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LARGE DEGREES AND COMPONENTS OF SCALE-FREE RANDOM CONNECTION MODELS

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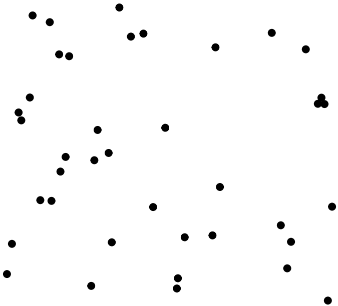
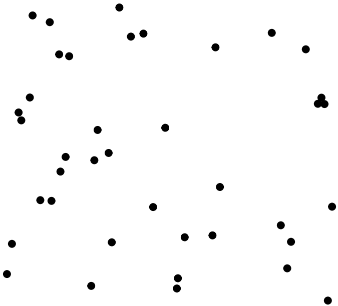
Joint works with Chinmoy Bhattacharjee (University of Hamburg) and Matthias Lienau (Hamburg University of Technology)

September 13, 2024



- Stationary Poisson process η in $\mathbb{R}^d$ with intensity one as nodes

Random connection model (RCM)



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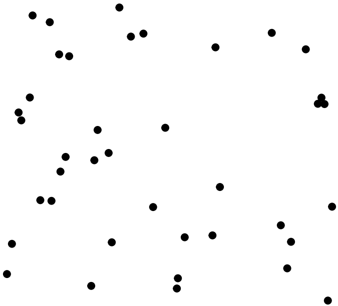

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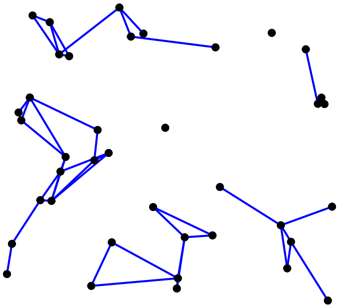

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Random connection model (RCM)

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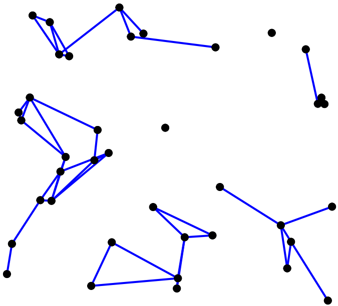

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- Penrose (1991)

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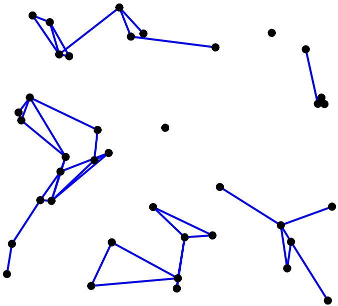

- $\varphi(x, y) = \exp(-c\|x - y\|^2)$

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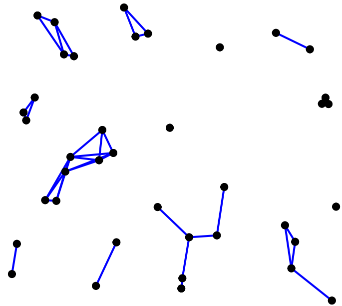

- $\varphi(x, y) = \exp(-c\|x - y\|^2)$
- $\varphi(x, y) = 1\{\|x - y\| \leq r\}$ with $r > 0$
  $\Longrightarrow$ random geometric graph

Degree of the typical vertex of the RCM

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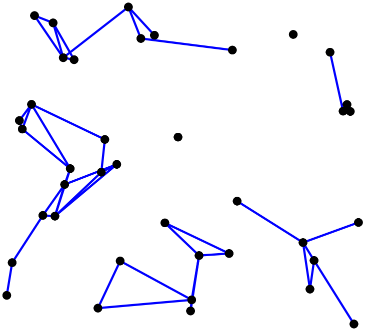

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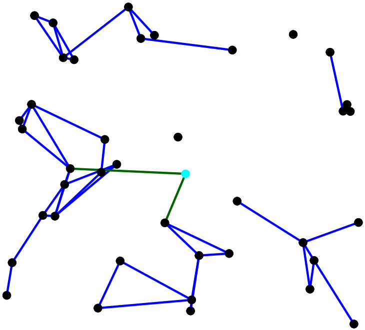
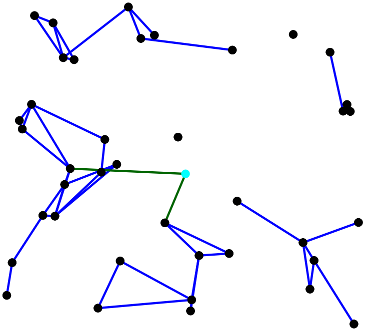


The degree D of the typical vertex has the same distribution as the degree of 0 if we add 0 as additional point to the RCM.

