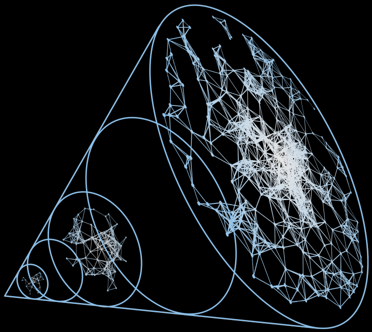
Condensation in scale-free geometric graphs with excess edges

Pim van der Hoorn

Joint work with: Remco van der Hofstad, Céline Kerriou, Neeladri Maitra and Peter Mörters

Stochastic Geometry in Action

September 10, 2024



EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computer Science

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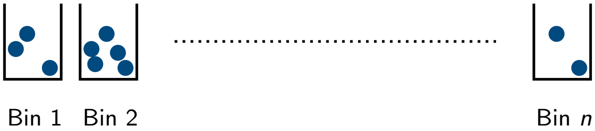

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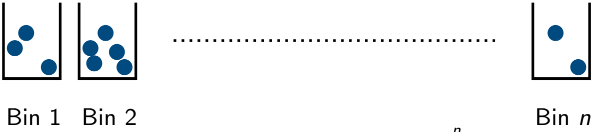

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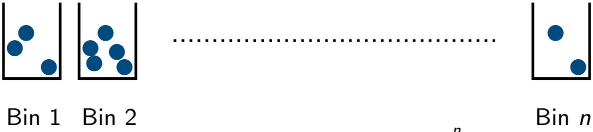

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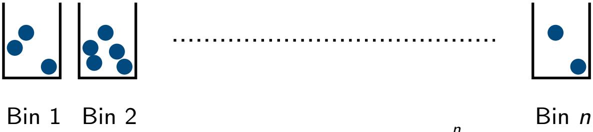

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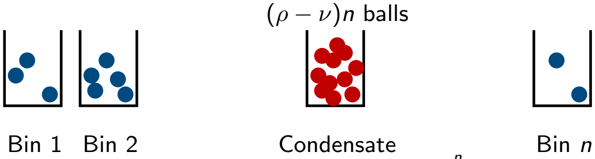

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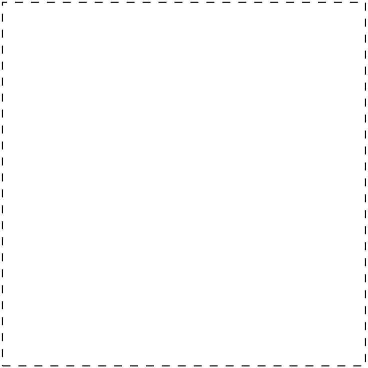

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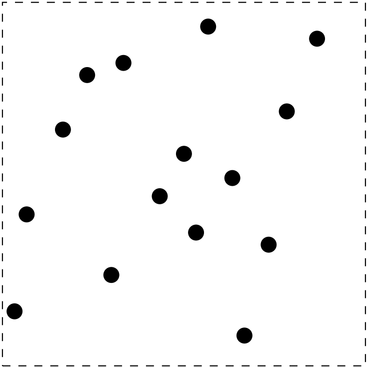

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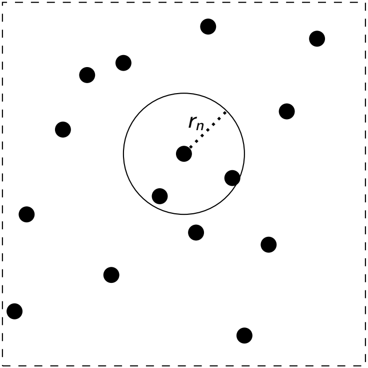

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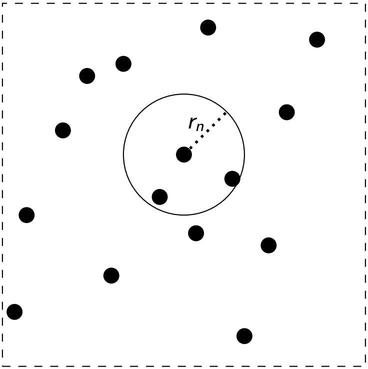

