of equilibrium efforts is $\frac{v}{\alpha}$.

Proposition 2 follows from two results. First, Corollary 1 states that an equilibrium in pure strategies exists if and only if $m \leq \hat{m}(N)$, in which case the sum of equilibrium efforts is $\frac{v}{\alpha}\frac{m}{\hat{m}(N)}$ by Proposition 1. Second, Ewerhart (2017) shows that if $m > \hat{m}(N)$, there is full rent dissipation in all mixed strategy equilibria, i.e., the sum of (expected) equilibrium efforts is given by $\frac{v}{\alpha}$.

Figure 2 shows an example of rent dissipation for all jury sizes. Note that the plot should consist of discrete points instead of a smooth curve, as the number of jurors is a discrete number. However, here and henceforth we use the Regularized Beta Function to plot continuous curves, as we believe this helps to visualize trends and compare different scenarios more easily.

Notice how rent dissipation is increasing with the jury size, and maximised when a pure strategy equilibrium is no longer possible. As shown in Proposition 2, this observation holds for all symmetric contests. However, this is not the case for asymmetric contests, where rent dissipation is non-monotonic on jury size because of the pivotality and uncertainty effects. We explore this non-monotonicity in the next subsection.

3.3 Effort Maximising Jury Size

Next, we explore how jury size affects the efforts of both players in the pure strategy equilibrium. Note that, since in equilibrium $e_2 = ke_1$ with $k > 0$, we have that

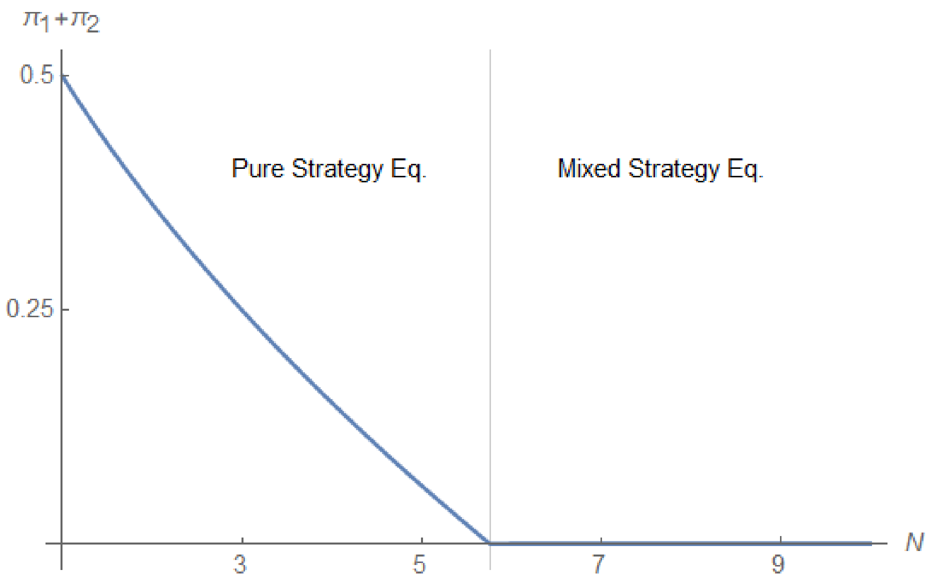

Figure 2: Sum of equilibrium profits when $v = \alpha_1 = \alpha_2 = n_1 = n_2 = m = 1$.

maximising $e_1 + e_2$ is equivalent to maximising e_1. Moreover, since in equilibrium $\pi_1 + \pi_2 = v - \alpha_1 e_1 - \alpha_2 e_2 = v - (\alpha_1 + \alpha_2 k)e_1$, maximizing the sum of efforts is equivalent to maximizing rent dissipation.

From (5), maximising e_1 is equivalent to maximising $A(N)$ given by

$$A(N) = \frac{N!}{\frac{N-1}{2}!^2} \left(\frac{\theta}{(1+\theta)}\right)^{\frac{N+1}{2}}.$$

Define $D(N) = A(N+2) - A(N)$. If $D(N) > 0$, then both players' equilibrium efforts increase when the jury size goes from N to $N+2$. We have,

$$D(N) \propto \frac{N+2}{N+1}\frac{4\theta}{(1+\theta)^2} - 1. \tag{6}$$

Note that $D(N)$ is decreasing in N. Moreover, $\frac{N+2}{N+1} > 1$ for all N, whereas

$$\frac{4\theta}{(1+\theta)^2} \leq 1$$

with equality only when $\theta = 1$. Therefore, ignoring equilibrium payoffs for now, if $\theta \neq 1$, then there exists an $N \geq 1$ that maximises equilibrium effort. The effort-maximising jury size is either the odd floor or the odd ceiling of the N^* that solves $D(N^*) = 0$.

Figure 3 shows how $e(N)$, i.e., the sum of equilibrium efforts for a given N, changes as the jury size changes. This figure also shows the function $D(N)$. In the example from this figure, the effort-maximising jury size is 5 (N is only allowed to take on odd values in this section). Figure 4 shows how players' profits change as jury size changes.

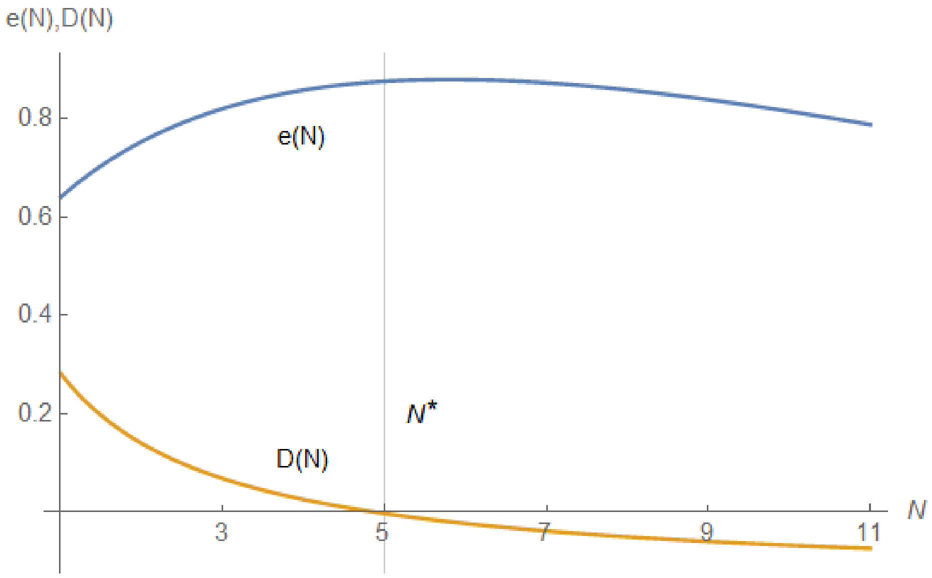

Figure 3: Example with $v = 1$, $\alpha_1 = 1$, $\alpha_2 = 0.2$, $n_1 = n_2 = 1$ and $m = 0.5$. Effort maximising jury size is $N = 5$.

The reason why there can be an optimal interior effort-maximising jury size is that there are two incentives at play. On the one hand, increasing the jury size reduces uncertainty, i.e. the player with a higher effort is more likely to win, which in contests with not too heterogeneous players leads to an increase in effort. On the other hand, increasing the jury size also leads to a reduction in the probability that a juror is pivotal and, thus, a reduction in effort. With low jury size, the uncertainty effect dominates, and increasing the number of jurors leads to more effort. However, as jury size increases, there comes a point where the pivotal effect starts to dominate, and increasing the number of jurors leads to a decrease in equilibrium effort.

