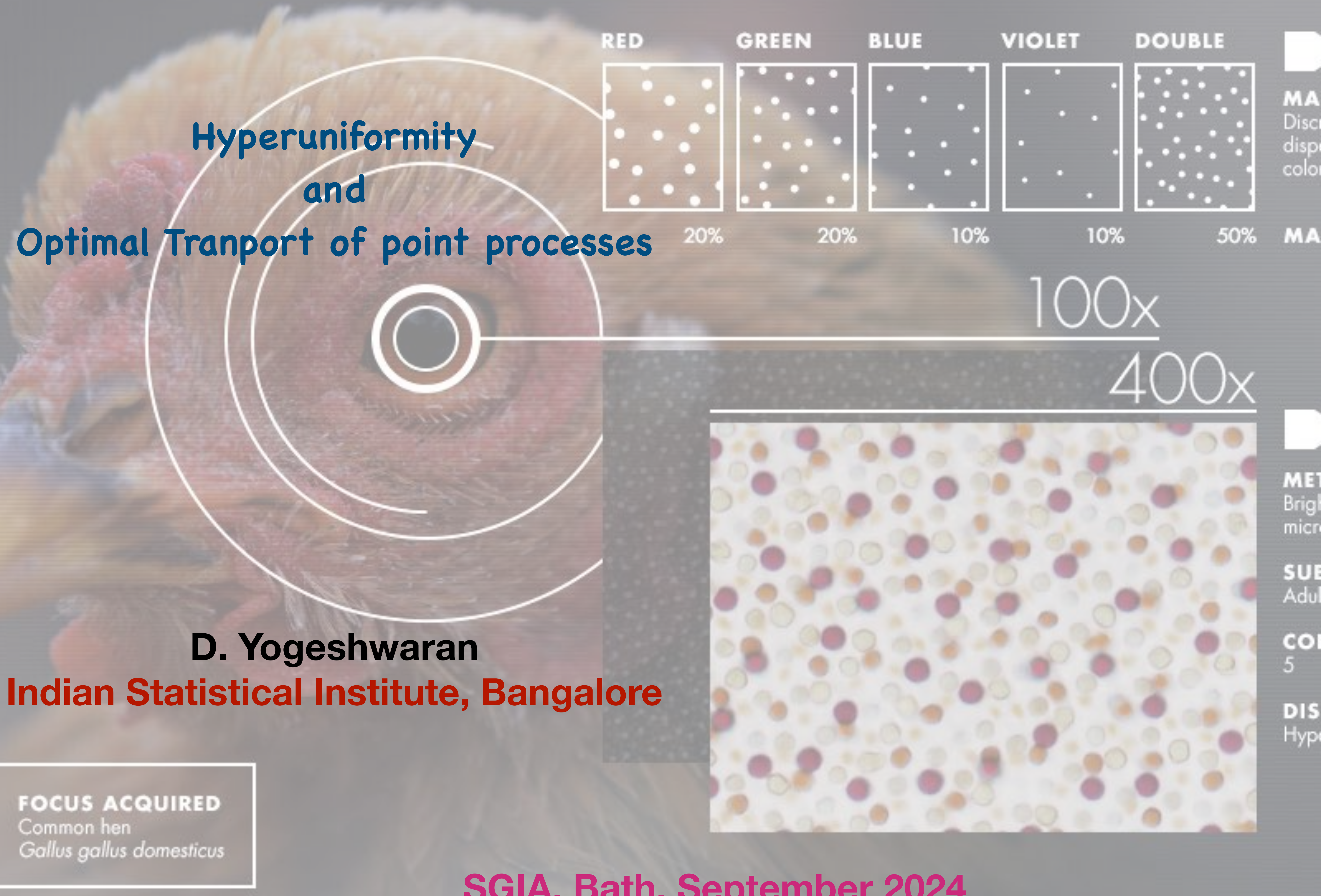
Hyperuniformity
and
Optimal Tranport of point processes



