Multivariate CLTs for Poisson Functionals

under Minimal Moment Assumptions

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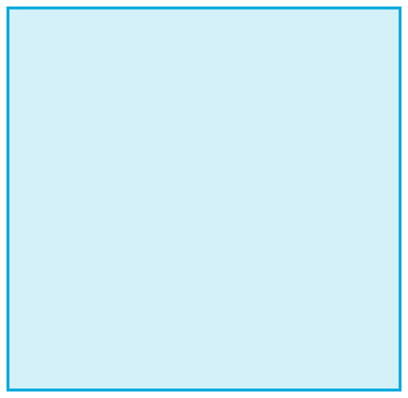
UNIVERSITY OF OXFORD

Motivation

- Convex body $W \subset \mathbb{R}^d$;



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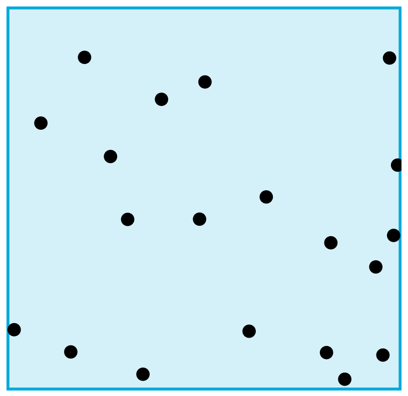

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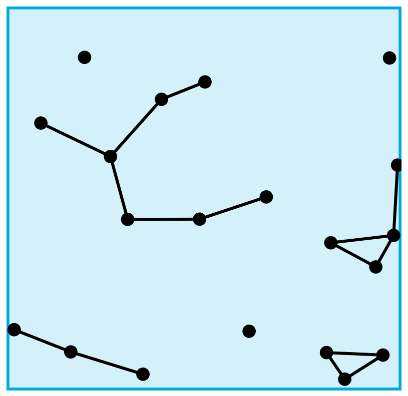

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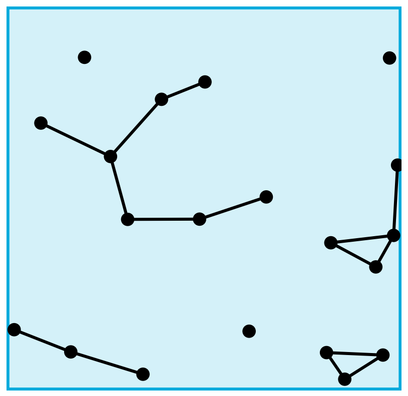

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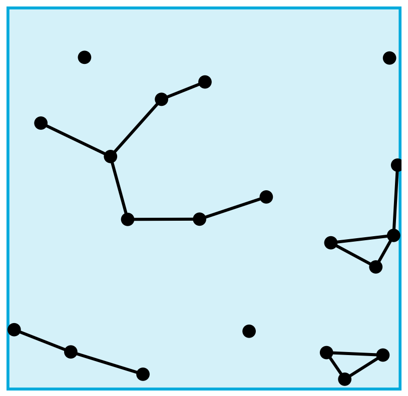

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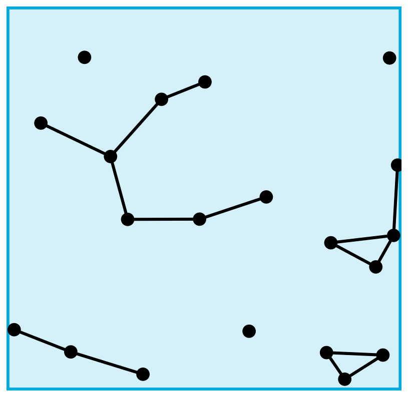

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$\rightarrow$ Reitzner, Schulte, and Thäle 2017

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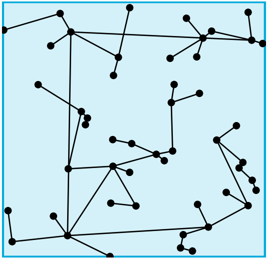