The contest considered in Klumpp and Polborn (2006) is such that $\alpha_1 = \alpha_2 = 1$, and $n_1 = n_2 = 1$. The situation in Klumpp and Polborn (2006) is a fully symmetric

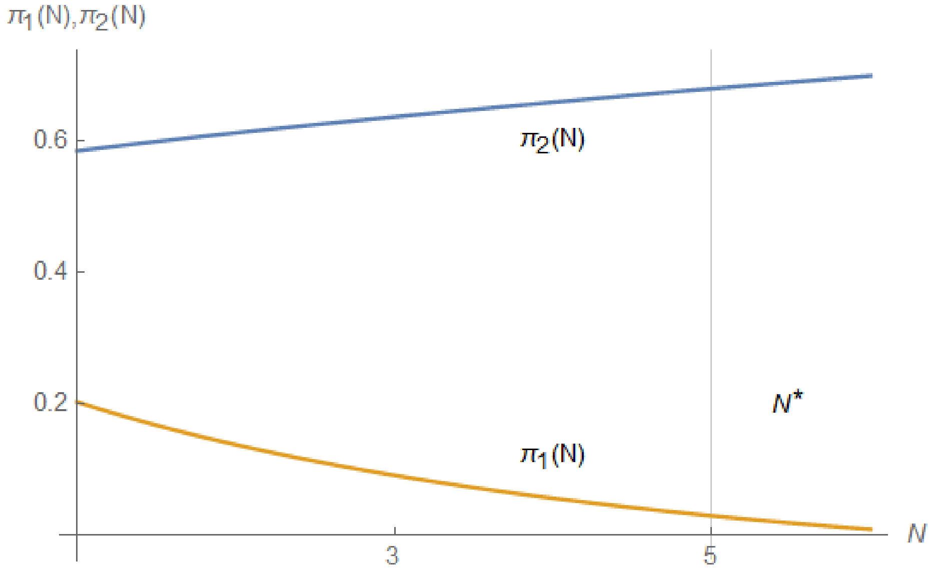

Figure 4: Example with $v = 1$, $\alpha_1 = 1$, $\alpha_2 = 0.2$, $n_1 = n_2 = 1$ and $m = 0.5$. Effort maximising jury size is $N = 5$.

case: players' costs are symmetric, and jurors treat players symmetrically. With their restrictions, we have $\theta = 1$, which means that $\frac{4\theta}{(1+\theta)^2} = 1$ for all $m > 0$, and thus $D(N)$ is always positive for all N. This is a special case that does not appear as soon as either players are not symmetric or jurors do not treat players symmetrically. In this paper, we drop all these symmetry assumptions (note that Klumpp and Polborn (2006) consider asymmetric players, but only for the limit case of $N \to \infty$).

The intuition for why the pivotality effect is not relevant in fully symmetric contests is that $\frac{4\theta}{(1+\theta)^2} = 1$ implies that in equilibrium $p_1 = p_2 = \frac{1}{2}$. When the contest is most contested, and only then, the probability that a juror is pivotal vanishes at speeds orders of magnitude lower compared to the uncertainty effect. In particular, we have for all $N \geq 1$, including in the limit, that $\frac{N+2}{N+1}\frac{4\theta}{(1+\theta)^2} > 1$. That is, the pivotality effect never dominates the uncertainty effect. In this case, the effort maximising jury sizes are those that lead to full rent dissipation as discussed in the previous section: all N such that $m > \hat{m}(N)$.

We collect the facts proven in this subsection in the next proposition:

Proposition 3. *Define* $p = \frac{1}{1+\theta}$. *In Tullock contests, the (pure-strategy) effort-maximising jury size is given by* $N^* = \min\{N_1, N_2\}$, *where* N_1 *is the maximum* N *such that both players have non-negative profits and* N_2 *is either the odd floor or the odd ceiling of the* $N \geq 1$ *that solves*

$$\frac{N+2}{N+1} 4p(1-p) \quad = \quad 1.$$

Moreover, changing the jury size either increases or decreases both equilibrium effort levels.

4 Extensions

4.1 Strategic Jurors

We now explore in more detail jurors' behaviour by considering the case where jurors can be strategic. A way to motivate jurors' strategic behaviour is by considering a situation where jurors want to choose the player that exerted more effort, i.e., they want player i to win the contest if $e_i > e_j$ for $i \in \{1, 2\}$, and are indifferent about who wins if $e_1 = e_2$.

However, in many real-life scenarios, the effort of each player, nor their type (i.e., its cost function) may be directly observable. Consider instead that each juror k has a symmetric prior about players (i.e., their prior distribution about e_1 and α_1 is the same as that about e_2 and α_2), and, on top of that, each juror receives an informative signal $\sigma_k \in \{1, 2\}$ about which player exerted the most effort.

Consider, for instance, that signals are independent and identically distributed and are given by

$$P\left(\sigma_k = i | e_1, e_2\right) \quad = \quad p_i, \tag{7}$$

where p_i is the TCSF given in Section 2. Since jurors' prior is symmetric, if $\sigma_k = i$, then the probability that player i exerted more effort than player j is higher than $\frac{1}{2}$, whereas if $\sigma_k = j$, then this probability is less than $\frac{1}{2}$.

Consider a situation where jurors want to choose the player that exerted the most effort and use their signal σ_k from equation 7 to do so. In the main text, it was assumed that jurors' vote follows p_i, which with strategic jurors is equivalent to stating that they vote according to their signal. The question is then whether a situation where each juror chooses a candidate according to their signal is a (Bayesian Nash) equilibrium. This is referred to in the voting literature as an informative equilibrium (see, for instance, Austen-Smith and Banks (1996)). The answer is positive.

Proposition 4. *Assume jurors are strategic, have a symmetric common prior, and receive an informative i.i.d. signal about effort choices given in (7), then there is an informative equilibrium where every juror votes informatively.*

The proof follows from Austen-Smith and Banks (1996) and is thus omitted. The intuition is that a juror must behave as if pivotal (i.e., the vote of the other jurors is split 50/50) because otherwise, their vote is irrelevant. If a juror is pivotal and if jurors vote according to their signal, then there are as many signals for candidate 1 as there are for candidate 2 when not counting a juror's own signal. Since the prior is symmetric and signals are i.i.d., the posterior when considering being pivotal and after observing a juror's own signal is simply the juror's own signal. Given this posterior, it is a best response to vote according to the signal as the signal has a more than 50% chance of being correct.

Note that, just as in Austen-Smith and Banks (1996), if we used a non-majority voting rule, then there are equilibria where some jurors are not sincere to offset whichever bias the voting rule might introduce. For example, if there are 7 jurors and player 1 wins if and only if this player is chosen by at least 6 jurors, there is no equilibrium where all jurors are sincere, but there is an equilibrium where any 4 jurors vote for player 1 and the other 3 vote sincerely. This contest is then equivalent to one where the voting rule is simple majority and there are only 3 jurors who are all sincere.

Since sincere voting is essentially what we assumed from the outset, the assumption is made without loss of generality if the appropriate informational setting and voting equilibrium is used.

4.2 Other Voting Rules and Even Number of Jurors

Suppose that N can take even values and the winner of the contest is player 1 if they are chosen by at least a fraction $q \in (0,1]$ of the jurors; otherwise, player 2 wins. We refer to q as the voting rule. For example, if $q = \frac{1}{2}$, then the voting rule is simple majority (this is the case we have looked at so far) where ties go to player 1 (alternatively, we could assume that ties imply that the contest ends in a draw, as in Vesperoni and Yildizparlak (2017), however, this is outside the scope of this paper). On the other hand, if $q = 1$, the voting rule is unanimity for 1 with player 2 as the status quo.

The payoff of player 1 as a function of effort levels is given by

$$\pi_1 \quad = \quad v \sum_{k=\lceil Nq \rceil}^{N} \binom{N}{k} p_1^k (1-p_1)^{N-k} - c_1(e_1),$$

and similarly for player 2.