Degree of the typical vertex of the RCM



The degree D of the typical vertex has the same distribution as the degree of 0 if we add 0 as additional point to the RCM. Thus,

$$D \overset{d}{=} \sum_{x \in \eta} \mathbb{1}\{0 \leftrightarrow x\} \overset{d}{=} \text{Poisson}\left(\int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}x \right).$$

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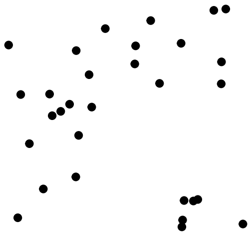
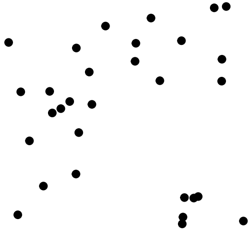


- W positive random variable such that $\mathbb{P}(W > u) = u^{-\beta} L(u)$, $u > 0$, with $\beta > 0$ and L slowly varying.

The scale-free RCM



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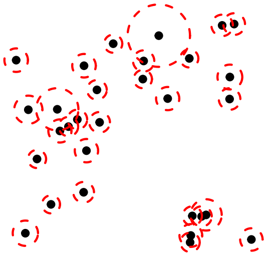

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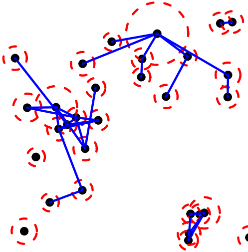

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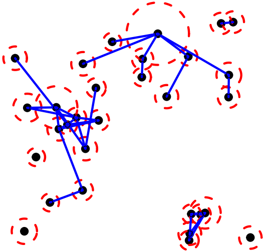
- Mark the points of η with i.i.d. copies $(W_x)_{x \in \eta}$ of W.
- Connect $x, y \in \eta$, $x \neq y$, independently with probability

$$\mathbb{P}(x \leftrightarrow y) = 1 - \exp\left(- \frac{\lambda W_x W_y}{\|x - y\|^\alpha} \right).$$

Typical degree of the scale-free RCM

The degree D of the typical vertex has the same distribution as the degree of 0 if we add $(0, W_0)$ with independent $W_0 \overset{d}{=} W$ to the scale-free RCM.



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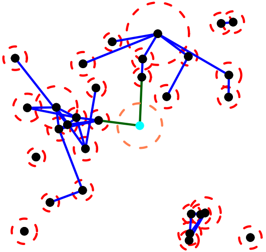

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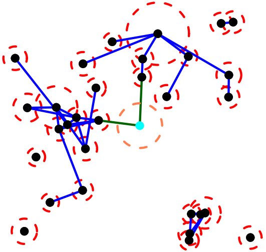


Theorem: Deprez/Wüthrich 2019

If $\min\{\alpha, \alpha\beta\} > d$, then

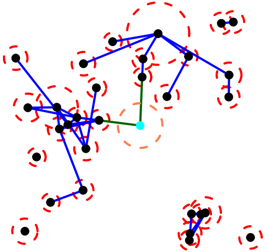
$$\mathbb{P}(D > u) = \ell(u) u^{-\alpha\beta/d}, \quad u > 0,$$

with a slowly varying function $\ell : (0, \infty) \to (0, \infty)$.

Since

$$D \overset{d}{=} \sum_{x \in \eta} 1\{0 \leftrightarrow x\},$$

for given weight W_0 of the typical vertex (or of 0),



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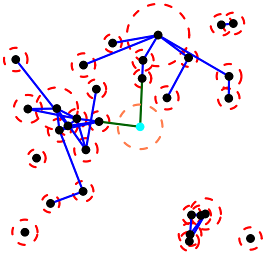

for given weight W_0 of the typical vertex (or of 0),

$$\mathbb{E}[D \mid W_0] = \mathbb{E}_W \int_{\mathbb{R}^d} 1 - \exp\left(-\frac{\lambda W_0 W}{\|x\|^\alpha}\right) \mathrm{d}x$$

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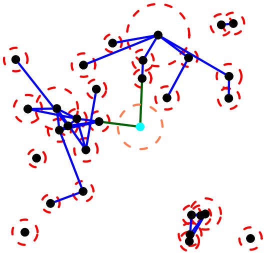

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$$= W_0^{d/\alpha} \lambda^{d/\alpha} \mathbb{E}[W^{d/\alpha}] \Gamma(1 - d/\alpha)$$