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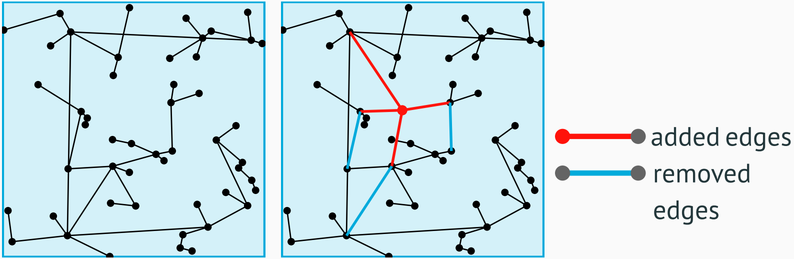

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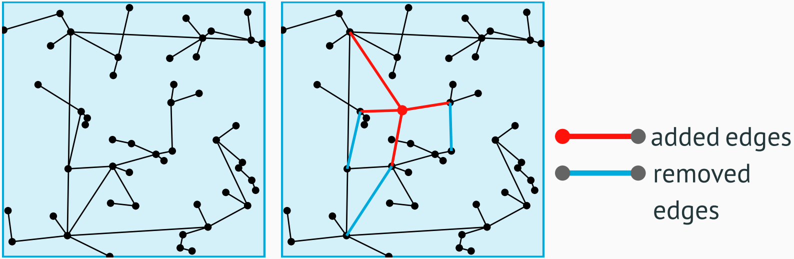

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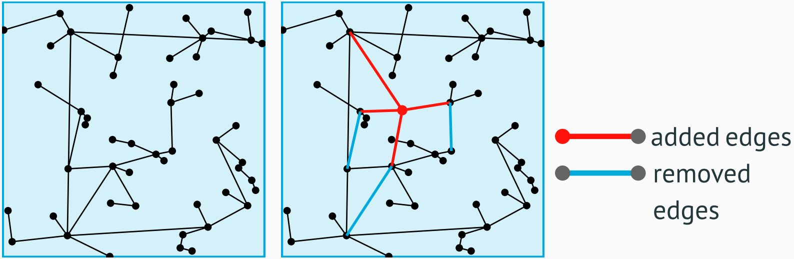

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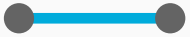
$= \sum |$  $|^\alpha - \sum |$  $|^\alpha$

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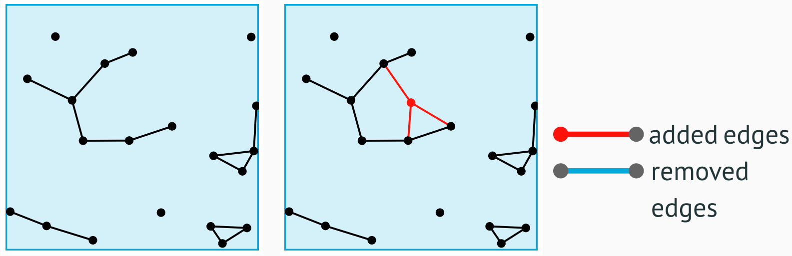

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F Poisson functional, $\mathbb{E}F = 0$, and N standard Gaussian.

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$$d_{\mathcal{H}}(F, N) = \sup_{h \in \mathcal{H}} |\mathbb{E}\, h(F) - \mathbb{E}h(N)\,|$$

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Malliavin Calculus: integration by parts formula

$$\mathbb{E}\, F f_h(F) = \mathbb{E} \int_{\mathbb{X}} \mathrm{D}_x \, \mathrm{L}^{-1} F \cdot \mathrm{D}_x f_h(F) \lambda(dx).$$

$\rightarrow$ Malliavin-Stein method (Nourdin and Peccati 2009; Nourdin, Peccati, and Reinert 2009)

Second-order Poincaré Inequality

Theorem (Last, Peccati, and Schulte 2016)

- *Poisson measure* η *on* σ*-finite space* $(\mathbb{X}, \lambda)$
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Moments of the Random Geometric Graph

Recall:

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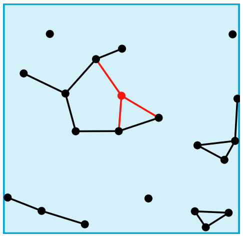

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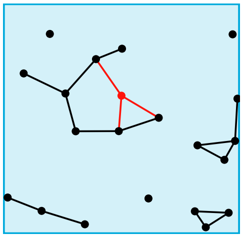

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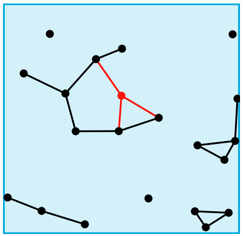

