Fluctuations of Dynamic Convex Hulls

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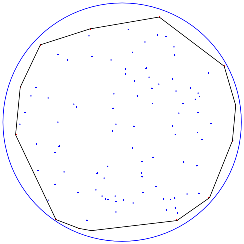
Stochastic Geometry in Action

Introduction

- K: $\mathcal{C}^3$ smooth convex body in $\mathbb{R}^d$.
- K_n: convex hull of n i.i.d. uniform points in K.
- $K := \mathbb{B}^2, n = 100$:



- Boundary: ∂K_n . As n increases, new points appear, creating new facets which may subsume existing facets.
- Dynamics: 'peaks' are smoothed, 'valleys' are filled in.

$K_n :=$ convex hull of n iid uniform pts

- $d = 2$. $K = \mathbb{B}^2$.
- $\mathcal{F}_n$: Facet chosen at random from the facets of K_n.
- $\mathrm{dist}(\mathcal{F}_n)$: distance between the hyperplane containing $\mathcal{F}_n$ and nearest parallel supporting hyperplane on K.
- **Proposition.**
  $\mathbb{E}[\mathrm{dist}(\mathcal{F}_n)] = Cn^{-\frac{2}{3}}(1 + o(1))$; $\mathbb{E}[\mathrm{length}(\mathcal{F}_n)] = Cn^{-\frac{1}{3}}(1 + o(1))$.
- Scaling exponents $\frac{1}{3}, \frac{2}{3}$ reflect locally parabolic behavior of ∂K.

$\frac{1}{3}, \frac{2}{3}$ scaling

- Is there any significance to $\frac{1}{3}, \frac{2}{3}$?
- These scaling exponents appear when describing the fluctuations of planar growth models:

a) the Brownian bridge constrained to lie above semi-circle,

b) Poissonian last passage percolation,

c) the random cluster model,...

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- These and other growth models, though seemingly different, are conjectured to exhibit similar behavior on large time and space scales. More precisely, ...

KPZ universality conjecture

- Planar random growth models have a two-parameter height function $h(t, x)$, with t the time and x the one-dimensional spatial position.
- KPZ universality conjecture: for a large class of models, under 1:2:3 scaling, for fixed $t > 0$ the re-scaled height function converges as $n \to \infty$

$$\frac{h(n^{\frac{3}{3}}t, n^{\frac{2}{3}}x) - C_n t}{n^{\frac{1}{3}}} \xrightarrow{\mathcal{D}} H(t, x; H_0)$$

  with $H(t, x; H_0)$ a model independent two-parameter field which depends only on initial data H_0.
- $C_n t$ non-random, determined by macroscopic limit

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- $C_n t$ non-random, determined by macroscopic limit
- $H(t,x;H_0)$ given by the variational formula

$$H(t,x;H_0) = \sup_{y\in\mathbb{R}}\{t^{1/3}A(t^{-2/3}x, t^{-2/3}y) + H_0(y) - \frac{(x-y)^2}{t}\}$$

with $A(x,\cdot)$ a stationary process which is locally Brownian in space.

KPZ universality

- KPZ universality class should not be confused with KPZ equation of Kardar, Parisi, and Zhang (1986)

$$\frac{\partial}{\partial t}h(t,x) = \nu\frac{\partial^2}{\partial x^2}h(t,x) + \lambda(\frac{\partial}{\partial x}h(t,x))^2 + \text{Gaussian white noise},$$

  the ill-posed canonical continuum equation for planar random growth.

- $h = h(t,x)$ belongs to KPZ universality class. (Hairer, Quastel 2014; Quastel, Sarkar 2023; Virag 2020)

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- TASEP (Matetski, Remenik, Quastel 2021)
- ASEP (Quastel, Sarkar 2022)

KPZ universality class

- Major unresolved problem: determine sufficient conditions guaranteeing that a growth model belongs to KPZ universality class.
- Loosely speaking, when the model has linear scale n, the height statistic $h(t, x)$ should exhibit:

  a) global parabolic constraints,

  b) fluctuations of the order $n^{1/3}$,

  c) spatial correlations of the order $n^{2/3}$,

  d) marginals converging to Tracy-Widom, and

  e) locally Gaussian fluctuations in space coordinate x.