Using the equivalence shown in the appendix, the critical points of π_i for $i \in \{1,2\}$ are given by

$$v \frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} p_i^{\lceil Nq \rceil - 1} (1-p_i)^{N - \lceil Nq \rceil} \frac{\partial p_i}{\partial e_i} \quad = \quad c_i'(e_i).$$

Best response effort levels are then related via the equation

$$\frac{c_1'(e_1)}{c_2'(e_2)} \quad = \quad \frac{\frac{\partial p_1}{\partial e_1}}{\frac{\partial p_2}{\partial e_2}}.$$

Note that this is the same equation as for simple majority, and that the voting rule q is not present. While N and q affect the equilibrium levels of effort, they do not affect how these effort levels are related.

Proceeding as in the proof if Proposition 1, a sufficient condition for an equilibrium in pure strategies is

$$m\left(\max\left\{\lceil Nq \rceil, N - \lceil Nq \rceil\right\} + 1\right) \quad \leq \quad 1. \tag{8}$$

In terms of effort-maximising jury size, we proceed as in Section 3.3. Increasing the jury size by 2 jurors (we choose 2 extra jurors instead of 1 to keep this section in line with the main model) increases the votes that i needs to win the contest by $a = \lceil q(N+2) \rceil - \lceil qN \rceil \in \{0, 1, 2\}$. Proceeding as in the case with $q = \frac{1}{2}$, we find after some algebra

$$D_i(N) \quad \propto \quad \frac{(N+2)(N+1)}{G(N,q)} 4p_i^a(1-p_i)^{2-a} - 1.$$

where

$$G(N,q) = \begin{cases} 4(N - \lceil Nq \rceil + 2)(N - \lceil Nq \rceil + 1) & \text{if } a = 0, \\ 4\lceil Nq \rceil (N - \lceil Nq \rceil + 1) & \text{if } a = 1, \\ 4(\lceil Nq \rceil + 1)\lceil Nq \rceil & \text{if } a = 2, \end{cases}$$

with $a = \lceil q(N+2) \rceil - \lceil qN \rceil$.

Note that the equation above is qualitatively the same as Equation (6), with the difference that increases in jury size have a varying effect on the sum of equilibrium efforts depending on the rounding of natural numbers. Thus, the incentives and results about effort-maximising jury size are the same as those already presented.

4.3 Effort maximising Voting Rule

A contest designer may be interested in maximising the effort choice of participants but constrained in the number of jurors it can employ. In such cases, the voting rule used to determined the winner of the contest can be leveraged to affect effort levels. In this subsection we explore, given a jury size, what is the voting rule that maximises the sum of efforts.

Given the complexities of dealing with a general voting rule whose outcome depends on the rounding of real numbers, we explore this issue numerically. Consider a Tullock contest with $v = \alpha_1 = n_1 = n_2 = 1$, and $\alpha_2 = \alpha > 0$. Assume that there are $N = 5$ jurors. From equation (8), we know that a necessary condition for an equilibrium in pure strategies for all $q \in (0, 1]$ is $m \leq 0.2$. We thus set $m = 0.2$.

Figure 5 shows how the sum of efforts changes for different voting rules for two different values of the cost function of player 2. When both players have the same cost, $\alpha = 1$, simple majority maximises the sum of efforts. On the other hand, when player 2 has a higher cost function, $\alpha = 5$, then the voting rule where unanimity is required for 1 to win outperforms other voting rules. That is, when the contest is unbalanced in that one player is better than the other, the effort-maximising voting rule is one where the better player needs more votes to win. This observation is consistent across different parameter values.

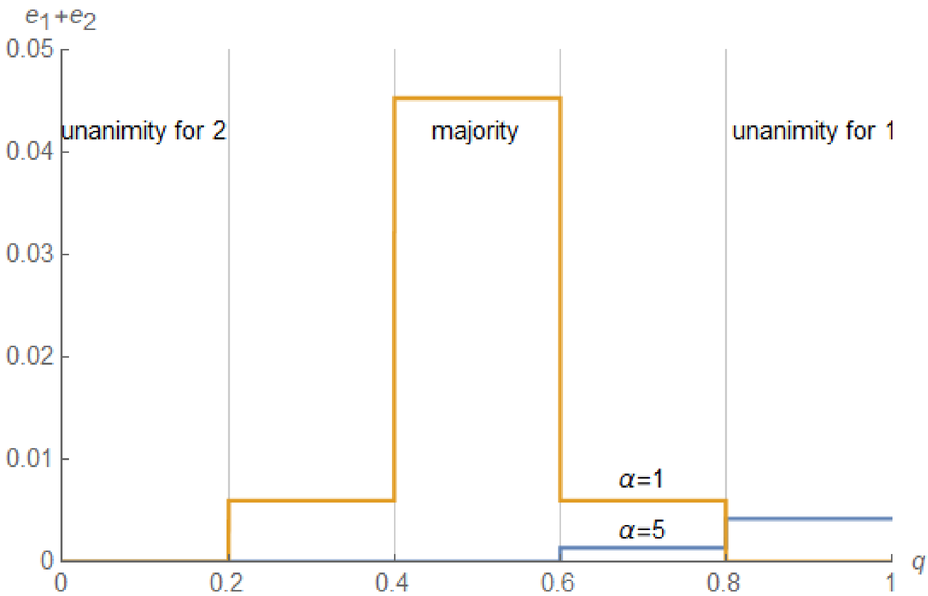

Figure 5: Example with $N = 5$, $v = 1 = \alpha_1 = n_1 = n_2 = 1$, $m = .2$ and $\alpha_2 = \alpha$.

The intuition behind this is that in contests where competition is uneven, aggregate effort is lower than in contests where there is even competition between players. The reason for this is that, first, the player with the higher costs has a lower return on effort given the high effort exerted by the other player, and second, the player with the lower costs does not need to exert a high level of effort to win the contest given the costs of the other player. Hence, a voting rule that balances the contest is the one that maximises aggregate effort. The way the balancing is achieved is by requiring the player with the lower cost function to obtain more votes to win than the other player.

4.4 More than Two Contestants

Assume for this section that there are $C \in \{3, \ldots\}$ players and the winner of the contest is the player that is chosen by most jurors, where for simplicity, in the case of a tie, a winner is chosen uniformly at random among tied players. Given effort levels, define a state of the world as a vector $\mathbf{v} \in \{1, \ldots, N\}^C$ such that $\sum_i v_i = N$. That is, v_i denotes the votes received by player i.

If there are more than two contestants, deriving analytical results is no longer possible. The reason is that with more than two contestants, the number of votes needed to win the contest depends on the whole distribution of votes that all players receive. As an example, if there are 3 players and 7 jurors, then player 1 wins in any of the following states of the world: $(3, 2, 2)$, $(4, 2, 1)$, $(4, 1, 2)$, $(5, 2, 0)$, $(5, 0, 2)$, $(5, 1, 1)$, $(6, 1, 0)$, $(6, 0, 1)$, $(7, 0, 0)$, and $(3, 3, 1)$ and $(3, 1, 3)$ with probability $\frac{1}{2}$. As the number of players and/or the number of jurors increases, the number of states increases exponentially and, more importantly, there is no known analytically tractable way to enumerate them for general N and C.

Intuitively, however, our results carry through to the case with more than two contestants, as both the uncertainty effect and pivotality effect are still present. Nevertheless, with three or more players, there is equilibrium multiplicity, and in particular, there can be equilibria where there is crowding out. Crowding out equilibria are those where a contestant, not necessarily the one with the lower costs, exerts zero effort in equilibrium.