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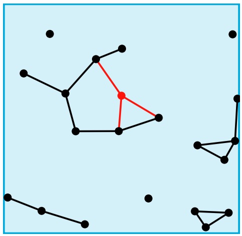

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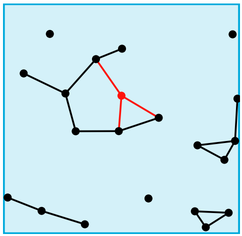

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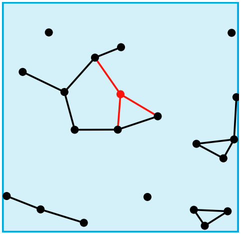

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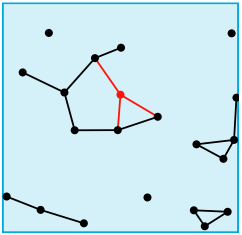

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Reitzner, Schulte, and Thäle 2017: Convergence up to $\alpha > -\frac{d}{2}$ (and beyond for different rescaling)

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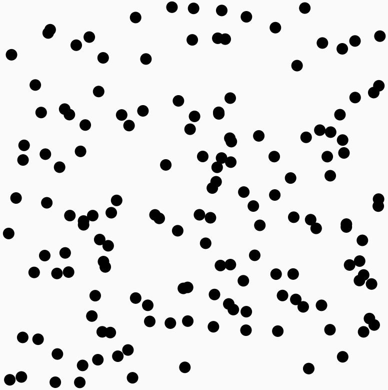

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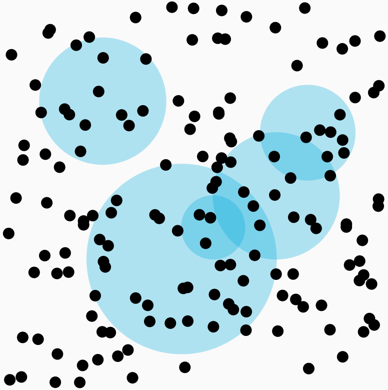

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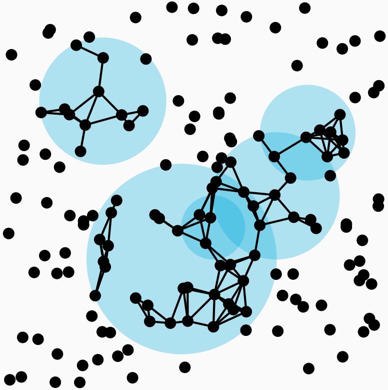

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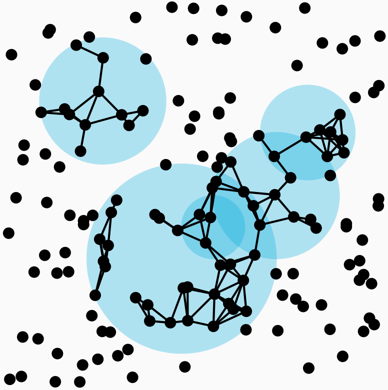

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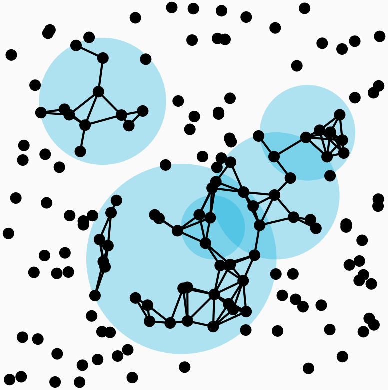

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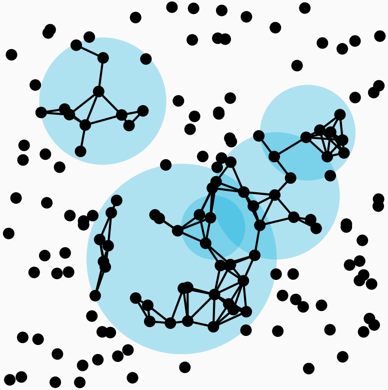

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- **Two methods:** Malliavin - Stein & Malliavin - interpolation

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Recall:

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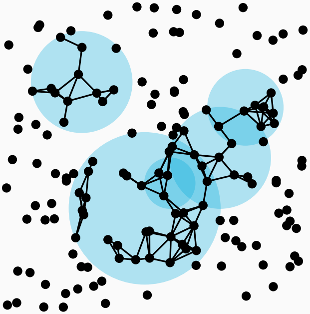