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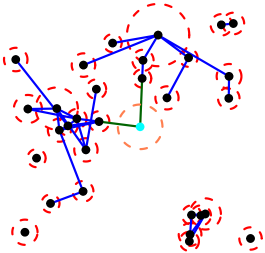

for given weight W_0 of the typical vertex (or of 0),

$$\begin{aligned}\mathbb{E}[D \mid W_0] &= \mathbb{E}_W \int_{\mathbb{R}^d} 1 - \exp\left(-\frac{\lambda W_0 W}{\|x\|^\alpha}\right) \mathrm{d}x \\ &= W_0^{d/\alpha}\lambda^{d/\alpha}\mathbb{E}[W^{d/\alpha}]\Gamma(1-d/\alpha)\end{aligned}$$

and

$$D \overset{d}{=} \text{Poisson}\left(W_0^{d/\alpha}\lambda^{d/\alpha}\mathbb{E}[W^{d/\alpha}]\Gamma(1-d/\alpha)\right).$$

Typical degree of the scale-free RCM

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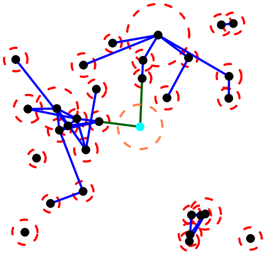

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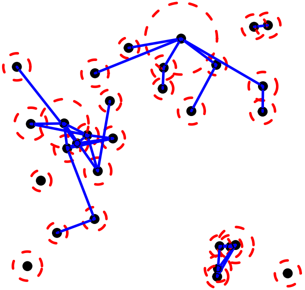
Thus, for large W_0, $D \approx W_0^{d/\alpha} \lambda^{d/\alpha} \mathbb{E}[W^{d/\alpha}] \Gamma(1 - d/\alpha)$.

How does

$$\max_{x \in \eta \cap [0, n^{1/d}]^d} \mathrm{Deg}(x)$$

behave as $n \to \infty$?



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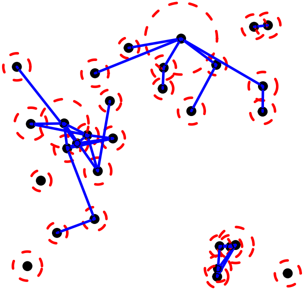

For random geometric graphs (see Penrose (2003)) there exist sequences $(a_n)_{n \in \mathbb{N}}$ with

$$\lim_{n \to \infty} \mathbb{P}(\text{maximum degree} \in \{a_n, a_n + 1\}) = 1.$$

Notation

- A random variable Z is Fréchet(γ)-distributed with $\gamma > 0$ if
$$\mathbb{P}(Z \leq y) = e^{-y^{-\gamma}}$$
for $y \geq 0$.
- Let $\mathcal{P}_\gamma$, $\gamma > 0$, be a Poisson process on $(0, \infty)$ with intensity measure ν_γ such that
$$\nu_\gamma((a, b]) = a^{-\gamma} - b^{-\gamma}$$
for $0 < a < b < \infty$.

Large degrees of the scale-free RCM

Let

$$q(t) := \inf\{s \geq 0 : \mathbb{P}(W^{d/\alpha} \leq s) \geq 1 - 1/t\}, \quad t \geq 1,$$

κ_d the volume of the d-dimensional unit ball and

$$\xi := \lambda^{d/\alpha} \kappa_d \Gamma(1 - d/\alpha) \mathbb{E} W^{d/\alpha}.$$

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Theorem: Bhattacharjee/S. 2022

For the scale-free RCM with $d < \min\{\alpha, \alpha\beta\}$, as $n \to \infty$,

$$\frac{1}{\xi q(n)} \max_{x \in \eta \cap [0, n^{1/d}]^d} \mathrm{Deg}(x) \xrightarrow{d} \text{Fréchet}(\alpha\beta/d)$$

and

$$\left\{ \frac{\mathrm{Deg}(x)}{\xi q(n)} : x \in \eta \cap [0, n^{1/d}]^d \right\} \cap (0, \infty) \xrightarrow{d} \mathcal{P}_{\alpha\beta/d}.$$

Proof idea

Compare the point processes

$$\mathcal{D}_n := \left\{ \frac{\mathrm{Deg}(x)}{\xi q(n)} : x \in \eta \cap [0, n^{1/d}]^d \right\}$$

and

$$\mathcal{E}_n := \left\{ \frac{W_x^{d/\alpha}}{q(n)} : x \in \eta \cap [0, n^{1/d}]^d \right\}$$

and use that $\mathcal{E}_n$ is a Poisson process.