To illustrate these observations, we consider a contest with 3 players and jury sizes $N = 1$ and $N = 3$. Assume for simplicity that $v = 1$, $\alpha_1 = n_1 = m_1 = 1$, $\alpha_2 = n_2 = m_2 = 1$, $n_3 = m_3 = 1$ and $\alpha_3 = \alpha > 0$. With these parameter values, we have that $p_i = \frac{e_i}{\sum_j e_j}$ for all $i \in \{1, 2, 3\}$, and $c_i(e_i) = e_i$ for all $i \in \{1, 2\}$ and $c_3(e_3) = \alpha e_3$.

Let v_i denote the votes received by player i. The payoff of each player is given by

$$\pi_1 = P(v_1 > v_2, v_1 > v_3) + \frac{1}{3}P(v_1 = v_2 = v_3) - e_1,$$
$$\pi_2 = P(v_2 > v_1, v_2 > v_3) + \frac{1}{3}P(v_1 = v_2 = v_3) - e_2,$$
$$\pi_3 = P(v_3 > v_1, v_3 > v_2) + \frac{1}{3}P(v_1 = v_2 = v_3) - \alpha e_3.$$

Note that with 3 players and jury sizes 1 and 3, there can never be a 2-way tie.

With a one-person jury ($N = 1$), it can be easily shown that for $\alpha \leq 2$, there is a unique pure strategy equilibrium given by $e_1 = e_2 = \frac{2\alpha}{(2+\alpha)^2}$ and $e_3 = \frac{2(2-\alpha)}{(2+\alpha)^2}$. If $\alpha \geq 2$, then there is an equilibrium where player 3 is crowded out and exerts no effort, whereas the other two players choose the same effort level (this is similar to the case in Franke (2012)). If $\alpha < 1$, i.e. player 3 has a lower cost function than players 1 and 2, then this player exerts more effort, whereas if $\alpha > 1$, then the opposite happens. This is a standard result in traditional Tullock contests without a jury.

However, if the jury size is instead $N = 3$, we have that there is equilibrium multiplicity. To begin with, there is an equilibrium where all three players exert positive effort. The equilibrium efforts are shown in Figure 6 for different values of α. Given the complexity of the explicit form of the equilibrium efforts we have chosen not to show them in the paper.

On top of that, with $N = 3$, there are also equilibria with crowding out where one of the players exerts zero effort in equilibrium. In terms of crowding out players 1 or 2, if $0.646 < \alpha < 1.194$, there are two equilibria where $e_3 = \frac{6\alpha}{(1+\alpha)^4}$ and either $(e_1, e_2) = \left(\frac{6\alpha^2}{(1+\alpha)^4}, 0\right)$ or $(e_1, e_2) = \left(0, \frac{6\alpha^2}{(1+\alpha)^4}\right)$. A lower bound on α is needed because otherwise player 3's cost function is significantly lower than that of player 1, and this player would obtain negative profits in equilibrium. On the other hand, an upper bound on α is needed because otherwise player 3's cost function is significantly higher than that of players 1 and 2, and player 2 (or player 1) finds it profitable to enter the competition and exert a positive effort level.

In terms of crowding out player 3, for $\alpha > 0.906$, there is an equilibrium where players 1 and 2 choose $e_1 = e_2 = \frac{3}{8}$, and player 3 chooses $e_3 = 0$. Notice that there

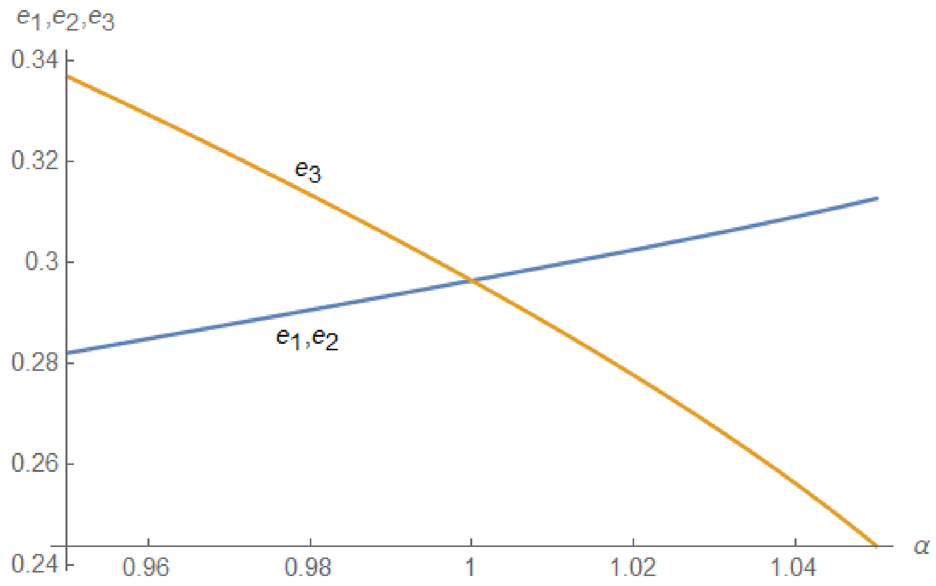

Figure 6: Example with three players and $N = 3$, $v = 1$, $\alpha_1 = n_1 = 1$, $\alpha_2 = n_2 = 1$, and $\alpha_3 = \alpha$, $n_3 = 1$ and $m = 1$.

is crowding out of player 3 even in cases where this player has a lower cost function than that of players 1 and 2. Such a situation is not possible in the traditional contest with a single juror.

The reason why there are equilibria with crowding out, even when the player crowded out is the one with the lowest cost function, is due to the pivotality effect. To see this, consider a situation where player 3 is not exerting any effort and players 1 and 2 are exerting positive effort equal to the two-player equilibrium, i.e. the equilibrium effort levels in a situation where there is no player 3. Player 3's marginal profit at zero effort level is negative. The reason is that increasing effort marginally only improves payoff in cases where there is a pivotal juror for player 3 (the pivotality effect), but since player 3 is currently exerting zero effort and the other two players are exerting positive effort, the probability of being in a pivotal state for player 3 is zero. Thus, a marginal increase in effort for player 3 will lead to negative profits.

This can be seen in Figure 7, where the payoff of player 3 for different effort levels is compared between the 2-player $N = 1$ equilibrium efforts ($e_1 = e_2 = \frac{1}{2}$), and the

3-player $N = 3$ crowding out equilibrium efforts ($e_1 = e_2 = \frac{3}{8}$). As can be noted, for low effort levels, with only 1 juror, player 3 finds it optimal to exert a positive effort, which makes an equilibrium where there is crowding out not possible (concave shape). With 3 jurors, exerting a positive effort leads to negative profits as that effort needs to swing two jurors, i.e. a marginal increase in effort equal to $\Delta > 0$ increases the probability of winning by a quantity proportional to Δ^2 (convex shape), i.e. a negligible amount given the cost.

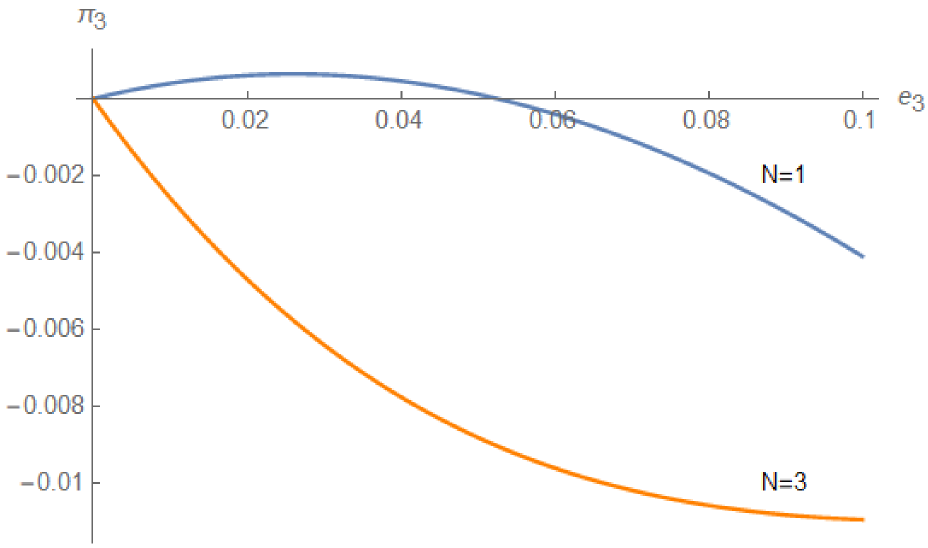

Figure 7: Profits of player 3 as a function of its own effort when $\alpha = 0.95$, $v = 1$, $\alpha_1 = n_1 = 1$, $\alpha_2 = n_2 = 1$, $\alpha_3 = \alpha$, $n_3 = 1$ and $m = 1$. For $N = 1$ we have $e_1 = e_2 = \frac{1}{2}$, whereas for $N = 3$ we have $e_1 = e_2 = \frac{3}{8}$.

