Radial growth in aggregation models

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joint work with Tillmann Bosch

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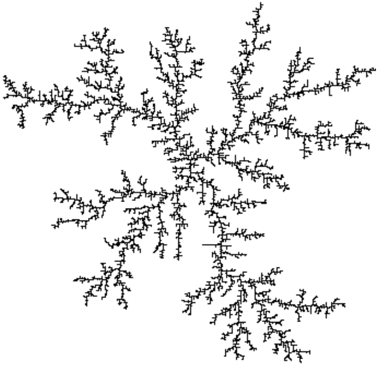
Diffusion-limited aggregation (DLA)

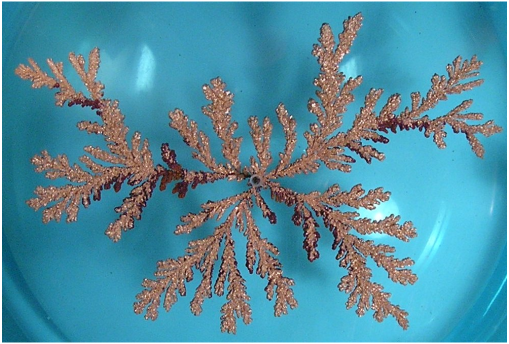


(Discrete) aggregation models: (for continuum models $\rightarrow$ ask Frankie)

- defined on $\mathbb{Z}^d$ (or on some other graph)
- growth starts from some initial cluster F_1 (e.g. a single particle at origin),
- particles arrive one after another and are attached where they hit the cluster for the first time
- DLA: particles perform random walks on $\mathbb{Z}^d$ (started at ∞)

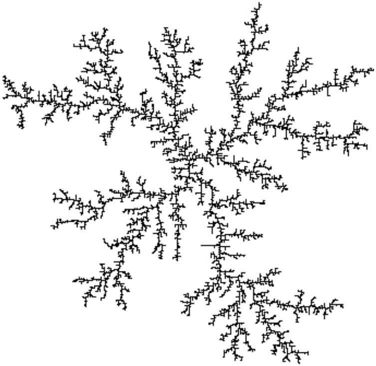
Diffusion-limited aggregation

- introduced by **[Witten, Sander ’81]** as a model for metal-particle aggregation
- similar clusters observed in many physical systems, e.g. in electrodeposition, mineral deposition or dielectric breakdown
- essential: particles aggregate irreversibly and diffusion (thermal motion) is the means of transport,
- $\sim$ 4500 citations (APS); many variants of the model; few rigorous results



Ballistic aggregation

How does the DLA model change, if the arriving particles do not perform random walks but travel along straight lines?



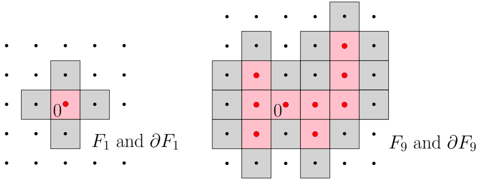
- known in physics as ballistic aggregation or Vold-Sutherland model: **[Vold '59, Sutherland '66, Bensimon, Domany, Aharony '81, Meakin '83]**
- considered a good model when particles can move freely such as molecules in a low density vapour
- There is also *ballistic deposition* **[Seppäläinen 00], [Penrose 08]**

A general framework

Let $\mathcal{P}_f^d$ be the family of finite subsets of $\mathbb{Z}^d$.

- We consider the nearest neighbor graph on $\mathbb{Z}^d$, i.e., the graph $(\mathbb{Z}^d, E)$ with edge set $E := \{\{x,y\} \subset \mathbb{Z}^d : \|x-y\| = 1\}$.
- $A \in \mathcal{P}_f^d$ is called connected, if the subgraph of $(\mathbb{Z}^d, E)$ generated by A is.
- For $A \in \mathcal{P}_f^d$, the (outer) boundary of A is the set

$$\partial A := \{y \in \mathbb{Z}^d \setminus A : \exists x \in A \text{ such that } \{x,y\} \in E\}.$$



- For $A \in \mathcal{P}_f^d \setminus \{\emptyset\}$, a random point in A is a measurable mapping $y_A : \Omega \to \mathbb{Z}^d$ with $\mathbb{P}(y_A \in A) = 1$. Denote by $\mathcal{D}(A)$ the family of all probability measures on A.
- A random finite set is a measurable mapping $F : \Omega \to \mathcal{P}_f^d$.