Tracy-Widom Distribution -1994

- Gaussian Unitary/Orthogonal Ensemble. Given $N \times N$ random Hermitian matrix with i.i.d. mean zero complex/real Gaussian entries with variance N, the largest eigenvalue $\lambda_{\max}$ satisfies

$$\frac{\lambda_{\max} - 2N^{1/2}}{N^{1/3}} \xrightarrow{\mathcal{D}} TW$$

- GUE: $\mathbb{P}(TW \geq s) = \exp(-\int_s^\infty (x-s)u(x)^2 dx),\ s \in \mathbb{R}$

  $u'' = 2u^3 + xu$, Painlevé II (similar for GOE)

- G0E: $\mathbb{P}(TW \geq s) = \exp(-\frac{2}{3}s^{\frac{3}{2}}(1+o(1)).$

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- Length of longest increasing subsequence of random permutation; Baik, Deift, Johansson - 1999
- Fluctuations of corner growth model.

The convex hull

- How does the convex hull of i.i.d. uniform points fit into this picture?
- When the linear scale is n, it turns out that the two-parameter height statistic $h(t, x)$ for convex hull model exhibits:

  a) global parabolic constraints,

  b) fluctuations of the order $n^{1/3}$,

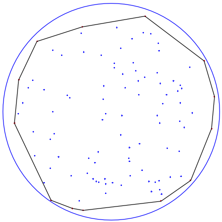
  c) spatial correlations of the order $n^{2/3}$,

  d) marginals converging to Tracy-Widom-like GOE distribution,

  e) but there are no locally Gaussian fluctuations.

The growth model: the convex hull of t i.i.d. pts

- Consider the convex hull of $\{X_i\}_{i=1}^t$, with X_i i.i.d. uniform in the unit disc $\mathbb{B}^2$.
- To make the linear scale t, we blow up by t: $\{tX_i\}_{i=1}^t$



- radial fluctuation of convex hull of $\{tX_i\}_{i=1}^t$ is of order $t^{1/3}$

KPZ scaling of height fct of convex hull

- $\{X_i\}_{i=1}^t$ i.i.d. uniform in the unit disc $\mathbb{B}^2$.
- $h(t, x)$ is the radius vector process (height) of $\mathrm{conv}(\{tX_i\}_{i=1}^t)$ in the direction $x \in t\mathbb{S}$.
- Object of interest: fluctuations of defect height process $t - h(t, x)$.

* convergence in $C([-L, L])$, L fixed.

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- Object of interest: fluctuations of defect height process $t - h(t,x)$.
- **Theorem** (convergence of defect height process*) Fix $t > 0$. At large time and length scales ($n \to \infty$), the re-scaled two-parameter height process satisfies

$$\left(\frac{nt - h(n^{\frac{3}{3}}t, n^{\frac{2}{3}}x)}{n^{\frac{1}{3}}}\right)_{|x|\le \pi n^{\frac{1}{3}}t} \xrightarrow{\mathcal{D}} (H(t,x))_{x\in\mathbb{R}}.$$

* convergence in $C([-L,L])$, L fixed.

