Binomial and Poisson Random Polytopes

Bath, September 2023

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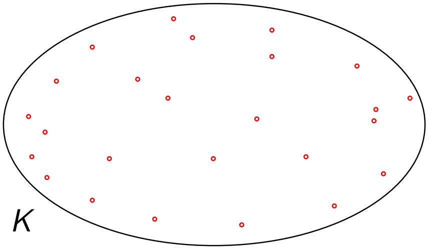
Random polytopes

Random points $X_1, \ldots, X_n$ uniformly in $K \subset \mathbb{R}^d$

$$P_n = [X_1, \ldots, X_n] \subset K$$



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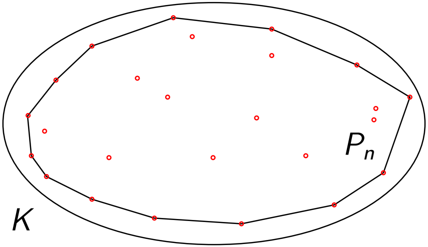

Random polytopes

Random points $X_1, \dots, X_n$ uniformly in $K \subset \mathbb{R}^d$

$$P_n = [X_1, \dots, X_n] \subset K$$

- $\vec{f}$-vector: $\vec{f}(P_n) = \begin{pmatrix} f_0(P_n) \\ f_1(P_n) \\ \vdots \\ f_{d-1}(P_n) \end{pmatrix} = \begin{pmatrix} \sharp \text{ vertices of } P_n \\ \sharp \text{ edges of } P_n \\ \vdots \\ \sharp \text{ facets of } P_n \end{pmatrix}$

- intrinsic volumes of P_n : $V_i(K) - V_i(P_n)$

Expectations: $K \in \mathcal{K}^d_{2+}$

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{E}\, \vec{f}(P_n) = \vec{c}\ \Omega_d(K)\, n^{1-\frac{2}{d+1}} + o(n^{1-\frac{2}{d+1}})$$

$$V_d(K) - \mathbb{E}\, V_d(P_n) = c_d\, \Omega_d(K)\, n^{-\frac{2}{d+1}} + o(n^{-\frac{2}{d+1}})$$

Bárány, Buchta, Efron, R. , Rényi, Schneider, Sulanke, Wieacker, ...

$$\Omega_d(K) = \int\limits_{\partial K} \kappa_K(x)^{\frac{1}{d+1}}\, dx$$

Expectations: $K \in \mathcal{K}^d_{2+}$

Random points $X_1, \ldots, X_n$ uniformly in $K \subset \mathbb{R}^d$

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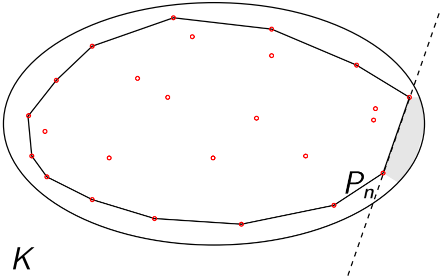


Expectations: $K \in \mathcal{P}^d$

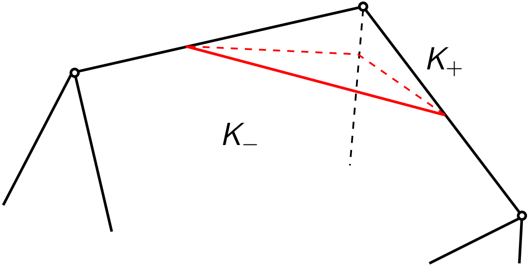
$K \in \mathcal{P}^d$:

$$\mathbb{E}\,\vec{\boldsymbol{f}}(P_n) = \tilde{\vec{\boldsymbol{c}}}\ T(K)\ln^{d-1} n + o(\ln^{d-1} n)$$

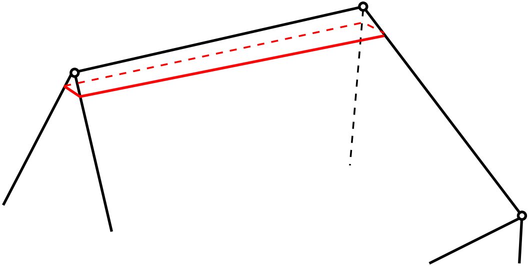
$$V_d(K) - \mathbb{E}\,V_d(P_n) = \tilde{c}_d\,T(K)\,n^{-1}\ln^{d-1} n + o(n^{-1}\ln^{d-1} n)$$

Affentranger, Bárány, Buchta, Efron, R. , Schneider , Schütt, ...