D. Yogeshwaran
Indian Statistical Institute, Bangalore

SGIA, Bath, September 2024

Pre-print available on arXiv: 2402.13705

Joint work with

Raphaël Lachièze-Rey,
INRIA Paris and Univ Paris Cité

Acknowledgements

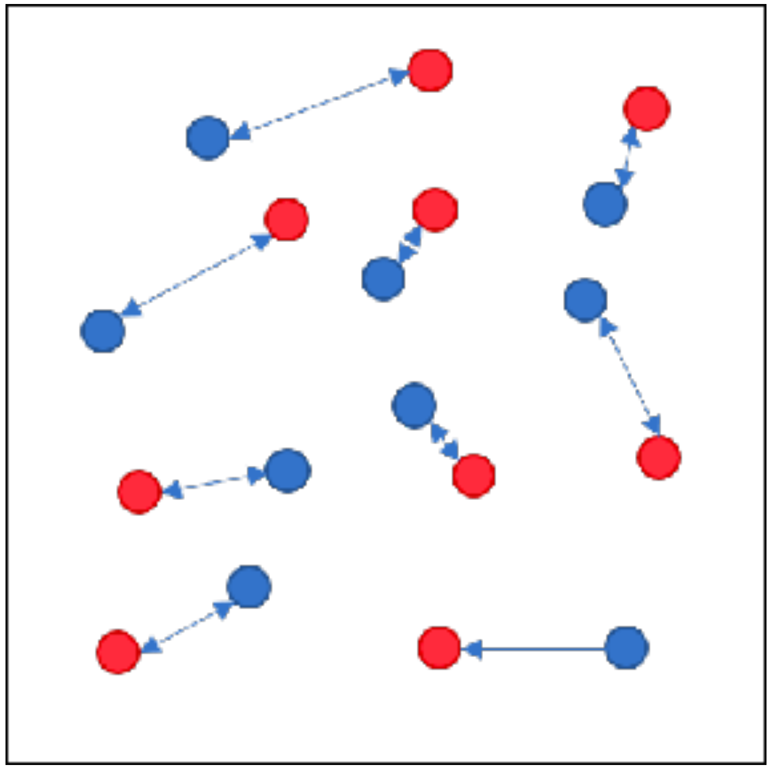
Raphaël Butez,
Manjunath Krishnapur,
Thomas Leblé and
David Dereudre.

Funding

SERB, India.
CNRS, France.

Optimal Matchings of i.i.d. points

$X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ i.i.d. points in $W_n := [0, n^{1/d}]^d$



Raphaël Lachièze-Rey

Empirical measures: $\mu_n = \frac{1}{n}\sum_i \delta_{X_i}$; $\nu_n = \frac{1}{n}\sum_i \delta_{Y_i}$

Matching Cost: $W_2^2(\mu_n, \nu_n) := \frac{1}{n} \min_{T:[n]\to[n]} \sum_i |X_i - Y_{T(i)}|^2$

Motivation – CS, comb. optimization, statistics, probability, Statistical Physics, ergodic theory, Optimal transport + PDE.

AKT (Ajtai, Komlós & Tusnády, 1984) Theorem: W.H.P. and in Expectation,

$W_2^2(\mu_n, \nu_n) \approx n$, $d = 1$; $W_2^2(\mu_n, \nu_n) \approx \ln n$, $d = 2$; $W_2^2(\mu_n, \nu_n) \approx C_d$, $d \geq 3$.

Typical matching distances – $\sqrt{n}$, $d = 1$; $\sqrt{\ln n}$, $d = 2$, ; $\sqrt{C_d}$, $d \geq 3$.

$\mathbb{P}(\nu_n \cap B_{X_1}(1) = \emptyset) = (1 - \frac{\theta_d}{n})^n \approx e^{-\theta_d}$ implies that $\mathbb{E}W_2^2(\mu_n, \nu_n) \geq C_d$

Original proof via a transportation algorithm; later Majorizing measures (Talagrand…),
Optimal Transport + PDE (Ambrosio, Stra & Trevisan, 2018) ;
'Magically simple' Fourier-analytic proof (Bobkov & Ledoux, 2019)

Optimal Matchings - Beyond AKT

$X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ i.i.d. points in $W_n := [0, n^{1/d}]^d$ $\quad \mu_n = \frac{1}{n}\sum_i \delta_{X_i}\,;\nu_n = \frac{1}{n}\sum_i \delta_{Y_i}$