With respect to a non-marginal increase in effort, since players 1 and 2 are exerting the crowding out equilibrium effort levels, player 3 exerting a positive effort level will lead to negative profits since the cost of effort required to have a significant probability of winning the contest is too high given the potential gains. Figure 8 shows the profits of player 3 as a function of their own effort level when $\alpha = 0.95$ and players 1 and 2 are exerting the equilibrium effort levels when there is crowding out of player 3. As can be noted, player 3's profits are negative for all effort levels and, thus, their optimal choice is to stay out of the contest even though they have a lower cost function than that of players 1 and 2.

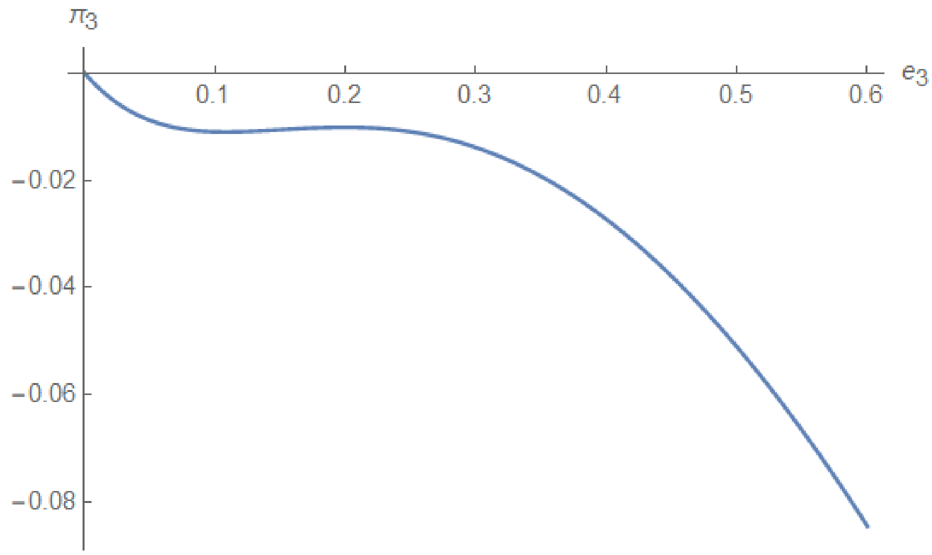

Figure 8: Profits of player 3 as a function of its own effort when $\alpha = 0.95$, $e_1 = e_2 = \frac{3}{8}$ and $N = 3$, $v = 1$, $\alpha_1 = n_1 = 1$, $\alpha_2 = n_2 = 1$, $\alpha_3 = \alpha$, $n_3 = 1$ and $m = 1$.

Increasing the player count past 3 or using different parameter values leads to the same observations as just made for three players. We collect these in the following proposition, whose proof is via the example just shown.

Proposition 5. *With three or more players, there can be equilibrium multiplicity. Moreover, there can be crowding out equilibria where the player with the lowest cost function exerts zero effort.*

4.5 Contests with Discrimination

In this subsection, we focus on contests where there is discrimination against a contestant and show how increasing the jury size, even when new jurors also show discrimination in the same way as existing jurors, can help mitigate the effects of discrimination if the player that is being discriminated against is better (in the sense that they have a lower cost function).

The key mechanism is that increasing the jury size can lead to both players exerting more effort, and since the cost of effort is lower for the player that is being discriminated against, this player increases their effort more than the other player. This extra increase in effort may not only offset the fact that there are now more jurors who discriminate against the player, but it can also lead to a net increase in the probability that the player being discriminated against wins.

With linear cost functions, players' equilibrium efforts are related linearly as per equation (4), and changes in the jury size do not change each individual juror's behaviour (i.e., p_i). To have players' equilibrium efforts not to be related linearly, it suffices to assume non-linear costs. Assume then that cost functions are given by $c_i(e_i) = \alpha_i e_i^{\beta_i}$ with $\beta_i \geq 1$ for $i \in 1, 2$. Given these cost functions, equation (3) implies $e_2 = \left(\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right)^{\frac{1}{\beta_2}} e_1^{\frac{\beta_1}{\beta_2}}$.

We use the following two definitions.

Definition 1. *We say that player i is **discriminated** against if, for all $a > 0$, whenever $e_1 = e_2 = a$, then $p_i < \frac{1}{2}$.*

That is, with discrimination against i, for all effort levels, when both players exert the same effort, player i has a lower probability of winning than player j. Given our assumptions on p_i, this means that if i is discriminated against, then $p_i \geq \frac{1}{2}$ implies $e_i > e_j$, whereas $p_j > \frac{1}{2}$ and $e_j < e_i$ are possible.

Definition 2. *We say that player i is **better** than player j if $\beta_i < \beta_j$.*

We focus on the case where, if a player is discriminated against, this player is also better than the other player. A way to represent this situation is by assuming $\alpha_1 = \alpha_2 = \beta_1 = n_1 = 1$, $\beta_2 > 1$, and $n_2 > 1$. Thus, $p_1 = \frac{e_1}{e_1 + n_2 e_2}$ with $n_2 > 1$. That is, player 1 is discriminated against. Moreover, the cost function of player 1 is $c_1(e_1) = e_1$, whereas that of player 2 is $c_2(e_2) = e_2^{\beta_2}$ with $\beta_2 > 1$. That is, player 1 is better than player 2.

Figure 9 shows an example where the probability that player 1 wins increases from less than 50% to more than 50% when the jury size is increased from 1 to

9. Proposition 1 can be easily adapted to non-linear cost functions to show that a sufficient condition for a pure strategy equilibrium is that $m^{\frac{N+1}{2}} \leq 1$. Therefore, for $N = 9$ we require $m \leq 0.2$, and, thus, we set $m = 0.2$.

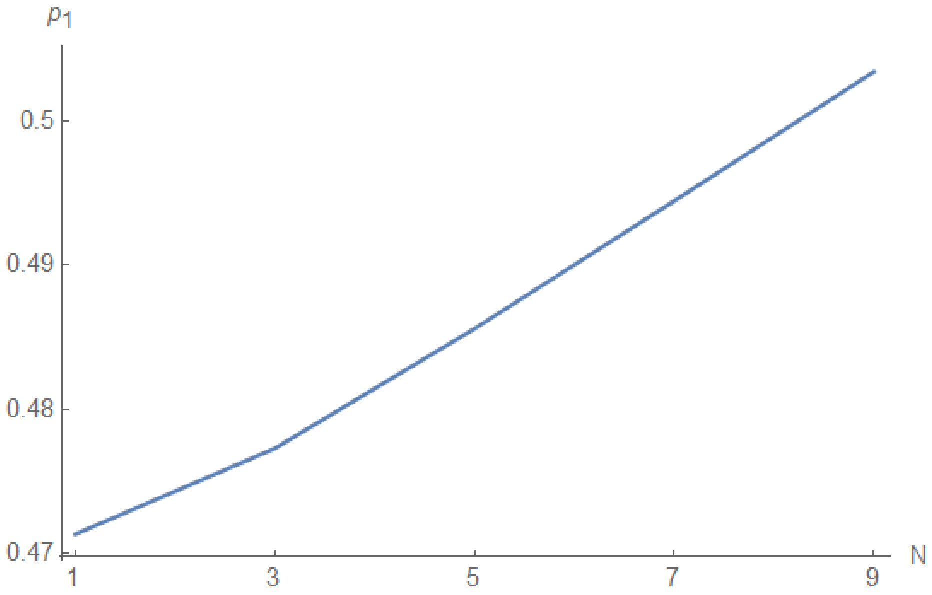

Figure 9: Probability that player 1 wins as jury size increases with $v = 10$, $\alpha_1 = \beta_1 = n_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 3$, $n_2 = 1.1$ and $m = 0.2$.