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and use that $\mathcal{E}_n$ is a Poisson process.

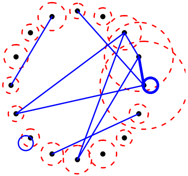
Our proof implies that, for any $k \in \mathbb{N}$,

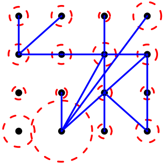
$$\lim_{n\to\infty} \mathbb{P}(\text{point with } k\text{-th largest weight in } [0, n^{1/d}]^d \text{ has } k\text{-th largest degree}) = 1.$$

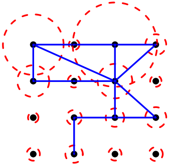
Extension to other random graphs

The results for the large degrees of the scale-free RCM follow from a more general result in Bhattacharjee/S. (2022) also applicable to:

- Norros-Reittu model
- Chung-Lu model
- Scale-free percolation model on $\mathbb{Z}^d$
- Ultra-small scale-free geometric networks







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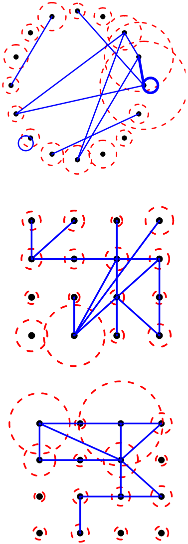
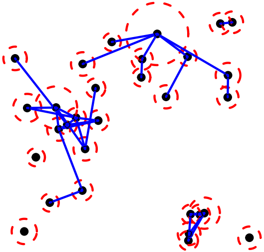
It is shown that the Hill estimator for the degree distribution is consistent.



Components of the scale-free RCM



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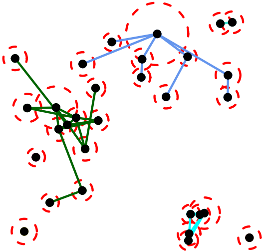

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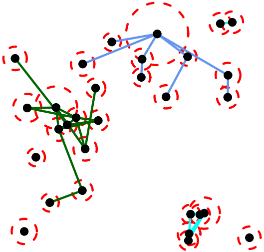
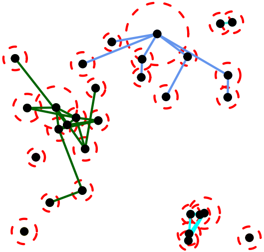


What are the sizes of the large components within $[0, n^{1/d}]^d$?



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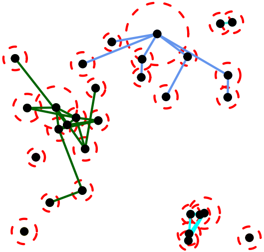
- Let $\mathcal{C}(x)$ be the component of $x \in \eta$ and $|\mathcal{C}(x)|$ its cardinality.
- Define $V_{\max} := \{x \in \eta : W_x \geq W_y \quad \text{for all} \quad y \in \mathcal{C}(x)\}$.

Components of the scale-free RCM



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We study

$$\max_{x \in \eta \cap [0, n^{1/d}]^d \cap V_{\max}} |\mathcal{C}(x)| \quad \text{and} \quad \left\{ |\mathcal{C}(x)| : x \in \eta \cap [0, n^{1/d}]^d \cap V_{\max} \right\}.$$

Large components of the scale-free RCM

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Theorem: Lienau/S. 2024+

Assume that $\alpha > d$, $\mathbb{E}[W^{3d/\alpha}] < \infty$ and

$$\varrho := \lambda^{d/\alpha} \kappa_d \Gamma(1 - d/\alpha) \mathbb{E}[W^{2d/\alpha}] < 1.$$

Then, there exists a constant $\zeta > 0$ such that, as $n \to \infty$,

$$\frac{1}{\zeta q(n)} \max_{x \in \eta \cap [0, n^{1/d}]^d \cap V_{\max}} |\mathcal{C}(x)| \xrightarrow{d} \text{Fréchet}(\alpha\beta/d)$$

and

$$\left\{ \frac{|\mathcal{C}(x)|}{\zeta q(n)} : x \in \eta \cap [0, n^{1/d}]^d \cap V_{\max} \right\} \cap (0, \infty) \xrightarrow{d} \mathcal{P}_{\alpha\beta/d}.$$

Component size of the typical vertex

Let $|\mathcal{C}_0|$ denote the size of the component of 0 in the RCM with the additional point $(0, W_0)$ with independent $W_0 \overset{d}{=} W$.