Expectations: $K \in \mathcal{P}^d$



Expectations: $K \in \mathcal{P}^d$



Variance

Efron-Stein Jackknife inequality

$$\mathbb{V}g(P_n) \leq (n+1)\mathbb{E}(g(P_{n+1}) - g(P_n))^2$$

$K \in \mathcal{K}^d_{2+}$:

$$\mathbb{V}f_i(P_n) \lessapprox n^{1-\frac{2}{d+1}}, \; \mathbb{V}V_d(P_n) \lessapprox n^{-1-\frac{2}{d+1}}$$

R.

Variance

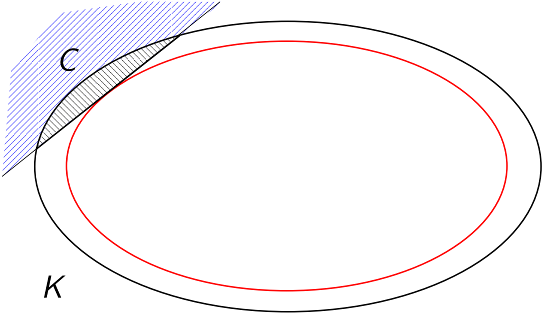
$K \in \mathcal{P}^d$:

$$\mathbb{V}\, f_i(\mathbb{P}_n) \lessapprox \ln^{d-1} n$$

$$V_d(K) - \mathbb{E}\, V_d(\mathbb{P}_n) \lessapprox n^{-2} \ln^{d-1} n$$

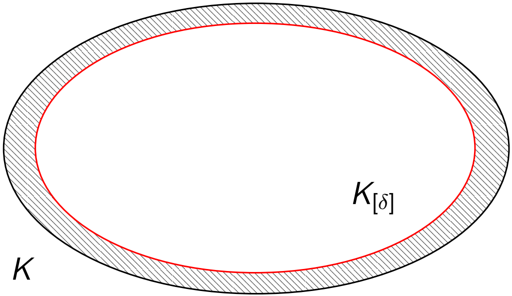
Bárány, R.

The floating body



$V_d(C) = \delta$

The floating body



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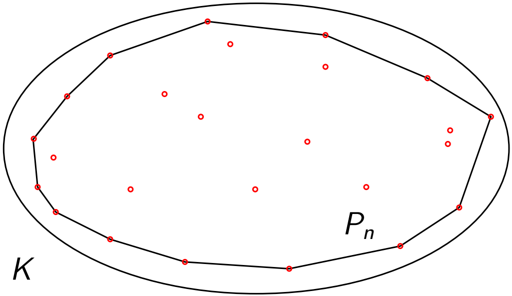

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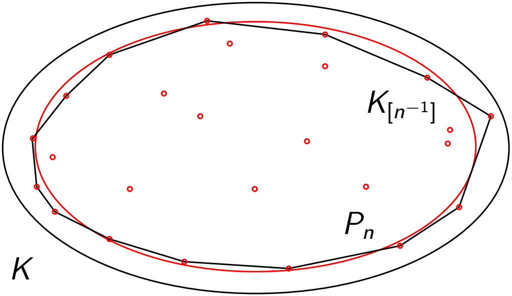

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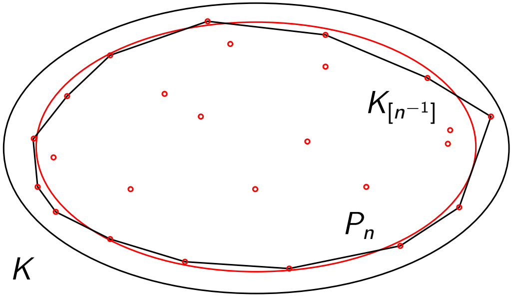

$$\mathcal{F}_0(P_n) \subset \{X_1, \ldots, X_n\} \cap K_{[n^{-1} \ln \cdots n]} \sim \mathrm{Bin}(n, V_d(K_{[n^{-1} \ln \cdots n]})) \to \pi(\cdot)$$