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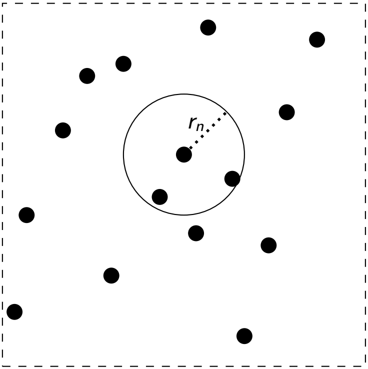

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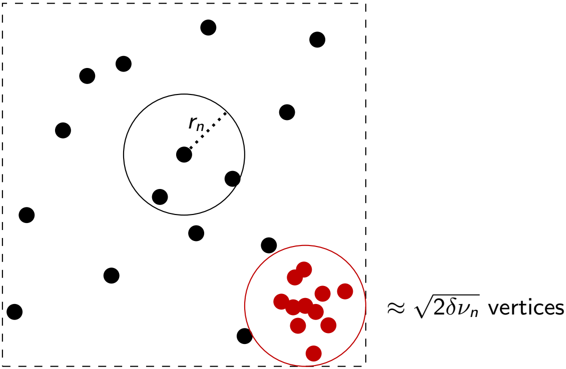

Outline of the rest of the talk

- Introduce model
- State results
  - Upper large deviation
  - Edge-length distribution
  - Conditional degree distribution
- Some ideas for the proof

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- Age-based preferential attachment [Gracar et al 2019]: $\phi(x) = 1 \wedge x^{-\alpha}$, $\alpha > 1$ and $\kappa(u, v) = (u \wedge v)^{\frac{1}{\gamma}-1}(u \vee v)$, $\gamma \in (0, 1)$

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Empirical edge density measure: $\mu_n := \frac{1}{n}\sum_{\{x,y\}\in E_n}\delta_{d_n(x,y)}$

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Empirical edge-length distribution [vdH et al. 2024+]

Same assumptions as before. For any non-integer $\rho > 0$ and continuous bounded function f with compact support. Conditional on the event $|E_n| \geq (\rho+\nu)n$:

- $\int f(x)\, d\mu_n(x) \xrightarrow{\mathbb{P}} \frac{1}{2}\mathbb{E}[\lambda_f(W)]$,
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$$\lambda_f(w) = \begin{cases} \sum_{z\in\mathbb{Z}^d} f(\|z\|)\lambda(w,z) & \text{lattice case} \\ \int f(\|z\|)\lambda(w,z)\, dz & \text{Poisson case} \end{cases}$$

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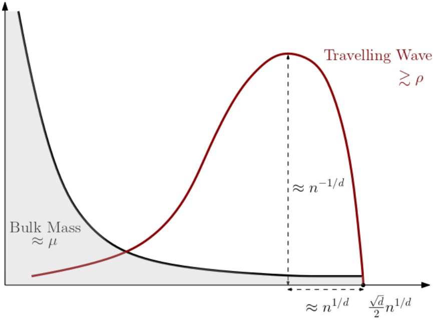

Result III: Degree distribution

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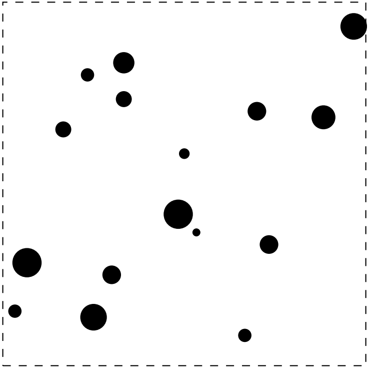

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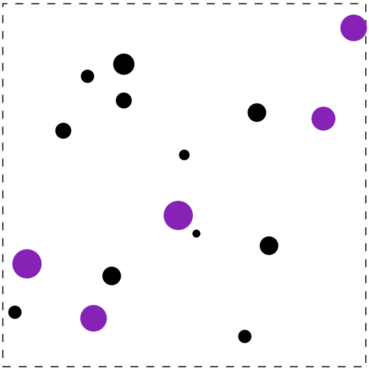

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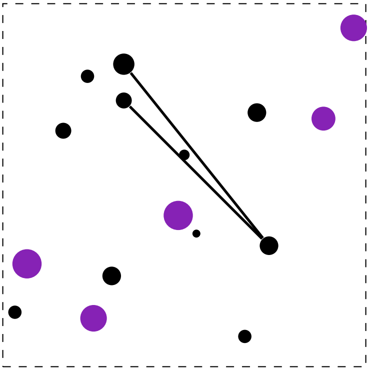