Matching Cost: $W_2^2(\mu_n, \nu_n) := \frac{1}{n} \min_{T:[n]\to[n]} \sum_i |X_i - Y_{T(i)}|^2$

AKT (Ajtai, Komlós & Tusnády, 1984) Theorem: W.H.P. and in Expectation,

$W_2^2(\mu_n, \nu_n) \approx n,\ d = 1\,;\ W_2^2(\mu_n, \nu_n) \approx \ln n,\ d = 2\,;\ W_2^2(\mu_n, \nu_n) \approx C_d, d \geq 3.$

MUCH MORE - Grid-matching, transport to Lebesgue, p-matching costs, large deviations, exact constants, mixing sequences…..

Caraciolo, Lucibello, Parisi & Sicuro (2014) ; Ambrosio, Stra & Trevisan (2018)

$$d = 2; \mathbb{E}W_2^2(\mu_n, \nu_n) \sim \frac{\ln n}{2\pi} + \text{"tight seq."}$$

Ajtai, Komlós & Tusnády (1984) ; Shor & Yukich (1991) ; Talagrand (1990s) ; ………..

Caracciolo, Lucibello, Parisi, & Sicuro (2014) ; Guilin & Fournier (2015) ; ………

Ambrosio, Stra & Trevisan (2019) ; Goldman, Huesmann & Otto (2023) ; ……….

Optimal Matchings of Poisson processes

Poisson process - $\mu = \sum_x \delta_x$ in $\mathbb{R}^d$; unit intensity.

Lattice Matching - $T : \mathbb{Z}^d \to \mu$, translation-invariant bijection. Matching cost- ???

Typical matching cost - $X := |T(0)|$ also

$$\mathbb{P}(X > r) = (2k+1)^{-d}\, \mathbb{E} \sum_{z \in \mathbb{Z}^d \cap [-k,k]^d} \mathbf{1}\left[|z - T(z)| > r \right]$$

- Can be defined for matching between stationary point processes μ and ν.

THEOREM (HPPS, 2009) There exists a matching T such that

$$\mathbb{E}\left[\frac{X^{d/2}}{\log(X)^{d/2}} \right] < \infty \quad \text{for } d = 1,2 \text{ and for some } c,\ \mathbb{E}[e^{cX}] < \infty \text{ for } d \geq 3.$$

- Infinite version of **AKT theorem** - AKT + relative compactness. Bounds are tight.

- SD of $\mu \cap B(r) \approx r^{d/2}$ ('excess points') vs $\mu-$points near the boundary $\approx r^{d-1}$

Hoffman, Holroyd & Peres (2006) ; Holroyd, Pemantle, Peres & Schramm (2009) ; Huesmann & Sturm (2013).

ARE THERE POINT PROCESSES THAT DO BETTER OR WORSE ?

Do not listen to the prophets of doom who preach that every point process will eventually be found out to be a Poisson process in disguise!

\- *Gian-Carlo Rota*.

Greedy matching idea of HPPS

- Tile $\mathbb{R}^d$ by dyadic cubes $W_k = [-2^{k-1}, 2^{k-1}]^d, k \geq 1$ shifted randomly

- Keep matching greedily within each cube W_k and move onto the next scale W_{k+1}.

- Set $r = 2^k$. $\mathbb{P}(X > r) \leq r^{-d}\mathbb{E}\left(| \#\mu \cap W_r - \#\nu \cap W_r | \right) \leq r^{-d} SD(\mu \cap W_r)$

- $\Omega(r^{d-1}) = VAR(\mu \cap W_r) = o(r^{2d})$

- TYPICAL μ, $VAR(\mu \cap W_r) = O(r^d)$, volume-order variance decay; `Weak mixing'

- TYPICAL $\mu \Rightarrow \mathbb{P}(X > r) \leq C r^{-d/2}$ gives that

$$\mathbb{E}\left[\frac{X^{d/2}}{\log(X)^{d/2}} \right] < \infty$$

- Good in d = 1,2 and d > 4 but no finite second moments in d = 3,4.