Expectations: $K \in \mathcal{K}^d_{2+}$

Poisson point process η_n in $K \subset \mathbb{R}^d$

$$\Pi_n = [\eta_n] = P_N, \; N \sim \pi(n)$$

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{E}\, \vec{f}(\Pi_n) = \mathbb{E}\vec{c}\; \Omega_d(K) N^{1-\frac{2}{d+1}} + \mathbb{E} o(N^{1-\frac{2}{d+1}})$$

$$V_d(K) - \mathbb{E}\, V_d(\Pi_n) = c_d\, \Omega_d(K)\, n^{-\frac{2}{d+1}} + o(n^{-\frac{2}{d+1}})$$

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Expectations: $K \in \mathcal{P}^d$

$K \in \mathcal{P}^d$:

$$\mathbb{E}\, \vec{f}(\Pi_n) = \tilde{\vec{c}}\ T(K)\, \ln^{d-1} n + o(\ln^{d-1} n)$$

$$V_d(K) - \mathbb{E}\, V_d(\Pi_n) = \tilde{c}_d\, T(K)\, n^{-1} \ln^{d-1} n + o(n^{-1} \ln^{d-1} n)$$

Variance

Poincaré inequality

$$\mathbb{V} g(\eta_n) \leq n\, \mathbb{E}(g(\eta_n + \delta_X) - g(\eta_n))^2$$

Nualart und Vives, Last und Penrose, ...

Variance

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{V} f_i(\Pi_n) \lesssim n^{1-\frac{2}{d+1}}, \;\; \mathbb{V} V_d(\Pi_n) \lesssim n^{-1-\frac{2}{d+1}}$$

$K \in \mathcal{P}^d$:

$$\mathbb{V}\, f_i(\Pi_n) \lesssim \ln^{d-1} n + o(\ln^{d-1} n), \;\; \mathbb{V} V_d(\Pi_n) \lesssim n^{-2} \ln^{d-1} n$$

Variances

$K \in \mathcal{K}^d_{2+}$

$$\mathbb{V} f_i(\Pi_n) = c_i \Omega_d(K) n^{1-\frac{2}{d+1}} + \dots$$

$$\mathbb{V} V_d(\Pi_n) = c_d \Omega_d(K) n^{-1-\frac{2}{d+1}} + \dots$$

Calka, Yukich

Variances

$K \in \mathcal{K}^d_{2+}$

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Calka, Yukich

$K \in \mathcal{P}^d$ simple

$$\mathbb{V} f_i(\Pi_n) = c_i f_0(K) \ln^{d-1} n + \dots$$

$$\mathbb{V} V_d(\Pi_n) = c_d f_0(K) n^{-2} \ln^{d-1} n + \dots$$

Calka, Yukich

de-Poissonization

$K \in \mathcal{K}^d_{2+}$:

$$|\mathbb{V} f_k(P_n) - \mathbb{V} f_k(\Pi_n)| \lessapprox n^{1-\frac{4}{d+1}}$$
$$|\mathbb{V} V_d(P_n) - \mathbb{V} V_d(\Pi_n)| \lessapprox n^{-1-\frac{4}{d+1}}$$

$K \in \mathcal{P}^d$:

$$|\mathbb{V} f_k(P_n) - \mathbb{V} f_k(\Pi_n)| \lessapprox n^{-1} \ln^{2d^2+d} n$$
$$|\mathbb{V} V_d(P_n) - \mathbb{V} V_d(\Pi_n)| \lessapprox n^{-2-1/2} \ln^{3d+1} n$$