The general model: incremental aggregation

Definition

Let $\mathcal{M} := (\mu_A)_{A \in \mathcal{P}_f^d}$ be a family of distributions s.t. $\mu_A \in \mathcal{D}(A)$ for each $A \in \mathcal{P}_f^d$.
A sequence $(F_n)_{n \in \mathbb{N}}$ of random finite sets $F_n \subset \mathbb{Z}^d$ is called incremental aggregation (with distribution family $\mathcal{M}$), if it satisfies the following conditions:

(i) $F_1 := \{y_1\}$, where $y_1 := 0 \in \mathbb{Z}^d$;

(ii) for any $n \in \mathbb{N}$, $F_{n+1} := F_n \cup \{y_{n+1}\}$, where y_{n+1} is a random point in $\mathbb{Z}^d$ whose conditional distribution given F_n is $\mu_{\partial F_n}$, i.e.,

$$\mathbb{P}\left(y_{n+1} = y \mid F_n = A\right) := \mu_{\partial A}(y) \qquad \text{for any } A \in \mathcal{P}_f^d \text{ and } y \in \mathbb{Z}^d.$$

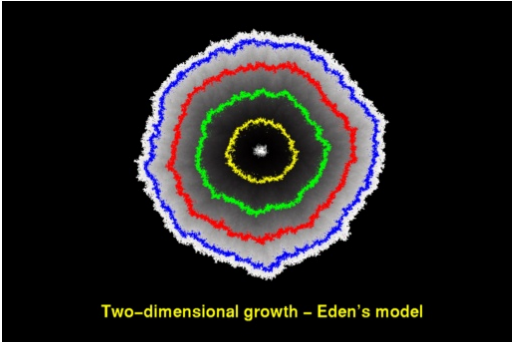
F_n is called cluster or aggregate at time n.
Observe that

- $0 \in F_1 \subset F_2 \subset F_3 \subset \ldots \subset \mathbb{Z}^d$;
- for any $n \in \mathbb{N}$, a.s. F_n is connected and $\#F_n = n$;
- $(F_n)_n$ is a Markov chain, in particular, F_{n+1} depends on F_n, but not on the order, in which the points have been added to F_n.

A simple example of incremental aggregation

Example

- Let μ_A be the uniform distribution on $A \in \mathcal{P}_f^d$. The resulting incremental aggregation is known as Eden growth model. **[Eden '61]**
- Clusters look ball-like with a rough "boundary" and few holes.
- generalization: internal DLA **[Lawler, Bramson, Griffeath 92, ...]**



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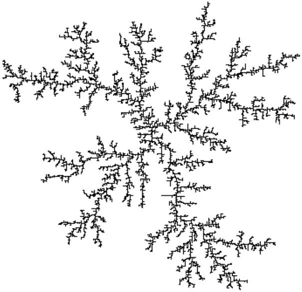
http://algorithmicbotany.org/vmm-deluxe/Plates.html#eden

DLA as incremental aggregation

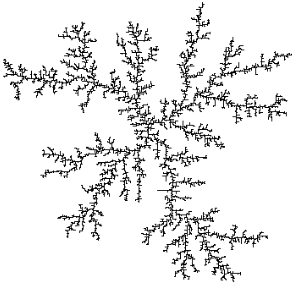


Diffusion-limited aggregation is incremental aggregation with distribution family $(h_A)_{A\in\mathcal{P}_f^d}$ where h_A is the harmonic measure on A.

- If $(S_t)_{t\in\mathbb{N}}$ is a symmetric random walk on $\mathbb{Z}^d$ started 'at ∞' (or very far away) and conditioned to visit the set A, then, for any $z \in A$,

  $h_A(z)$ is the probability that z is the first point in A visited by (S_t).