Lemma:

Under the assumptions of the previous theorem, there exists a constant $\zeta > 0$ such that

$$\lim_{w \to \infty} \frac{\mathbb{E}[|\mathcal{C}_0| \mid W_0 = w] - \zeta w^{d/\alpha}}{w^{d/\alpha}} = 0.$$

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Proof idea of the theorem: Compare the point processes

$$\mathcal{D}_n := \left\{ \frac{|\mathcal{C}(x)|}{\zeta q(n)} : x \in \eta \cap [0, n^{1/d}]^d \cap V_{\max} \right\}$$

and $\mathcal{E}_n := \{ W_x^{d/\alpha} / q(n) : x \in \eta \cap [0, n^{1/d}]^d \}$.

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Component size of the typical vertex

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Percolation of the scale-free RCM

Recall that

$$\mathbb{P}(x \leftrightarrow y) = 1 - \exp\left(- \frac{\lambda W_x W_y}{\|x - y\|^\alpha} \right)$$

and define $\lambda_c := \inf\{\lambda > 0 : \mathbb{P}(|\mathcal{C}_0| = \infty) > 0\}$.

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Theorem: Deprez/Wüthrich (2019)

Let $d \geq 2$ and assume that $\min\{\alpha, \alpha\beta\} > d$.

a) If $\alpha\beta < 2d$, then $\lambda_c = 0$.

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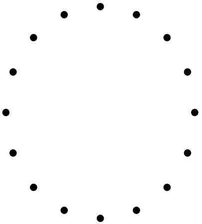
There are some $\lambda < \lambda_c$ with $\varrho \geq 1$. For those it is open if our result holds.

Norros-Reittu model



- W positive random variable such that $\mathbb{P}(W > u) = u^{-\beta} L(u)$, $u > 0$, with $\beta > 0$ and L slowly varying

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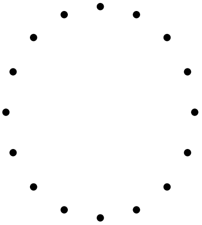

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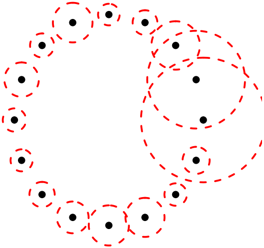

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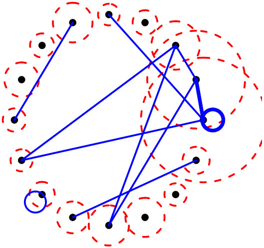

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- Mark the nodes $[n] := \{1, \ldots, n\}$ with i.i.d. copies of W.
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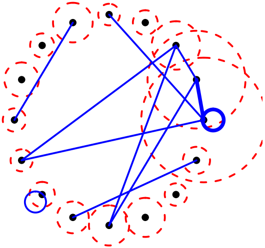

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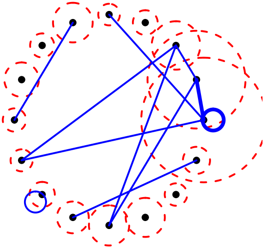

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Subcritical Norros-Reittu model

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Define

$$\tilde{q}(t) := \inf\{s \geq 0 : \mathbb{P}(W \leq s) \geq 1 - 1/t\}, \quad t \geq 1,$$

and $\tilde{\zeta} := \mathbb{E}[W]/(\mathbb{E}[W] - \mathbb{E}[W^2])$.

Theorem: Lienau/S. 2023+

Assume that $\beta > 2$ and that $\mathbb{E}[W^2] < \mathbb{E}[W]$. Then, as $n \to \infty$,

$$\frac{1}{\tilde{\zeta}\tilde{q}(n)} \max_{i \in [n]} |\mathcal{C}(i)| \xrightarrow{d} \text{Fréchet}(\beta)$$

and

$$\left\{ \frac{|\mathcal{C}(i)|}{\tilde{\zeta}\tilde{q}(n)} : i \in [n] \text{ and } W_j \geq W_i \ \forall\, j \in \mathcal{C}(i) \right\} \cap (0, \infty) \xrightarrow{d} \mathcal{P}_\beta.$$

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Thank you!

C. Bhattacharjee, M. Schulte (2022): Large degrees of scale-free inhomogeneous random graphs, *Ann. Appl. Probab.* **32**, 696–720.

M. Lienau, M. Schulte (2023+): Large components in the subcritical Norros-Reittu model, arXiv:2311.17606.

M. Lienau, M. Schulte (2024+): Large components in the subcritical scale-free random connection model, in preparation.

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