Variances

$K \in \mathcal{K}^d_{2+}$

$$\mathbb{V} f_i(P_n) = c_i \Omega_d(K) n^{1-\frac{2}{d+1}} + \dots$$

$$\mathbb{V} V_d(P_n) = c_d \Omega_d(K) n^{-1-\frac{2}{d+1}} + \dots$$

$K \in \mathcal{P}^d$ simple

$$\mathbb{V} f_i(P_n) = c_i f_0(K) \ln^{d-1} n + \dots$$

$$\mathbb{V} V_d(P_n) = c_d f_0(K) n^{-2} \ln^{d-1} n + \dots$$

CLT for Π_n

Theorem

For $K \in \mathcal{K}^d_{2+}$ or K a polytope:

$$\left| \mathbb{P}\left(\frac{f_i(\Pi_n) - \mathbb{E} f_i(\Pi_n)}{\sqrt{\mathbb{V} f_i(\Pi_n)}} \leq x \right) - \Phi(x) \right| \leq \varepsilon(n) \to 0$$

$$\left| \mathbb{P}\left(\frac{V_d(\Pi_n) - \mathbb{E} V_d(\Pi_n)}{\sqrt{\mathbb{V} V_d(\Pi_n)}} \leq x \right) - \Phi(x) \right| \leq \varepsilon(n) \to 0$$

Bárány, Bräker, Cabo, Calka, Groeneboom, Hsing, Pardon, R., Schreiber, Thäle, Yukich, Vu

$$K \in \mathcal{K}^d_{2+} : \ \varepsilon(n) \approx n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{\cdots} n$$

$$K \in \mathcal{P}^d : \ \varepsilon(n) \approx \ln^{-\frac{d-1}{2}} \ln\ln^{\cdots} n$$

de-Poissonization

$K \in \mathcal{K}^d_{2+}$:

$$|F_{f_k(P_n)}(x) - F_{f_k(\Pi_n)}(x)| \lessapprox n^{-\frac{2}{d+1}} \ln^{\frac{2}{d+1}} n$$

$$|F_{V_d(P_n)}(x) - F_{V_d(\Pi_n)}(x)| \lessapprox n^{-\frac{2}{d+1}} \ln^{\frac{2}{d+1}} n$$

$K \in \mathcal{P}^d$:

$$|F_{f_k(P_n)}(x) - F_{f_k(\Pi_n)}(x)| \lessapprox \ln^{-4d^2}$$

$$|F_{V_d(P_n)}(x) - F_{V_d(\Pi_n)}(x)| \lessapprox \ln^{-4d^2}$$

CLT

Theorem

For $K \in \mathcal{K}^d_{2+}$ or K a polytope:

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$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E} V_d(P_n)}{\sqrt{\mathbb{V} V_d(P_n)}} \leq x \right) - \Phi(x) \right| \leq \varepsilon(n) \to 0$$

Bárány, Bräker, Cabo, Calka, Groeneboom, Hsing, Pardon, R., Schreiber, Thäle, Yukich, Vu

$$K \in \mathcal{K}^d_{2+} : \ \varepsilon(n) \approx n^{-\frac{1}{2} + \frac{1}{d+1}} \ln^{\cdots} n$$

$$K \in \mathcal{P}^d : \ \varepsilon(n) \approx \ln^{-\frac{d-1}{2}} \ln\ln^{\cdots} n$$

CLT for $\mathcal{K}^d_{2+}$

K sufficiently smooth:

$$\left|\mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathbb{V}V_d(P_n)}} \le x\right) - \Phi(x)\right| \le c(K) n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+\frac{2}{d+1}} n$$

R., Vu

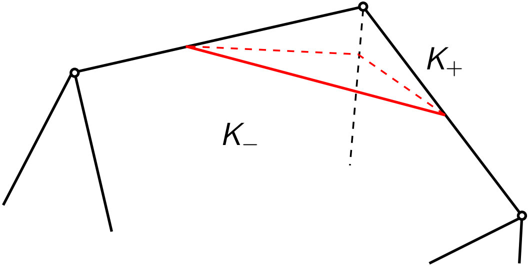
CLT for $\mathcal{K}^d_{2+}$

K sufficiently smooth:

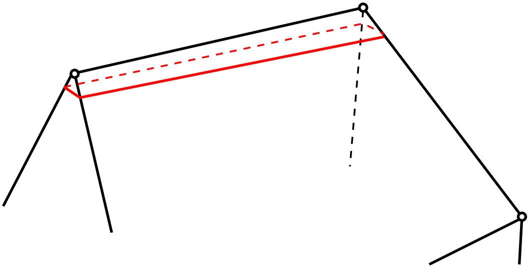
$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathbb{V}V_d(P_n)}} \leq x \right) - \Phi(x) \right| \leq c(K) n^{-\frac{1}{2}+\frac{1}{d+1}}$$

R., Vu, Lachieze-Rey & Schulte & Yukich
stabilization!