Radius, diameter and cardinality



For any finite $A \subset \mathbb{Z}^d$ (with $0 \in A$), denote by

- $\mathrm{rad}\,(A) := \max_{x \in A} \|x\|$ its radius;
- $\#A$ its cardinality.

Observe that

- $\mathrm{rad}\,(A) = \inf\{r > 0 : A \subset B(0, r)\}$
- If A is connected and $0 \in A$, then

$$(\#A)^{-d} \lesssim \mathrm{rad}\,(A) \leq \#A.$$

Growth rate and fractal dimension

Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathbb{Z}^d$.

- growth rate α of the radii:

$$\mathrm{rad}\,(A_n) \sim (\#A_n)^\alpha \qquad \text{as } n \to \infty$$

The lower and upper growth rate of the sequence (A_n) are defined by

$$\underline{\alpha}_f := \underline{\alpha}_f((A_n)_n) := \liminf_{n\to\infty} \frac{\log(\mathrm{rad}\,(A_n))}{\log(\#A_n)} \text{ and } \overline{\alpha}_f := \limsup_{n\to\infty} \frac{\log(\mathrm{rad}\,(A_n))}{\log(\#A_n)}.$$

Similarly, the lower and upper fractal dimension are defined by

$$\underline{\delta}_f := \liminf_{n\to\infty} \frac{\log(\#A_n)}{\log(\mathrm{rad}\,(A_n))} \text{ and } \overline{\delta}_f := \limsup_{n\to\infty} \frac{\log(\#A_n)}{\log(\mathrm{rad}\,(A_n))}.$$

Simple observations:

- $\underline{\delta}_f = 1/\overline{\alpha}_f$ and $\overline{\delta}_f = 1/\underline{\alpha}_f$
- If the sets A_n are connected and $0 \in A_1$, then

$$1 \le \underline{\delta}_f \le \overline{\delta}_f \le d \qquad \text{and hence} \qquad 1/d \le \underline{\alpha}_f \le \overline{\alpha}_f \le 1.$$

- Incremental aggregation: F_n is random and thus $\mathrm{rad}\,(F_n)$, $\underline{\delta}_f$ and $\overline{\delta}_f$ are random variables. The above relations hold almost surely.

Kesten's result for DLA

Let $(F_n)_{n\in\mathbb{N}}$ be DLA in $\mathbb{Z}^d$, $d \geq 2$.

Theorem [Kesten 87 and 90], [Lawler 91], [Benjamini,Yadin 17]

There exists a constant $c > 0$ such that a.s. for n sufficiently large

$$\mathrm{rad}\,(F_n) \leq \begin{cases} c\, n^{2/3}, & \text{if } d = 2, \\ c\, n^{1/2}(\ln n)^{1/4}, & \text{if } d = 3, \\ c\, n^{2/(d+1)}, & \text{if } d \geq 4. \end{cases}$$

For the lower fractal dimension $\underline{\delta}_f$ of DLA in $\mathbb{Z}^2$ this implies e.g.

$$\underline{\delta}_f = \liminf_{n\to\infty} \frac{\log n}{\log(\mathrm{rad}\,(F_n))} \geq \lim_{n\to\infty} \frac{\log n}{\log(cn^{2/3})} = \frac{3}{2}. \qquad \left(\textit{Conjecture}: \delta_f = \frac{5}{3}\right)$$

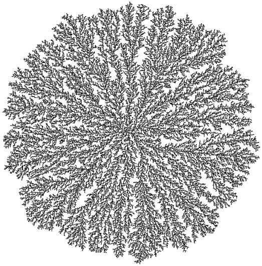
Corollary

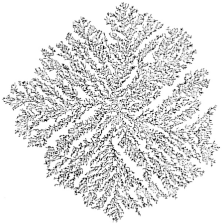
For DLA $(F_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $d \geq 2$ one has almost surely