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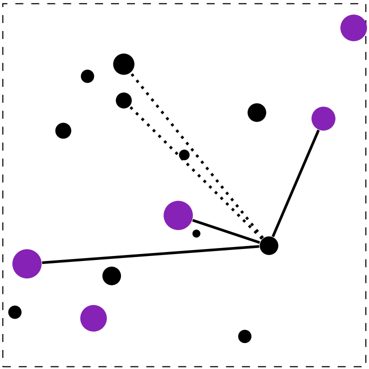

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Degree sequence of bulk and hubs [vdH et al. [2024+]

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$$\left(\left(\frac{D_{X_{(i)}}}{n} \right)_{1 \leq i \leq k}, (\pi^{(n)}_{a,b})_{0 \leq a,\, 0 \leq b \leq k} \right) \xrightarrow{d} ((\Lambda(Y_i))_{1 \leq i \leq k}, (\pi_{a,b})_{0 \leq a,\, 0 \leq b \leq k})$$

and $\frac{1}{n} D_{X_{(i)}} \xrightarrow{\mathbb{P}} 0$ for all $i > k$.

$$\mathbb{P}(Y_1 \geq x_1, \ldots, Y_k \geq x_k) = \frac{(\beta-1)^k}{F(\rho)} \iiint_0^\infty \mathbb{1}_{\{\Lambda(y_1)+\cdots+\Lambda(y_k)>\rho\}} \prod_{i=1}^{k} \mathbb{1}_{\{y_i \geq x_i \vee y_{i+1}\}} y_i^{-\beta}\, dy_i$$

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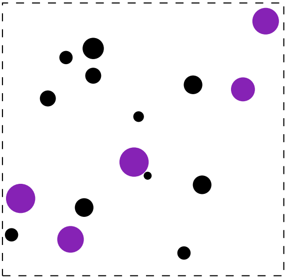

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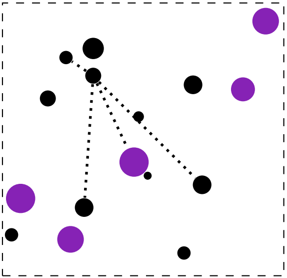

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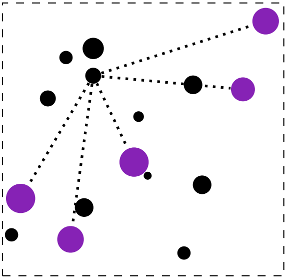

Condensation II

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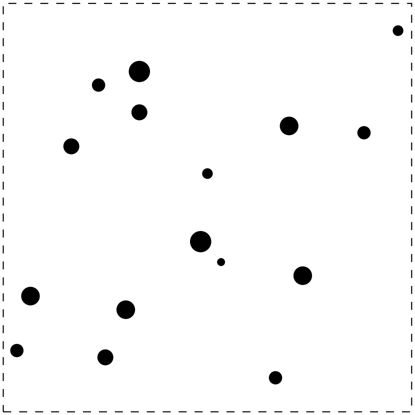
weights $\sim W$



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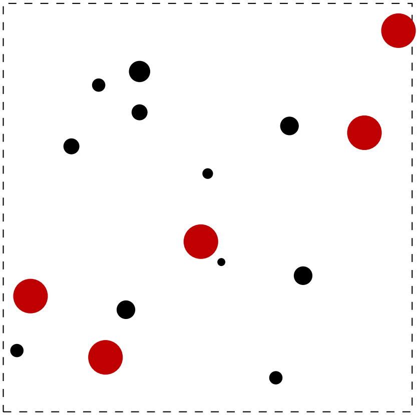

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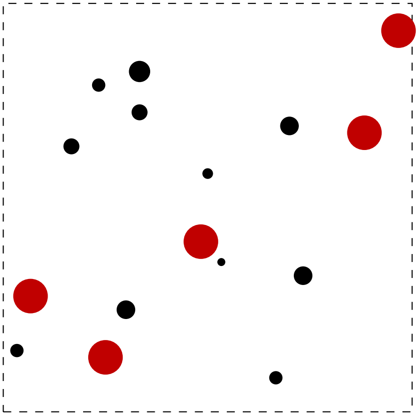