As shown in Figure 9, with one juror (i.e. the standard Tullock contest), the player who is discriminated against has less than a 50% probability of winning. As the jury size increases, this probability rises until it exceeds 50% with 9 jurors. This shows that increasing the jury size can help address the effects of discrimination. This is our next result, which we have illustrated through this example:

Proposition 6. *Consider a scenario where a player is discriminated against. Increasing the jury size, even if the additional jurors also discriminate against that player, can increase the probability that the discriminated player wins.*

Note that there are situations where discriminating against the better player in favour of the weaker player may be desirable on normative grounds. For papers that explore traditional contest settings with positive discrimination, see Franke (2012), Calsamiglia et al. (2013), Fu and Wu (2020) and Chowdhury et al. (2023)).

4.6 Sequential Effort Choice

In this section, we explore the scenario where players choose their effort levels sequentially. Player 1 chooses their effort e_1 first and, after observing e_1, player 2 chooses their effort level e_2. To simplify, we consider a Tullock contest with $v = 1$, $\alpha_1 = n_1 = m_1 = 1$, $n_2 = m_2 = 1$, and $\alpha_2 = \alpha > 0$. With these parameter values, we have $p_i = \frac{e_i}{\sum_j e_j}$, $c_1(e_1) = e_1$, and $c_2(e_2) = \alpha e_2$.

Using backwards induction, the best response of player 2 to the effort level chosen by player 1 is equivalent to the case with simultaneous choice of effort:

$$e_2 = \frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N+1}{2}} (1-p_1)^{\frac{N+1}{2}} \frac{1}{\alpha}. \tag{9}$$

Since, in equilibrium, e_2 depends on the effort level chosen by player 1, we can use implicit differentiation to obtain the following result after some algebra:

$$\frac{\mathrm{d}e_2}{\mathrm{d}e_1} = \frac{\frac{N+1}{2}(1-2p_1)\frac{1-p_1}{p_1}}{1+\frac{N+1}{2}(1-2p_1)}. \tag{10}$$

The first order condition of player 1 is

$$\frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N-1}{2}} (1-p_1)^{\frac{N-1}{2}} \left[\frac{\partial p_1}{\partial e_1} + \frac{\partial p_1}{\partial e_2}\frac{\mathrm{d}e_2}{\mathrm{d}e_1}\right] = 1,$$

where $\frac{\mathrm{d}e_2}{\mathrm{d}e_1}$ is given in equation (10). Substituting we obtain

$$e_1 = \frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N+1}{2}} (1-p_1)^{\frac{N+1}{2}} \left[1 - \frac{\frac{N+1}{2}(1-2p_1)}{1+\frac{N+1}{2}(1-2p_1)}\right]$$

Combining this with the best response of player 2 from equation (9), we obtain the following equilibrium condition:

$$\frac{1}{\alpha}\frac{p_1}{1-p_1} = \frac{1}{1+\frac{N+1}{2}(2p_1-1)}.$$

Solving the equation above yields the equilibrium value for p_1:

$$p_1 = \frac{1+\alpha+\frac{N+1}{2} - \sqrt{(1+\alpha+\frac{N+1}{2})^2 - 8\alpha\frac{N+1}{2}}}{4\frac{N+1}{2}},$$

the positive root leads to a minimum in the player's optimization problem and is thus ignored.

In the standard Tullock contest with a single juror ($N = 1$), the probability of player 1 winning is given by $p_1 = \frac{\alpha}{2}$ (note that for $\alpha > 2$, a pure strategy equilibrium does not exist). This implies that the player with the lower cost exerts more effort and consequently has a higher probability of winning, regardless of which player moves first. In other words, there is no first-mover advantage. This result is well-documented in the contests literature; see, for example, Linster (1993) or Morgan (2003).

When $N > 1$, the equilibrium value of p_1 is real if and only if $\alpha \leq 1$. Consequently, for some values of N and $\alpha > 1$, there is no equilibrium in pure strategies. The reason is that when player 1 moves first and has a lower cost function, the jury's presence leads player 1 to choose an effort level such that player 2 prefers not to compete at all, resulting in a best response of $e_2 = 0$. Therefore, in these cases, equilibrium can only be achieved in mixed strategies.

If $\alpha \leq 1$, our findings still hold. Figure 10 shows the sum of equilibrium efforts as a function of jury size when $\alpha = 0.95$. As with simultaneous competition, for small jury sizes, increasing the jury size raises the sum of efforts because the uncertainty effect dominates the pivotality effect. For large jury sizes, the pivotality effect dominates, and equilibrium efforts decrease.

Comparing sequential competition and simultaneous competition is a complex issue. The type of contest that leads to more aggregate effort depends non-trivially on the characteristics of the contests, such as the players and the contest success function. For more information, see, for example, Morgan (2003), Serena (2017), or Hinnosaar (2024).

5 Conclusions

We considered contests in which the winner is chosen by a jury. A jury introduces two distinct effects on a contest: an uncertainty effect and a pivotality effect. The

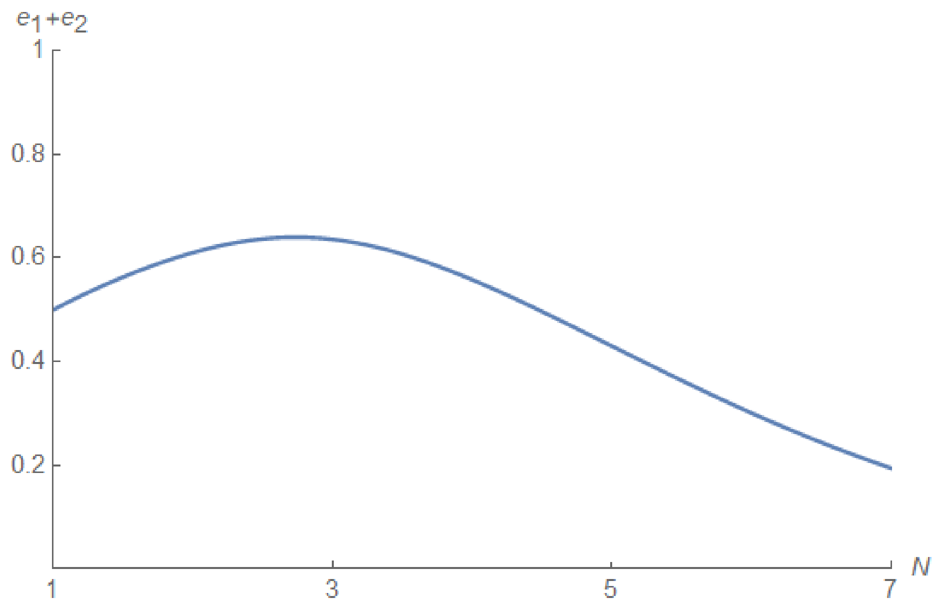

Figure 10: Sum of efforts with sequential competition, player 1 chooses first, with $\alpha_1 = n_1 = 1$, $n_2 = 1$, $\alpha_2 = \alpha = 0.95$ and $m = 1$. Effort maximising jury size is $N = 3$.

pivotality effect occurs because marginal increases in effort increase marginal profit only when there is a pivotal juror. As the jury size increases, the likelihood of a pivotal juror decreases, which in turn reduces the returns to effort.