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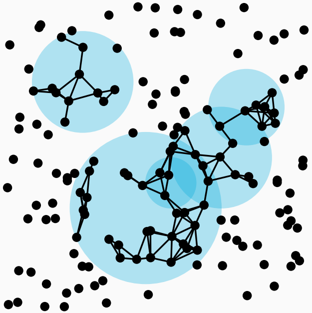

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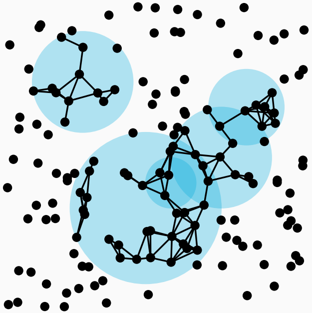

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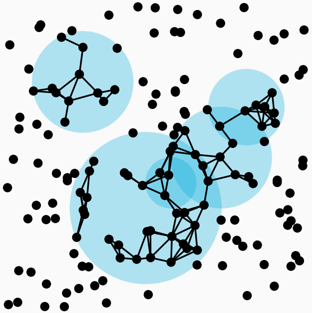

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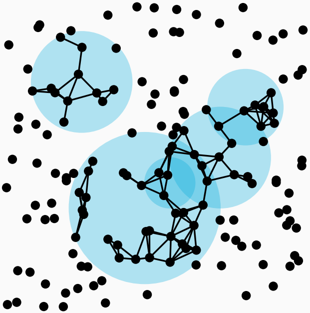

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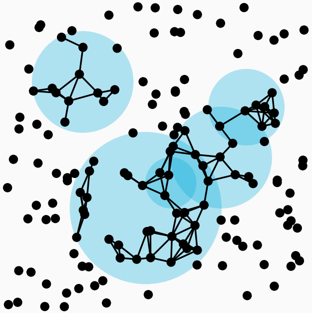

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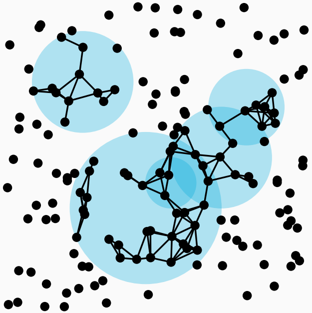

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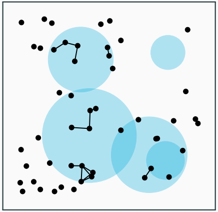

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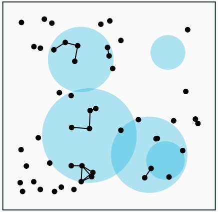

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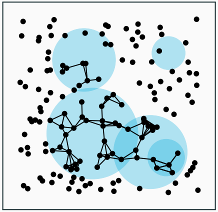

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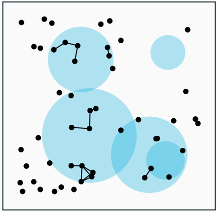

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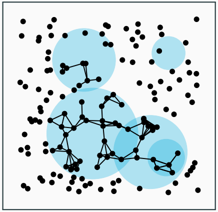

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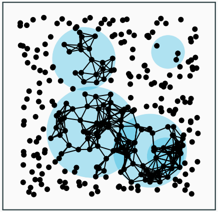

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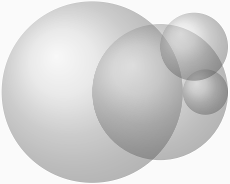

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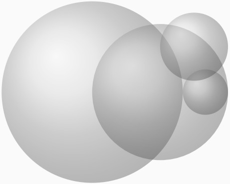

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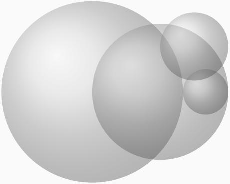

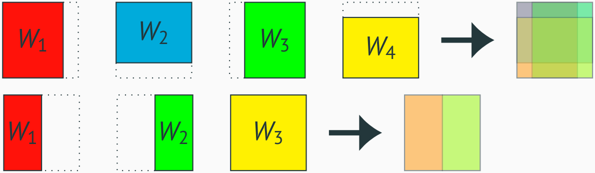

The End

Thank you!

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