Some point process notions

- $\mu = \sum_{i \geq 1} \delta_{x_i}$ - Stationary point process in $\mathbb{R}^d, (d \geq 1)$ with unit intensity; $\mathbb{E}\mu(A) = |A|$.

- Reduced Pair Correlation Measure (RPCM)

Heuristically, $\beta(\mathrm{d}x) = \dfrac{\mathbb{P}\left(\mathrm{d}x \in \mu \mid 0 \in \mu\right)}{\mathrm{d}x} - 1$ ($\beta \equiv 0$ for Poisson process)

Formally, for compactly supported φ

$$\mathbb{E}\Big[\sum_{x \neq y \in \mu} \varphi(x)\psi(y)\Big] = \int \varphi(x)\,\psi(y)\,\mathrm{d}x\,\mathrm{d}y + \int \varphi(x)\,\psi(x+z)\,\mathrm{d}x\,\beta(\mathrm{d}z)$$

- Variance formula - If β is integrable, $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = n\big(1 + \beta(\mathbb{R}^d)\big) + o(n)$; $|\Lambda_n| = n$.

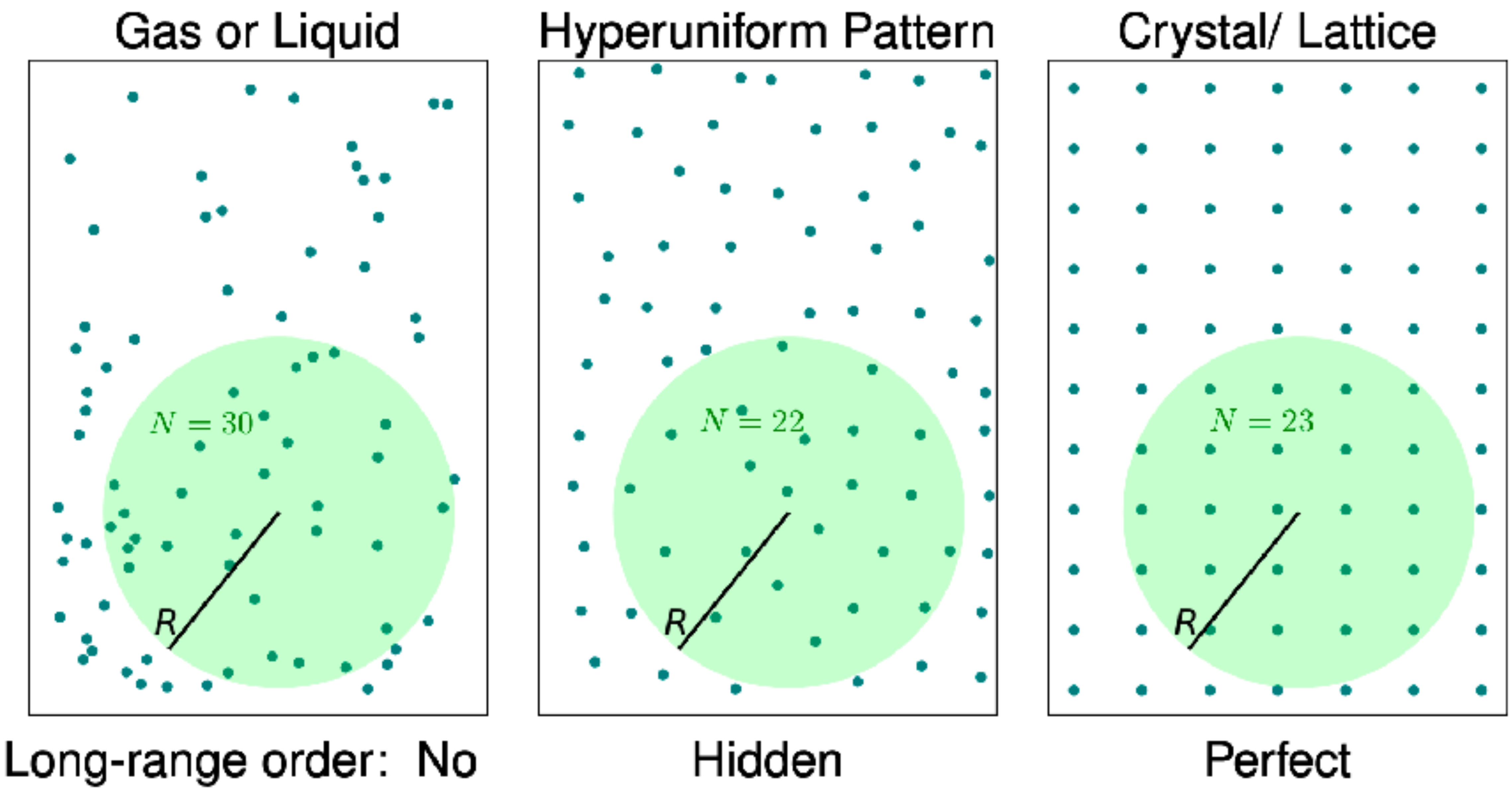
- Structure Factor - $S(k) = 1 + \int e^{-ik\cdot x}\,\beta(\mathrm{d}x),\ k \in \mathbb{R}^d$; well-defined if β is integrable.

- Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(n)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$. Integrable β

Gabrielli, Joyce, Labini (2002) ; Torquato-Stillinger (2003) ; Torquato (2018)

- Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(n)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$. Integrable β

'Large-scale suppression of fluctuations' or `local disorder and hidden long-range order'



$VAR(\mu(B_R)) \approx R^d$	$VAR(\mu(B_R)) = o(R^d)$	$VAR(\mu(B_R)) = O(R^{d-1})$
Volume-order variance	Lower than Volume-order variance	Surface-order variance

Figure due to Michael Klatt

Point processes and Empirical Structure Factors

Empirical approximation of $S + \delta_0$

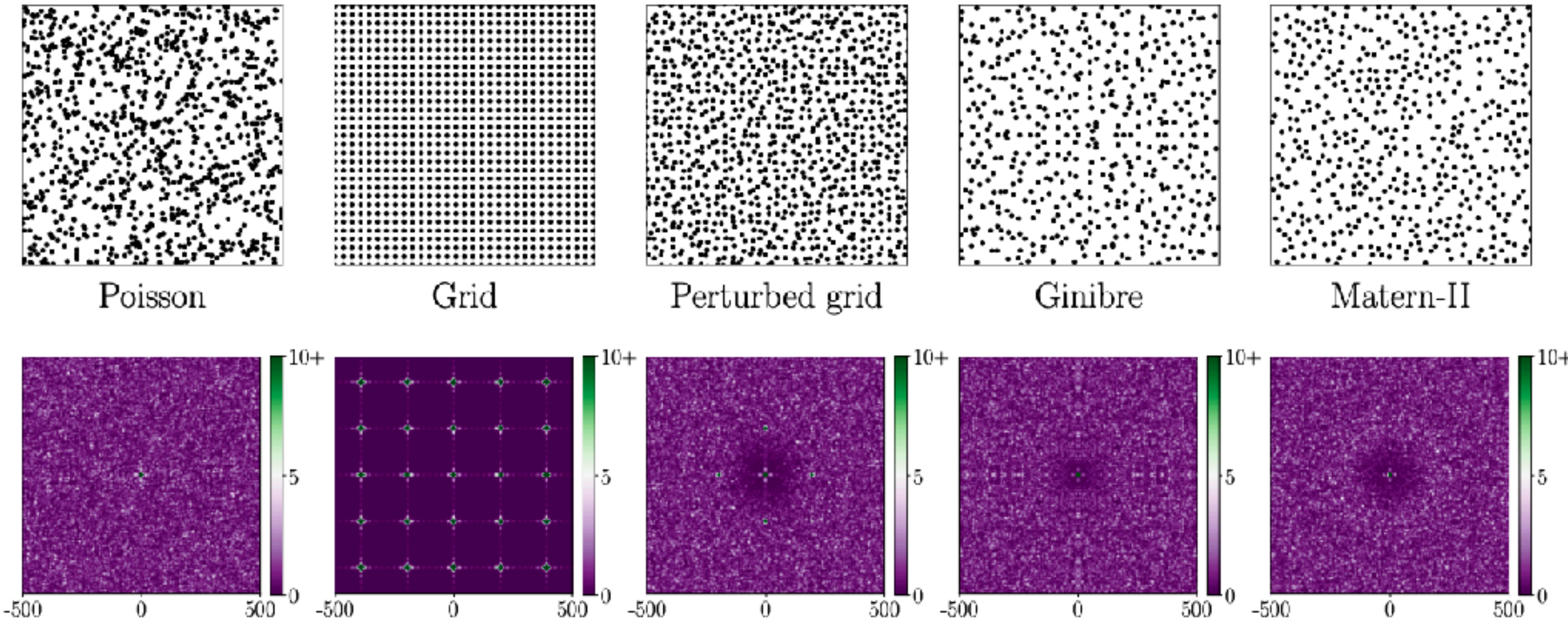


Figure due to Simon Coste

CAN TYPICAL POINT PROCESSES DO WORSE THAN POISSON ?

YES

CAN PLANAR HU POINT PROCESSES DO BETTER THAN POISSON ?

NO

Optimal Matchings of point processes

μ - Unit intensity stationary point process with integrable RPCM β

$T : \mathbb{Z}^d \to \mu$, a matching ; $X = |T(0)| -$ Typical transport/matching cost.

THEOREM (Lachièze-Rey & Y., 2024) There exists a matching T such that

$$\mathbb{E}\left[\frac{X^{d/2}}{\log(X)^{d/2}}\right] < \infty \quad \text{for } d = 1,2 \text{ and } \mathbb{E}[X^2] < \infty \text{ for } d \geq 3.$$

Integrable β - weaker than many mixing conditions typically assumed.

From the proof of Holroyd, Pemantle, Peres and Schramm (2009)

$$\mathbb{E}\left[\frac{X^{d/2}}{\log(X)^{d/2}}\right] < \infty \quad \text{for all } d.$$

Equals our bounds in d = 1,2 ; Worser than ours in d = 3,4

and Better than ours in d > 4.

`Almost all' HU processes are L^2-perturbed lattices