Among other findings, we have shown that there is an effort-maximising jury size and that the presence of a jury can mitigate discrimination against contestants. Furthermore, we found that in contests where one candidate is better than the others, it may be beneficial to use a voting rule requiring the better player to secure more votes to win. Additionally, in contests with three or more players, there may be crowding out of the more efficient players.

Our results have implications for contest design, rent dissipation, and discrimination, among other areas. Future research could explore both experimentally and empirically the pivotality effect and, more broadly, how the presence of a jury affects a contest across its various dimensions. Future work could also look at heterogeneous jurors and determine what is the optimal jury composition from the point of view of effort choice, or at strategic jurors where each juror has a bias towards either candidate and receives information about effort choices that is not independent and identically distributed.

References

Abramowitz, M. and I. Stegun (1965): Handbook of Mathematical Functions. New York, NY: Dover.

Alcalde, J. and M. Dahm (2007): "Tullock and Hirshleifer: A Meeting of the Minds", *Review of Economic Design* 11, 101-124.

Altmann, S., A. Falk and M. Wibral (2012): "Promotions and Incentives: The Case of Multistage Elimination Tournaments", *Journal of Labor Economics* 30 (1), 149-174.

Arbatskaya, M. and H. Mialon (2010): "Multi-Activity Contests". *Economic Theory* 43, 23-43

Baye, M.R., D. Kovenock and C.G. de Vries(1996): " The All-Pay Auction with Complete Information", *Economic Theory* 8, 291-305.

Calsamiglia, C., J. Franke and P. Rey-Biel (2013): "The Incentive Effects of Affirmative Action in a Real-Effort Tournament", *Journal of Public Economics* 98, 15-31.

Clark, D. and C. Riis (1998): "Competition over More than One Prize", *The American Economic Review* 88 (1), 276-289.

Chowdhury, S., P. Esteve-González and A. Mukherjee (2023): "Heterogeneity, Leveling the Playing Field, and Affirmative Action in Contests", *Southern Economic Journal* 89 (3), 924-974.

Chowdhury, S., D. Kovenock and R. Sheremeta (2013): "An Experimental Investigation of Colonel Blotto Games". *Economic Theory* 52, 833-861.

Chowdhury, S., D. Kovenock, D. Rojo Arjona and N. Wilcox (2021): "Focality and Asymmetry in Multi-Battle Contests", *The Economic Journal* 131 (636), 1593-1619.

Corchón, L., and Dahm, M. (2010): "Foundations for Contest Success Functions", *Economic Theory* 43 (1), 81-98.

Corchón, L., and M. Serena (2018): "Contest Theory", In Handbook of Game Theory and Industrial Organization, Volume II 125-146. Edward Elgar Publishing.

Deku, C., S. Srangi and M. Wiser (2015): "An Experimental Investigation of Simultaneous Multi-Battle Contests with Strategic Complementarities", *Journal of Economic Psychology* 63, 117-134.

Ewerhart, C. (2015): "Mixed Equilibria in Tullock Contests", *Economic Theory* 60, 59-71.

Ewerhart, C. (2017): "The Lottery Contest is a Best-Response Potential Game", *Economics Letters* 155, 168-171.

Feng, X., Q. Jiao and Z. Kuang (2024): "Optimal Prize Design in Team Contests with Pairwise Battles", *Journal of Economic Theory* 215, 1057-1065.

Feng, x. and J. Lu (2017): "Uniqueness of Equilibrium in Two-Player Asymmetric Tullock Contests with Intermediate Discriminatory Power", *Economic Letters* 159, 61-64.

Franke, J. (2012): "Affirmative Action in Contest Games", *European Journal of Political Economy* 28, 105-118.

Fu, Q., Q. Jiao and J. Lu (2015): "Contests with Endogenous Entry", *International Journal of Game Theory* 44, 387-424.

Fu, Q., J. Lu, and Y. Pan (2015): "Team Contests with Multiple Pairwise Battles", *American Economic Review* 105 (7), 2120-2140.

Fu, Q. and Z. Wu (2020): "On the Optimal Design of Biased Contests", *Theoretical Economics* 15, 1435-1470.

Harris, C. and J. Vickers (1987): "Racing with Uncertainty", *Review of Economic Studies* 54 (1), 1-21.

Hinnosaar, T. (2024): "Optimal Sequential Contests", *Theoretical Economics* 19, 207-244.

Hirshleifer, J. (1989): "Conflict and Rent-Seeking Success Functions: Ratio vs. Difference Models of Relative Success", *Public Choice* 63 (2), 101-112.

Hortala-Vallve, R. and A. Llorente-Saguer (2012): "Pure Strategy Nash Equilibria in Non-Zero Sum Colonel Blotto Games", *International Journal of Game Theory* 41, 331-343.

Hortala-Vallve, R. and A. Llorente-Saguer (2015): "An Experiment on Non-Zero Sum Colonel Blotto Games", *Working Paper.*

Klumpp, T. and M. Polborn (2006): "Primaries and the New Hampshire Effect", *Journal of Public Economics* 90, 1073-1114.

Knight, B. and N. Schiff (2010): "Momentum and Social Learning in Presidential Primaries", *Journal of Political Economy* 118 (6), 1110-1150.

Konrad, K. (2009). "Strategy and Dynamics in Contests", Oxford University Press.

Kovenock, D., and B. Roberson (2010): "Conflicts with Multiple Battlefields", in M. R. Garfinkel and S. Skaperdas (Eds.), Oxford Handbook of the Economics of Peace and Conflict. Empirical and Theoretical Methods.

Linster, B. (1993): "Stackelberg Rent-Seeking", *Public Choice* 77, 307-321.

Mago, S. D., and Sheremeta, R. M. (2017): "Multi-battle Contests: An Experimental Study", *Southern Economic Journal* 84 (2), 407-425.

Malueg, D. and A. Yates (2010): "Testing Contest Theory: Evidence from Best-of-Three Tennis Matches", *Review of Economics and Statistics* 92 (3), 689-692.

Mengel, F. and J. Rivas (2017): "Common Value Elections with Private Information and Informative Priors: Theory and Experiments", *Games and Economic Behavior* 104, 190-221.

Moldovanu, B., and A. Sela (2001): "The Optimal Allocation of Prizes in Contests." *American Economic Review* 91 (3), 542-558.

Morgan, J. (2003): “Sequential Contests”, *Public Choice* 116, 1-18.

Morgan, J. D. Sisak and F. Várdy (2018): “The Ponds Dilemma”, *The Economic Journal* 128 (611), 1634-1682.

Palfrey, T. and H. Rosenthal (1983): “A Strategic Calculus of Voting”, *Public Choice* 41 (1), 7-53.

Rai, B. and R. Sarin (2009): “Generalized Contest Success Functions”. *Economic Theory* 40, 139-149.

Serena, M. (2017): “Sequential Contests Revisited”, *Public Choice* 173, 131-144.

Skaperdas, S. (1996): “Contest Success Functions”, *Economic Theory* 7 (2), 283-290.

Snyder, J. (1989): “Election Goals and the Allocation of Campaign Resources”, *Econometrica* 57 (3), 637-660.

Stephenson, D. (2024): “Multi-battle Contests over Complementary Battlefields”, forthcoming in *Review of Economic Design*.