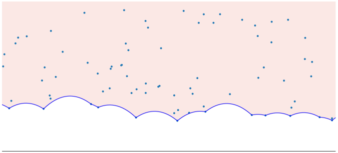
Scaling limit $H(t,x)$ given by variational formula

- Down parabola with apex at $(x',h') \in \mathbb{R} \times \mathbb{R}^+$:

$$\Pi_t^{\downarrow}(x',h') := \{(x,h) \in \mathbb{R} \times \mathbb{R},\ h \leq h' - \frac{|x-x'|^2}{2t}\}.$$

- $\mathcal{P}_{\frac{1}{t}}$: rate $\frac{1}{t}$ Poisson pt process on $\mathbb{R} \times \mathbb{R}^+$.
- Variational formula for H:

$$H(t,x) := \sup_{(x',h') \in \mathbb{R}\times\mathbb{R}^+,\ \Pi_t^{\downarrow}(x',h') \cap \mathcal{P}_{\frac{1}{t}} = \emptyset} (h' - \frac{|x-x'|^2}{2t}).$$



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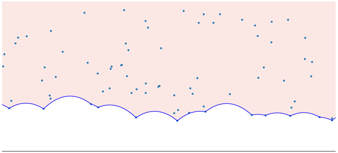
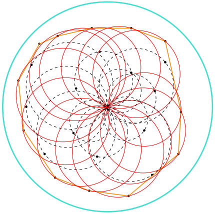


- Recall KPZ variational formula:

$$H(t,x;H_0) = \sup_{y\in\mathbb{R}}\{t^{1/3}A(t^{-2/3}x, t^{-2/3}y) + H_0(y) - \frac{(x-y)^2}{t}\}$$

KPZ scaling for support function of convex hull

- $X_i := (|X_i|, x_i), 1 \leq i \leq t$, i.i.d. uniform in $\mathbb{B}^2$; $|x_i| \leq \pi$.



Support process of convex hull of $\{tX_i\}_{i=1}^t$:

$$s(t,x) := \max_{1 \leq i \leq t} t|X_i| \cos\left(|\frac{x}{t} - x_i|\right), \ |x| \leq \pi t$$

- Object of interest: fluctuations of defect support process $t - s(t,x)$.
- Scale time by n, space by $n^{2/3}$, and fluctuations by $n^{1/3}$:

$$S_n(t,x) := \frac{nt - s(n^{\frac{3}{3}}t, n^{\frac{2}{3}}x)}{n^{\frac{1}{3}}}, \quad |x| \leq \pi n^{\frac{1}{3}} t$$

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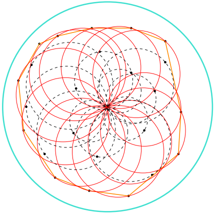

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- **Theorem** (convergence of defect support process) Fix $t > 0$. As $n \to \infty$

$$(S_n(t,x))_{|x| \leq \pi n^{\frac{1}{3}} t} \xrightarrow{\mathcal{D}} (S(t,x))_{x \in \mathbb{R}}.$$

KPZ for support process of convex hull

- The support process for the convex hull of $\{tX_i\}_{i=1}^{t}$ displays $1:2:3$ scaling and process-level convergence to a limit field $(S(t,x))_{x\in\mathbb{R}}$ whose parabolas have apices at the lowest pts of $\mathcal{P}_{\frac{1}{t}}$.
- Variational formula for S:

$$S(t,x) := \inf_{(x',h')\in\mathcal{P}_{\frac{1}{t}}} (h' + \frac{|x-x'|^2}{2t}).$$

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- Height process and support process for dynamic convex hull tK_t belongs to a KPZ sub-universality class.
- They would belong to KPZ universality class if the respective limit fields also contained a stationary process which was locally Brownian in the space variable.