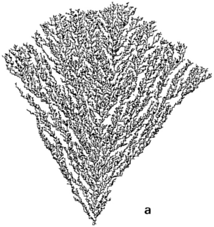
$$\underline{\delta}_f \geq \frac{d+1}{2}. \qquad \left(\text{Physicist's Conjecture: } \delta_f = \frac{d^2+1}{d+1}\right)$$

Ballistic aggregation

- In the ballistic model, the distributions μ_A of clusters A are determined by random lines (details in a moment).
- variations in the physics literature: lines have a
  - uniform direction
  - uniform axis parallel direction, e.g. **[Bensimon, Shraiman, Liang '83]**
  - fixed direction **[Bensimon, Shraiman, Liang '83, Vicsek '89]**







- general observation: clusters are much denser than DLA clusters
- conjectured dimension: $\delta_f = d$ in $\mathbb{Z}^d$ **[Meakin '83, Bensimon, Shraiman, Liang '83, Ball, Witten '84]**

Radial growth in the ballistic model

Let $(F_n)_{n\in\mathbb{N}}$ be ballistic aggregation in $\mathbb{Z}^d$.

Theorem [Bosch, W. 24]

There exists a constant $c = c(d) > 0$ such that a.s. for n sufficiently large

$$\mathrm{rad}\,(F_n) \le c\, n^{1/2}.$$

Corollary (resolution of physicist's conjecture for $d = 2$)

For the ballistic model in $\mathbb{Z}^2$, $\delta_f = 2$ almost surely (and $\underline{\delta}_f \ge 2$ in $\mathbb{Z}^d$, $d \ge 3$).

Proof. On the one hand, $\overline{\delta}_f \le 2$ a.s. in $\mathbb{Z}^2$. On the other hand

$$\underline{\delta}_f = \liminf_{n\to\infty} \frac{\log n}{\log(\mathrm{rad}\,(F_n))} \ge \lim_{n\to\infty} \frac{\log n}{\log(cn^{1/2})} = 2. \quad \square$$

Corollary (positive volume)

The ballistic model in $\mathbb{Z}^2$ satisfies almost surely

$$\liminf_{n\to\infty} \frac{\#F_n}{(\mathrm{rad}\,(F_n))^2} \ge \liminf_{n\to\infty} \frac{n}{c^2 n} = c^{-2} > 0.$$

Ballistic aggregation

How to choose the distributions μ_A, $A \in \mathcal{P}_f^d$ in the ballistic model?

- stochastic geometry: isotropic random lines (IRL)
- Let $A(d,1)$ be the space of lines in $\mathbb{R}^d$ (affine Grassmannian of 1-flats) equipped with the usual σ-algebra $\mathcal{A}(d,1) := \sigma(\{[K] : K \in \mathcal{K}^d\})$, where

$$[K] := \{L \in A(d,1) : L \cap K \neq \emptyset\}.$$

- There is a unique Euclidean motion-invariant Radon measure μ_1 on $A(d,1)$ such that

$$\mu_1([B_d]) = \kappa_{d-1},$$

where B_n is the unit ball in $\mathbb{R}^n$ and $\kappa_n = \lambda_n(B_n)$ its volume.

- For compact $K \subset \mathbb{R}^d$ with $\mu_1([K]) > 0$, an IRL through K is a measurable mapping $L : \Omega \to A(d,1)$ with distribution given by

$$\mathbb{P}(L \in \mathcal{A}) := \mathbb{P}^K(\mathcal{A}) := \frac{\mu_1(\mathcal{A} \cap [K])}{\mu_1([K])}, \qquad \mathcal{A} \in \mathcal{A}(d,1).$$

- by the Crofton formula (for $K \in \mathcal{K}^d$, i.e. K compact and convex):

$$\mu_1([K]) = \int_{A(d,1)} V_0(K \cap L)\mu_1(dL) = c_d V_{d-1}(K)$$

Ballistic aggregation

To define it, we need to specify a distribution b_A for each $A \in \mathcal{P}_f^d \setminus \{\emptyset\}$.