Szentes, B. and R. Rosenthal (2003a): “Three-Object Two-Bidder Simultaneous Auctions: Chopsticks and Tetrahedra”, *Games and Economic Behavior* 44 (1), 114-133.

Szentes, B. and R. Rosenthal (2003b): “Beyond Chopsticks: Symmetric Equilibria in Majority Auction Games”, *Games and Economic Behavior* 45 (2), 278-295.

Tullock, G. (1980): “Efficient Rent-Seeking”, in Buchanan, J. R. Tollison and G. Tullock (Eds.), Toward a Theory of the Rent-Seeking Society (pp. 97-112). College Station: Texas A&M University Press.

Vesperoni, A. and A. Yildizparlak (2017): “Contests with Draws: Axiomatization and Equilibrium”, *Economic Inquiry* 57, 1597-1616.

Wadsworth, G. (1960): Introduction to Probability and Random Variables. New York: McGraw-Hill.

Appendix

Proof of Equation (1)

We want to show the following equality for all $N \in \{1, 2, \ldots\}$, $p \in [0, 1]$ and $q \in (0, 1]$:

$$\sum_{k=\lceil Nq \rceil}^{N} \binom{N}{k} p^k (1-p)^{N-k} \quad = \quad \frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} \int_0^p x^{\lceil Nq \rceil - 1}(1-x)^{N-\lceil Nq \rceil} \mathrm{d}x.$$

Define $S = \sum_{k=\lceil Nq \rceil}^{N} \binom{N}{k} p^k (1-p)^{N-k}$. In the right-hand side in the equation above, set $u = (1-x)^{N-\lceil Nq \rceil}$ and $\mathrm{d}v = x^{\lceil Nq \rceil - 1}\mathrm{d}x$, then integrate by parts to obtain

$$\frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} \quad \int_0^p x^{\lceil Nq \rceil - 1}(1-x)^{N-\lceil Nq \rceil}\mathrm{d}x = \frac{N!}{\lceil Nq \rceil!(N - \lceil Nq \rceil)!} p^{\lceil Nq \rceil}(1-p)^{N-\lceil Nq \rceil} + \frac{N!}{\lceil Nq \rceil!(N - \lceil Nq \rceil - 1)!} \int_0^p x^{\lceil Nq \rceil}(1-x)^{N-\lceil Nq \rceil - 1}\mathrm{d}x.$$

Notice how the first term on the right-hand size equals the first term in the series S. In the integral on the right-hand side, integrate by parts again setting $u = (1-x)^{N-\lceil Nq \rceil - 1}$ and $\mathrm{d}v = x^{\lceil Nq \rceil}\mathrm{d}x$ to obtain

$$\frac{N!}{\lceil Nq \rceil!(N - \lceil Nq \rceil - 1)!} \quad \int_0^p x^{\lceil Nq \rceil}(1-x)^{N-\lceil Nq \rceil - 1}\mathrm{d}x = \frac{N!}{(\lceil Nq \rceil + 1)!(N - \lceil Nq \rceil - 1)!} p^{\lceil Nq \rceil + 1}(1-p)^{N-\lceil Nq \rceil - 1} + \frac{N!}{(\lceil Nq \rceil + 1)!(N - \lceil Nq \rceil - 2)!} \times \int_0^p x^{\lceil Nq \rceil + 1}(1-x)^{N-\lceil Nq \rceil - 2}\mathrm{d}x.$$

The first term on the right-hand size equals the second term in the series S. Continue in this fashion until, after the s-th integration by parts with $0 < s < N - \lceil Nq \rceil$, we arrive at the integral

$$\frac{N!}{(\lceil Nq \rceil + s - 1)!(N - \lceil Nq \rceil - s)!} \int_0^p x^{\lceil Nq \rceil + s - 1}(1-x)^{N-\lceil Nq \rceil - s}\mathrm{d}x.$$

Integrate by parts setting $u = (1-x)^{N-\lceil Nq\rceil-s}$ and $\mathrm{d}v = x^{\lceil Nq\rceil+s-1}\mathrm{d}x$ to obtain

$$\begin{aligned}\frac{N!}{(\lceil Nq\rceil+s-1)!(N-\lceil Nq\rceil-s)!} \quad & \int_0^p x^{\lceil Nq\rceil+s-1}(1-x)^{N-\lceil Nq\rceil-s}\mathrm{d}x = \\ & \frac{N!}{(\lceil Nq\rceil+s)!(N-\lceil Nq\rceil-s)!} \\ & \times p^{\lceil Nq\rceil+k}(1-p)^{N-\lceil Nq\rceil-s} \\ & +\frac{N!}{(\lceil Nq\rceil+s)!(N-\lceil Nq\rceil-s-1)!} \\ & \times\int_0^p x^{\lceil Nq\rceil+s}(1-x)^{N-\lceil Nq\rceil-s-1}\mathrm{d}x.\end{aligned}$$

Notice that the first term on the right-hand side equals the $(s+1)$-th term in the series S. Continue in this manner until, after performing $(N-\lceil Nq\rceil)$ integrations by parts, we are left with the integral $\frac{N!}{(N-1)!}\int_0^p x^{N-1}\mathrm{d}x$, which evaluates to p^N, i.e. the last term in the series S.

Proof of Proposition 1

For the effort levels in (5) to be an equilibrium, the second order conditions must be satisfied. The second derivative of the profit function for $i \in \{1,2\}$ is given by

$$\frac{\partial^2\pi_i}{\partial^2 e_i} \;=\; v\frac{N!}{\frac{N-1}{2}!2}p_i^{\frac{N-3}{2}}(1-p_i)^{\frac{N-3}{2}}\left(\frac{N-1}{2}(1-2p_i)\frac{\partial p_i}{\partial e_i}^2+p_i(1-p_i)\frac{\partial^2 p_i}{\partial^2 e_i}\right)-c_i''(e_i).$$

Substituting the functions p_i and c_i we obtain

$$\frac{\partial^2\pi_i}{\partial^2 e_i} \;\propto\; (1-2p_i)m\frac{N+1}{2}-1.$$

Therefore, if $m\frac{N+1}{2} \leq 1$, the profit function is concave as desired. If $m\frac{N+1}{2} > 1$, there exists an $\hat{e} > 0$ such that the profit function of player i is convex for $e_i < \hat{e}$ and concave for $e_i > \hat{e}$. Consequently, there is a single concave region. This implies that if the profit function is concave at (e_1^*, e_2^*), then e_1^* is a global maximiser of player 1's profit function when player 2's effort is e_2^*, and vice versa.

By Equation (4), we have that $p_1 = \frac{1}{1+\theta}$. Thus, the second order conditions are met if and only if

$$\begin{aligned}\left(1 - 2\frac{1}{1+\theta}\right) m\frac{N+1}{2} - 1 &< 0, \\ \left(1 - 2\frac{\theta}{1+\theta}\right) m\frac{N+1}{2} - 1 &< 0.\end{aligned}$$

Moreover, for the effort levels in Equation (5) to constitute an equilibrium, both players must have non-negative profits. Specifically, given the effort levels in Equation (5), it must be that

$$v \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} - c_i(e_i) \;\geq\; 0$$

for both $i \in \{1,2\}$. Substituting the functions p_i and c_i and using Equation (5) we obtain

$$\begin{aligned}v \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} \left(\frac{1}{1+\theta}\right)^k \left(\frac{\theta}{1+\theta}\right)^{N-k} - \frac{N!}{\frac{N-1}{2}!^2} \left(\frac{\theta}{(1+\theta)^2}\right)^{\frac{N+1}{2}} m &\geq 0, \\ v \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} \left(\frac{\theta}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)^{N-k} - \frac{N!}{\frac{N-1}{2}!^2} \left(\frac{\theta}{(1+\theta)^2}\right)^{\frac{N+1}{2}} m &\geq 0.\end{aligned}$$