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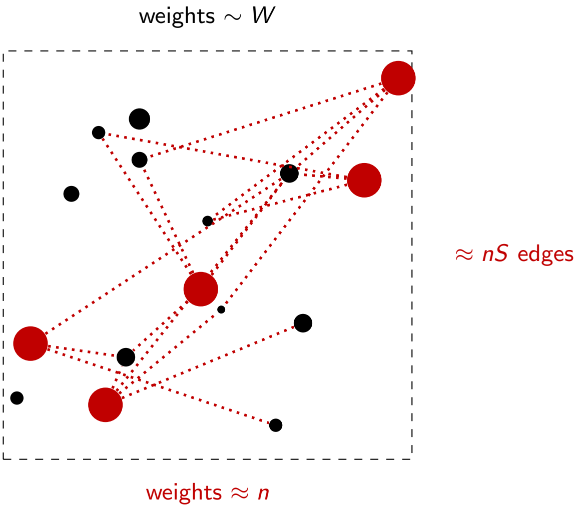

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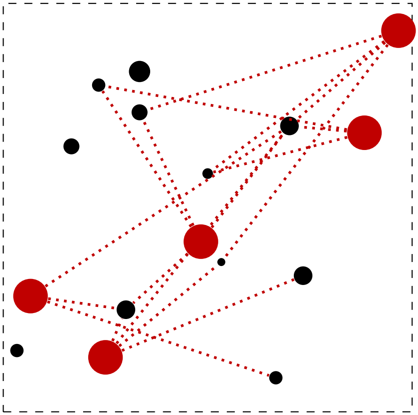

$\approx nS$ edges

weights $\approx n$

$$\mathbb{P}(S > s) = \frac{F(s)}{F(\rho)} \quad \text{for all } \rho \leq s < k.$$

Up next

Time for some proof ideas.

Proof for upper large deviation I

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Rewrite the number of high edges:

$$|E_{\text{high}}| = \sum_{x \in V_n} D_x \mathbb{1}_{\{W_x \geq n^a\}} - \sum_{\{x,y\} \in E_n} \mathbb{1}_{\{W_x \wedge W_y \geq n^a\}}$$

Proof for upper large deviation II

To show: $\mathbb{P}\left(|E_{\text{high}}| \geq \rho n\right) = (F(\rho) + o(1))n^{-k(\beta-2)}$

Rewrite the number of high edges:

$$|E_{\text{high}}| = \sum_{x\in V_n} D_x \mathbb{1}_{\{W_x \geq n^a\}} - \sum_{\{x,y\}\in E_n} \mathbb{1}_{\left\{W_x \wedge W_y \geq n^a\right\}}$$

$$= \sum_{x\in V_n} \left(D_x - \frac{1}{2} \sum_{y\in V_n} \mathbb{1}_{\left\{\{x,y\}\in E_n\, W_y \geq n^a\right\}} \right) \mathbb{1}_{\{W_x \geq n^a\}}$$

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$$\approx \binom{n}{k} (\beta - 1)^k \int_{\varepsilon n}^{\infty} \cdots \int_{\varepsilon n}^{\infty} \mathbb{1}_{\{\lambda_n(t_1) + \cdots + \lambda_n(t_k) \geq \rho n\}} \prod_{i=1}^{k} t_i^{-\beta} \, dt_i$$

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$$\frac{1}{n} \lambda_n(nw) \to \Lambda(w) := \int_{[-1/2,1/2]^d} \mathbb{E}\left[\phi\left(\frac{\|z\|^d}{\mathcal{K}(w, W)}\right)\right] dz$$

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$$\approx n^{-k(\beta-2)} \frac{(\beta-1)^k}{k!} \int_0^\infty \cdots \int_0^\infty \mathbb{1}_{\{\Lambda(y_1) + \cdots + \Lambda(y_k) > \rho\}} \prod_{i=1}^{k} y_i^{-\beta} \, dy_i$$

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Summary

Summary:

- We study rare event $|E_n| \geq (\rho + \nu)n$ in large class of spatial graphs
- Results obtained (asymptotics)
  - Upper large deviation probability,
  - Edge-length distribution (traveling wave)
  - Degree distribution (condensation large weight vertices)
- Rare event is attained by $k = \lceil \rho \rceil$ vertices of weight n
- Result are universal for non-integer $\rho > 0$
- At integers optimal strategy depends on precise asymptotics

Some open challenges:

- Analyze optimal strategies at integer $\rho = k$
- Study structure for other rare events (e.g. triangles)

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Condensation in scale-free geometric graphs with excess edges
arXiv preprint: 2405.20425