Convergence of marginals

- Recall: $h(t,x)$ is radius vector process in direction $x \in t\mathbb{S}$ for convex hull of $\{tX_i\}_{i=1}^t$, with X_i iid in $\mathbb{B}^2$.
- Recall previous process convergence theorem: for fixed $t > 0$

$$\left(\frac{nt - h(n^{\frac{3}{3}}t, n^{\frac{2}{3}}x)}{n^{\frac{1}{3}}} \right)_{|x| \le n^{\frac{1}{3}} t} \xrightarrow{\mathcal{D}} (H(t,x))_{x \in \mathbb{R}}.$$

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- **Theorem** (convergence of marginals to Tracy Widom-like distribution) Fix $t = 1$ and let $x \in n^{\frac{1}{3}}\mathbb{S}$. Then as $n \to \infty$

$$\frac{n - h(n^{\frac{3}{3}}, n^{\frac{2}{3}}x)}{n^{\frac{1}{3}}} \xrightarrow{\mathcal{D}} h_\infty$$

GOE tails : $\mathbb{P}(h_\infty \ge \frac{s}{2}) \sim \exp(-\frac{2}{3}s^{\frac{3}{2}}(1+o(1))), \quad s \to \infty$

Maximal radial fluctuations follow Gumbel law

Theorem

- Let $K \subset \mathbb{R}^d$ be $\mathcal{C}^3$ smooth, convex, with curvature bounded away from zero. K_t is convex hull of $\{X_i\}_{i=1}^t$.
- There are constants $a_i := a_i(K)$, $i \in \{0,1,2,3\}$, such that if

$$u_t(\tau, K) := t^{-\frac{2}{d+1}}[a_0(a_1 \log t + a_2 \log(\log t) + a_3 + \tau)]^{\frac{2}{d+1}}, \ \tau \in (-\infty, \infty),$$

then as $t \to \infty$

$$\mathbb{P}\left(d_H(K_t, K) \le u_t(\tau, K)\right) \to \exp(-\theta(d)e^{-\tau}), \ \tau \in (-\infty, \infty).$$

- $\theta(2) = \frac{3}{4}$, $\theta(d) = d^2 2^{-d} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})} \frac{\Gamma(\frac{d^2+1}{2})}{\Gamma(\frac{d^2}{2}+1)}$.
- Not many (any?) results for maximal fluctuations of growth models.

Extensions

- Process convergence for defect height and support functions were stated for convex hulls of iid pts in unit disc $\mathbb{B}^2$.
- Disc may be replaced by any $\mathcal{C}^3$ smooth convex $K \subset \mathbb{R}^2$, with curvature bounded away from zero.

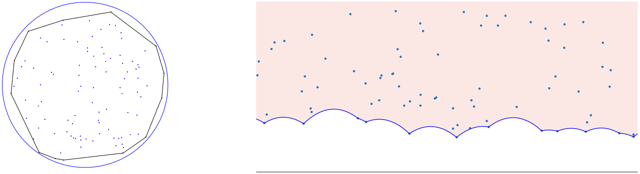
- Process convergence for height statistic $h(t,x)$ extends to all dimensions:

  · $1 : 2 : 3$ scaling replaced by $d-1$, d, $d+1$ scaling,

  · the triplet Fluctuations: Space: Time has scaling coefficients

$$n^{\frac{d-1}{d+1}},\ n^{\frac{d}{d+1}},\ n^{\frac{d+1}{d+1}}$$

Proof ideas: process convergence for radius vector fct



- Down parabola with apex at $(x', h') \in \mathbb{R} \times \mathbb{R}^+$:

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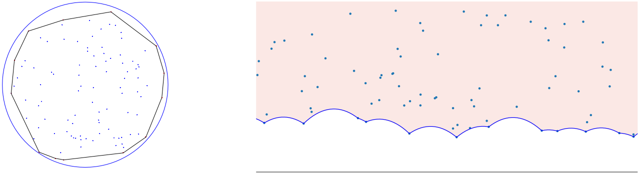
- $\mathcal{P}_{\frac{1}{t}}$: rate $\frac{1}{t}$ Poisson pt process on $\mathbb{R} \times \mathbb{R}^+$. Burgers' festoon H:

$$H(t, x) := \sup_{(x',h') \in \mathbb{R}\times\mathbb{R}^+,\ \Pi_t^{\downarrow}(x',h') \cap \mathcal{P}_{\frac{1}{t}} = \emptyset} (h' - \frac{|x - x'|^2}{2t}).$$

- Parabolas are the re-scaled asymptotic images of the edges in convex hull K_t.