μ - Unit intensity stationary point process in $\mathbb{R}^2$; integrable RPCM β & hyperuniform+.

Hyperuniformity (HU) - $\mathbb{VAR}\big(\mu(\Lambda_n)\big) = o(n)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$.

(HU+) Condition - $\int ln\,(1 + |x|)\,|\beta|\,(\mathrm{d}x) < \infty$; Nearly same as $\mathbb{VAR}\big(\mu(\Lambda_n)\big) = o(\frac{n}{\log n})$

THEOREM (Lachièze-Rey & Y., 2024)

There exists a matching $T : \mathbb{Z}^2 \to \mu$ such that $\mathbb{E}[X^2] < \infty$.

HU+ processes - Ginibre process, Zeros of Gaussian analytic functions…. `Almost all' stationary HU processes.

I.I.D. Perturbed lattices - Not HU+ but have L^2 matching if perturbations are L^2 !!!

Huesmann, Leblé (2024) - Example of HU process with ∞ L^2 matching !!!

RELATED WORKS

Prod'Homme (2021) - L^2-Matching for Ginibre process.

Sodin, Tsirelson (2006) - Exponential Matching for Zeros of Gaussian analytic functions.

Butez, Dallaporta, García-Zelada (2024) - Bounds on $\mathbb{E}[X^p]$ assuming more on μ.

Huesmann, Leblé (2024) - Approach via finiteness of Coulomb energy.

Koivusalo, Lagace, ++ (2024) - Finite BL distance to lattice if $Disc = o(\frac{n}{\log n})$

`Almost all' HU processes are L^2-perturbed lattices

μ - Unit intensity stationary point process in $\mathbb{R}^2$; integrable RPCM β & hyperuniform+.

Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(n)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$.

(HU+) Condition - $\int ln\,(1 + |x|)\,|\beta|\,(\mathrm{d}x) < \infty$; Nearly same as $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(\frac{n}{\log n})$

THEOREM (Lachièze-Rey & Y., 2024)

There exists a matching $T : \mathbb{Z}^2 \to \mu$ such that $\mathbb{E}[X^2] < \infty$.