- For $A \in \mathcal{P}_f^d \setminus \{\emptyset\}$ let

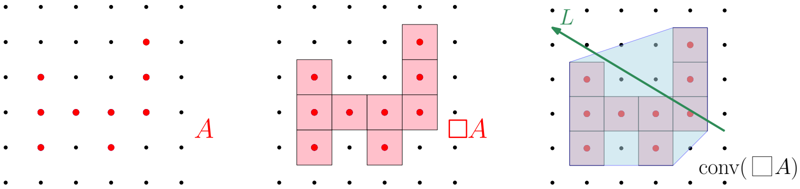
$$\Box A := \bigcup_{z \in A} C_z, \qquad \text{where } C_z = \left[-\frac{1}{2}, \frac{1}{2}\right]^d + z.$$

  Then an IRL through $\Box A$ is well defined ($\mu_1[\Box A] > 0$).

- Let L be a directed IRL through $\Box A$ (i.e., an IRL, on which one of the two directions is chosen uniformly). Then, for any $z \in A$,

$$b_A(z) := \mathbb{P}(C_z \text{ is the first box in } \Box A \text{ visited by } L).$$

- Ballistic aggregation on $\mathbb{Z}^d$ is incremental aggregation with distribution family $(b_A)_{A \in \mathcal{P}_f^d}$. (Note: $b_{\partial A} = b_{A \cup \partial A}$!)



Tool: Bounding the local speed of growth

The following statement generalizes Kesten's strategy in his proof for DLA.

Theorem (Kesten's method) [Bosch, W. 24]

Let $\mathcal{M} = (\mu_A)_{A \in \mathcal{P}_f^d}$ be some family of distributions. Suppose there exists some constants $q, C > 0$ such that for all $r > 1$, any connected set $A \in \mathcal{P}_f^d$ with $0 \in A$ and $\mathrm{rad}\,(A) \geq r$ and any $z \in A$,

$$\mu_A(z) \leq C\, r^{-q}.$$

Then there is a constant c, such that an incremental aggregation $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ with distribution family $\mathcal{M}$ satisfies almost surely

$$\mathrm{rad}\,(F_n) \leq c\, n^{1/(q+1)}$$

for n sufficiently large.

Hitting a location in the ballistic model

Let $(\mu_A)_{A\in\mathcal{P}_f^d}$ denote the family of distributions defining the ballistic model in $\mathbb{Z}^d$.

Theorem [Bosch, W. 24]

There is a $C_d > 0$ such that for any $r > 1$, any connected set $A \in \mathcal{P}_f^d$ with $0 \in A$ and $\mathrm{rad}\,(A) \geq r$ and any $z \in A$,

$$b_A(z) \leq C_d r^{-1}.$$

- Kesten's method yields (with $q = 1$ and $\mu_A = b_A$):

  $$\mathrm{rad}\,(F_n) \leq cn^{1/(q+1)} = cn^{1/2}$$

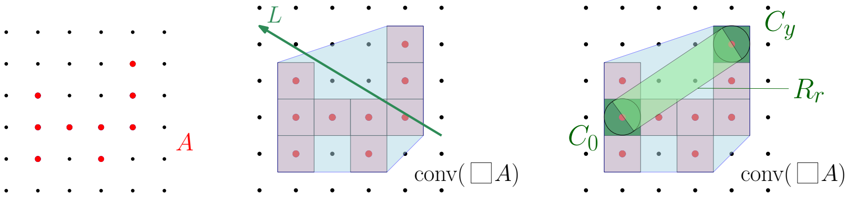
  and thus $\delta_f = 2$ for $d = 2$ and $\underline{\delta}_f \geq 2$ for any $d \geq 3$ (Conj.: $\delta_f = d$).
- The exponent $q = 1$ is optimal, i.e. such an estimate is not true for any $q > 1$. (This will be clear from the next slide). Therefore, one cannot expect better bounds for δ_f from Kesten's method.

Idea of proof for $d = 2$

Let $r \geq 1$, $A \in \mathcal{P}_f^2$ connected with $0 \in A$ and $\mathrm{rad}\,(A) \geq r$, and $z \in A$.