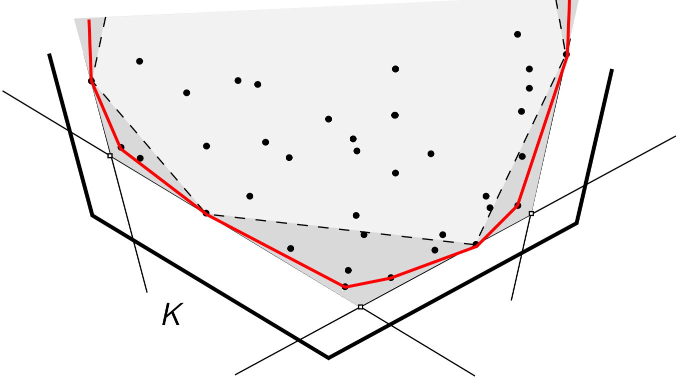
CLT for $\mathcal{P}^d$



CLT for $\mathcal{P}^d$



CLT for $\mathcal{P}^2$



Distribution function for rd chains (Buchta)
CLT for rd chains (Gusakova & Thäle) $\longrightarrow$ Poissonization
CLT for rd polygons (Gusakova & R & Thäle) $\longrightarrow$ de-Poissonization

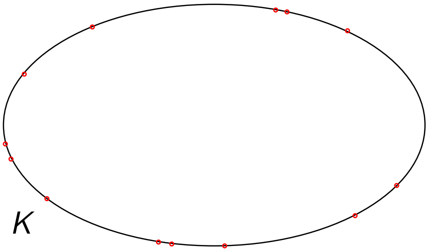
$K \in \mathcal{P}^2$:

$$\left| \mathbb{P}\left(\frac{f_0(P_n) - \mathbb{E} f_0(P_n)}{\sqrt{\mathbb{V} f_0(P_n)}} \leq x \right) - \Phi(x) \right| \leq c \ln^{-\frac{1}{2}} n$$

Random polytopes with vertices on ∂K

random points $X_1, \ldots, X_n$ uniformly on ∂K

$$P_n = [X_1, \ldots, X_n] \subset K$$



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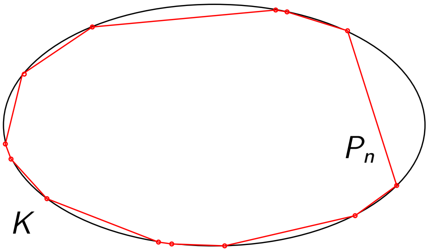

Expectations

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{E}\, f_0(P_n) = n$$

$$V_d(K) - \mathbb{E} V_d(P_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}}\, dx\, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

Affentranger, Böröczky, Buchta, Fodor, Hug, Gruber, Müller, R. , Schneider, Schütt, Tichy, Werner, . . .

Expectations

$$K \in \mathcal{K}^d_{2,+}: \qquad \mathbb{E}\, f_0(P_n) = n$$

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Affentranger, Böröczky, Buchta, Fodor, Hug, Gruber, Müller, R. , Schneider, Schütt, Tichy, Werner, . . .

$d \geq 2: \quad f_0(P_n) = n$

$d = 2: \quad f_0(P_n) = f_1(P_n) = n$

$d = 3: \quad f_0(P_n) = n,\; f_1(P_n) = 3n - 6,\; f_2 = 2n - 4$

Expectations

$$K \in \mathcal{K}^d_{2,+}: \qquad \mathbb{E}\, f_0(P_n) = n$$

$$V_d(K) - \mathbb{E} V_d(P_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}}\, dx\, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

Affentranger, Böröczky, Buchta, Fodor, Hug, Gruber, Müller, R. , Stemeseder, Schneider, Schütt, Tichy, Werner, . . .

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$d = 3: \quad f_0(P_n) = n,\ f_1(P_n) = 3n - 6,\ f_2 = 2n - 4$

$d \geq 4: \quad f_i(P_n)$ is a random variable, $i \geq 1$ Stemeseder

Expectations

$$K \in \mathcal{K}^d_{2,+}: \qquad \mathbb{E} f_i(P_n) = c_i n + o(n)$$

$$V_d(K) - \mathbb{E} V_d(P_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}} \, dx \, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

Affentranger, Böröczky, Buchta, Fodor, Hug, Gruber, Müller, R. , Stemeseder, Schneider, Schütt, Tichy, Werner, . . .