Dereudre, Flimmel, Huesmann, Leblé (2024) - Stationary L^2-perturbed lattices are HU.

COROLLARY $\sum_{x \in \mu} \delta_{x+Y(x)}$ is HU if $Y(x)$ is stationary with finite second moments.

HU process constructions - Matchings of lattice / Transport of Lebesgue measure

Peres, Sly (2014); Kim,Torquato (2019) ; Klatt, Last & Y. (2020);
Klatt, Last, Lotz & Y. (2024+)

Proof Sketch

Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(n)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$.

(HU+) Condition - $\int ln\,(1 + |x|)\,|\beta|(\mathrm{d}x) \;<\; \infty$; Nearly same as $\mathbb{V}AR\big(\mu(\Lambda_n)\big) = o(\frac{n}{\log n})$

THEOREM There exists a matching $T : \mathbb{Z}^2 \to \mu$ such that $\mathbb{E}[X^2] < \infty$.

- Rescale $\mu \cap n^{1/d}\Lambda$ to a prob. measure μ_n on $\Lambda = [-\pi, \pi]^d$ and same to Lebesgue - $\mathscr{L}_1$.
- Show that $\tilde{W}_2^2(\mu_n, \mathscr{L}_1) \leq c\, n^{-1}$ under Toroidal metric on Λ and then tile space with matchings in boxes $n^{1/d}\Lambda$.
- Match equi-spaced points to $\mathscr{L}_1$ and use triangle inequality + Birkhoff-von Neumann + relative compactness to get infinite matching.
- Upper bound $\tilde{W}_2^2(\mu_n, \mathscr{L}_1)$ using the Bobkov-Ledoux (2021) Fourier-analytic method.
- Quantify approximation of empirical structure factor $S_n(k)$ to structure factor $S(k)$ at small frequencies $k \approx n^{-1/d}$.
- HU $\Leftrightarrow S(0) = 0$. Our challenge - Quantify convergence at 'small frequencies'.

Bobkov-Ledoux Fourier-analytic method

- $P-$ prob measure on $\Lambda, \mathscr{L}_1-$ Unif. Measure. $f_P(\cdot)-$ Fourier transform of P; $f_{\mathscr{L}_1} = \delta_0$

- $\tilde{W}_2^2(P, \mathscr{L}_1) \lesssim \sum_{0<|m|<T} |m|^{-2} |f_P(m)|^2 + T^{-2}$, $m \in \mathbb{Z}^d$; 2nd term is smoothing error.

- $\mu_n(\cdot) = N^{-1}\mu(\mathbf{1}_{\Lambda_n} n^{1/d} \cdot)$; $N = \mu(\Lambda_n)$. $\Lambda_n = [-\pi n^{1/d}, \pi n^{1/d}]^d$.

- i.i.d. / mixing case - Use $\mathbb{E}|f_{\mu_n}(m)|^2 < cn^{-1}$ with $T = n^{1/d}$ and so

$$\mathbb{E}\tilde{W}_2^2(P, \mathscr{L}_1) \leq cn^{-1} \sum_{0<|m|<n^{1/d}} |m|^{-2} + cn^{-2/d} \leq cn^{-1} \sum_{k=1}^{n^{1/d}} k^{d-1-2} + cn^{-2/d}$$

- PP View: $|f_{\mu_n - \mathscr{L}_1}(m)|^2 = |f_{\mu_n}(m)|^2 = N^{-1} S_n(mn^{-1/d}), \; m \in \mathbb{Z}^d$;

- KEY ESTIMATE: r $0 < k = mn^{-1/d} \leq c_0$, $\mathbb{E}[\frac{S_n(k)}{N}] = \frac{S(k)}{n} + \frac{\mathbb{V}AR(N)}{n^2} + O_\beta(\ldots)$

What Else ?

- Results extend to random measures with matching replaced by transport.
- Proofs also give AKT type finite-volume bounds.
- Finite-volume bounds also hold for point processes with non-integrable RPCM β
- Other proof approaches - Whitney-type decomposition, sub-additivity and error estimation via potential-theoretic or PDE techniques.