- Note that the connectedness in $\mathbb{Z}^2$ implies $[\square A] = [\mathrm{conv}(\square A)]$.
- $\square A$ contains C_0 and another unit size box C_y with $y \in \mathbb{Z}^2$ and $\|y\| \geq r$. Hence $\mathrm{conv}(\square A)$ contains a rectangle R_r with sidelengths r and 1.
- Therefore, $\mu_1([\square A]) = \mu_1([\mathrm{conv}(\square A)]) \geq \mu_1([R_r])$ and so

$$b_A(z) \leq \mathbb{P}^{\square A}([C_z]) = \frac{\mu_1([C_z])}{\mu_1([\square A])} \leq \frac{V_1(C_0)}{V_1(R_r)} = \frac{2}{1+r} \leq 2r^{-1}. \quad \square$$



Remark: The rate $q = 1$ is optimal (largest possible) for all $d \geq 2$, since for a row of r points $A = A_r := \{0, ..., r-1\} \times \{0\}$ and $z = 0$, one gets

$$b_A(z) \geq \frac{1}{2}\mathbb{P}^{\square A}([C_z]) = \ldots \geq \frac{1}{2}r^{-1}.$$

Idea of proof for $d \geq 3$

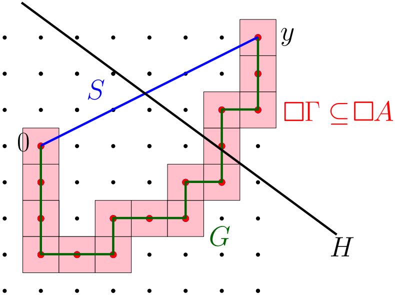
Let $r \geq 1$, $A \in \mathcal{P}_f^d$ connected with $0 \in A$ and $\mathrm{rad}\,(A) \geq r$, and $z \in A$.

- Now the connectedness of A does not imply $[\Box A] = [\mathrm{conv}(\Box A)]$.
- $\Box A$ contains C_0 and another unit size box C_y with $y \in \mathbb{Z}^d$ and $\|y\| \geq r$. There is a path $\Gamma \subset A$ connecting 0 and y.
- Let G be the shortest curve in $\mathbb{R}^d$ connecting these points in the given order. Then

$$G_{\oplus\frac{1}{2}} \subset \Box\Gamma \quad \text{and so } \mu_1([\Box A]) \geq \mu_1([\Box\Gamma]) \geq \mu_1([G_{\oplus\frac{1}{2}}]).$$

Therefore,

$$b_A(z) \leq \mathbb{P}^{\Box A}([C_z]) = \frac{\mu_1([C_z])}{\mu_1([\Box A])} \leq \frac{\mu_1([C_0])}{\mu_1([G_{\oplus\frac{1}{2}}])}$$



- needed: $\mu_1([G_{\oplus\frac{1}{2}}]) \geq cr$
- tool:
  $$\mu_1(\mathcal{A}) = \int_{A(d,d-1)} \mu_1^H(\mathcal{A})\mu_{d-1}(dH)$$
- $\mu_1^H([G_{\oplus\frac{1}{2}}]) \geq \tilde{c}\,\mathbb{1}\{H \cap S \neq \emptyset\}$.
- $\mu_{d-1}(\{H \cap S \neq \emptyset\}) > \hat{c}r!$

Conclusion and Outlook

- incremental aggregation, a framework for many aggregation models, and Kesten's method, a tool for lower bounds for radial growth of such models;
- ballistic aggregation: $\delta_f = 2$ in $\mathbb{Z}^2$; $\underline{\delta}_f \geq 2$ in $\mathbb{Z}^d$;
- analogous results for the variants when only axes parallel directions are allowed for the lines:

Some open questions:

- Can one improve Kesten's method by taking the volume of the cluster into account instead of the radius in dimension $d \geq 3$?
- What is the relation to continuum models (e.g. Hastings-Levitov models)?
- What is the relation to ballistic deposition? Can results be transferred from there?
- further questions: asymptotic shape; existence, structure and scaling properties of voids; anisotropic variants; sticking probabilities; ...

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