$d \geq 2: \quad f_0(P_n) = n$

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Expectation

$K \in \mathcal{K}^d_{2,+}$:

$$\begin{aligned}
\mathbb{E}\, f_i(\Pi_n) &= c_i n + o(n) \\
V_d(K) - \mathbb{E} V_d(\Pi_n) &= c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}}\, dx\, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}}) \\
\mathbb{V} f_i(\Pi_n) &\lessapprox c_i n \\
\mathbb{V} V_d(\Pi_n) &\lessapprox n^{-1-\frac{2}{d-1}}
\end{aligned}$$

Richardson, R., Stemeseder, Vu, Wu

CLT for $\mathcal{K}^d$: rd points on ∂K

Theorem

For $K \in \mathcal{K}^d_{2+}$:

$$\left| \mathbb{P}\left(\frac{f_i(\Pi_n) - \mathbb{E} f_i(\Pi_n)}{\sqrt{\mathbb{V} f_i(\Pi_n)}} \leq x \right) - \Phi(x) \right| \to 0$$

$$\left| \mathbb{P}\left(\frac{V_d(\Pi_n) - \mathbb{E} V_d(\Pi_n)}{\sqrt{\mathbb{V} V_d(\Pi_n)}} \leq x \right) - \Phi(x) \right| \to 0$$

Richardson, Stemeseder, Vu, Wu

$\longrightarrow$ no de-Possonization

CLT for $\mathcal{K}^d$: rd points on ∂K

Theorem

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Thäle,

CLT for $\mathcal{K}^d$: rd points on ∂K

Theorem

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$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E} V_d(P_n)}{\sqrt{\mathbb{V} V_d(\Pi_n)}} \le x \right) - \Phi(x) \right| \to 0$$

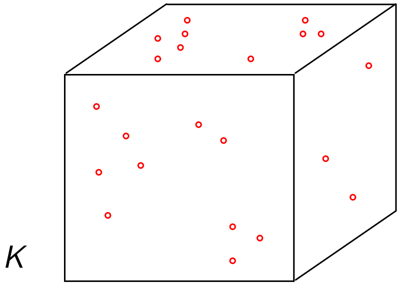
Thäle, R., Sonnleitner

…using Lachieze-Rey & Schulte & Yukich

Random polytopes with vertices on ∂K

$K \in \mathcal{P}^d$: random points $X_1, \ldots, X_n$ uniformly on ∂K

$$P_n = [X_1, \ldots, X_n] \subset K$$



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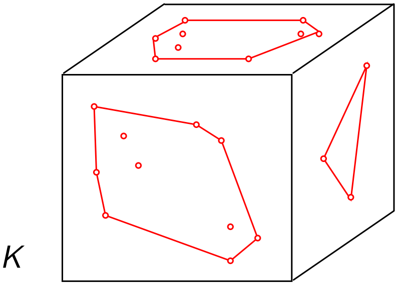

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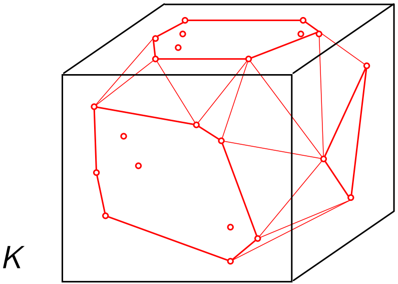

Expectations

$K \in \mathcal{P}^d$ a simple polytope:

$$
\begin{aligned}
\mathbb{E}\, f_0(P_n) &= c_d\, T(K)\, \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
\mathbb{E} f_{d-1}(P_n) &= c_d\, T(K)\, \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
V_d(K) - \mathbb{E}\, V_d(P_n) &= c(K) n^{-\frac{d}{d-1}} \quad + o(n^{-\frac{d}{d-1}})
\end{aligned}
$$

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Open Problems

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- optimal rates in the CLT for $K \in \mathcal{P}^d$
- random points on ∂K for K polytope: Var and CLT

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- High-dimensional limits: Pierre's talk on Friday!

Thank you!