We need to find the right scaling transform mapping $\partial(t \cdot K_t)$ to the upper half plane. The overly simplified approach has these ingredients:

- Define for each t and n a parabolic scaling transform $T^{(t,n)} : t\mathbb{B}^2 \to \mathbb{R} \times \mathbb{R}^+$



- $T^{(t,n)}$ maps the edges of $\partial(t \cdot K_t)$ to curves in $\mathbb{R} \times \mathbb{R}^+$ which are nearly parabolic and which become parabolic as $n \to \infty$.

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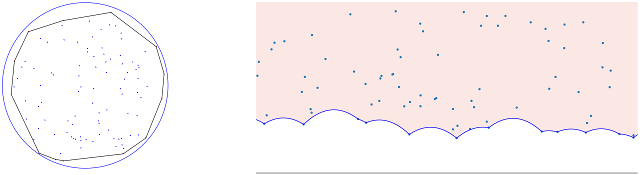

- $T^{(t,n)}$ maps the edges of $\partial(t \cdot K_t)$ to curves in $\mathbb{R} \times \mathbb{R}^+$ which are nearly parabolic and which become parabolic as $n \to \infty$.
- $T^{(t,n)}$ maps the t i.i.d. uniform points in $t\mathbb{B}^2$ to point process in rectangle $[-n^{\frac{1}{3}}t, n^{\frac{1}{3}}t] \times [0, n^{\frac{2}{3}}t]$.
- As $n \to \infty$ and using coupling, this point process converges to a Poisson point process in the upper half-plane of intensity $1/t$.

Proof ideas: convergence of facet distances

- **Typical facet:** $\mathcal{F}_\lambda$ is a typical facet of $K_{\mathrm{Po}(\lambda)}$ if for all non-neg. measurable g defined on the facets of K_λ we have

$$\mathbb{E}g(\mathcal{F}_\lambda) = \frac{1}{Z_\lambda}\mathbb{E}\sum_{F\in\mathrm{Facets}} g(F),$$

  Z_λ being the expected number of facets.

- **Mecke's formula**: $\mathbb{P}(g(\mathcal{F}_\lambda) \geq s) = \mathbb{E}[\mathbf{1}(g(\mathcal{F}_\lambda) \geq s)]$ equals

$$= \frac{1}{Z_\lambda}\mathbb{E}\Big[\sum_{\{x_1,\cdots,x_d\}\subset\mathcal{P}_\lambda^{\neq}} \mathbf{1}(g(\Delta(x_1,\cdots,x_d)) \geq s) \times \mathbf{1}(\Delta(x_1,\cdots,x_d) \text{ is a facet})\Big]$$

$$= \frac{1}{Z_\lambda}\frac{\lambda^d}{d!}\int_K \cdots \int_K \mathbf{1}(g(\Delta(x_1,...,x_d)) \geq s) \times e^{-\lambda \mathrm{Vol}_d(K\cap H^+(x_1,\cdots,x_d))}\mathrm{d}x_1\cdots\mathrm{d}x_d.$$

- if g is distance, $d=2$, then $e^{-\lambda\mathrm{Vol}_d(\dots)} \sim \exp(-\frac{2}{3}s^{\frac{3}{2}}(1+o(1)))$.

Summary

- Rescaled height process of dynamic convex hull tK_t converges as $n \to \infty$ to Burgers' festoon; the triplet 'fluctuations: space: time' exhibits $1:2:3$ scaling.
- The space-time height process of tK_t belongs to KPZ sub-universality class; same for support function process of tK_t.
- Marginal distribution of height process converges to an explicit limit distribution whose right-hand tail coincides with Tracy-Widom GOE tail.
- Maximal radial fluctuations converge to a Gumbel distribution.

Thank you!