Maximum cells in Poisson hyperplane mosaics

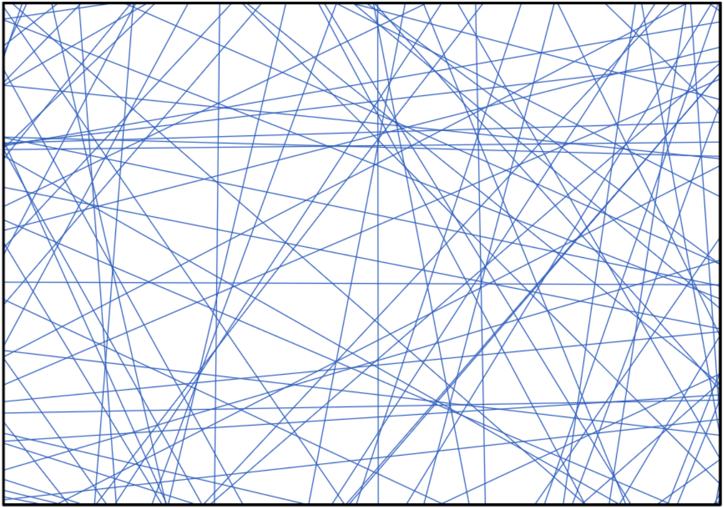
Moritz Otto (Leiden)

EJP 28 (2023), article no. 162

Workshop on Stochastic Geometry in Action
Bath, September 12, 2024

Universiteit Leiden

Poisson hyperplane mosaic



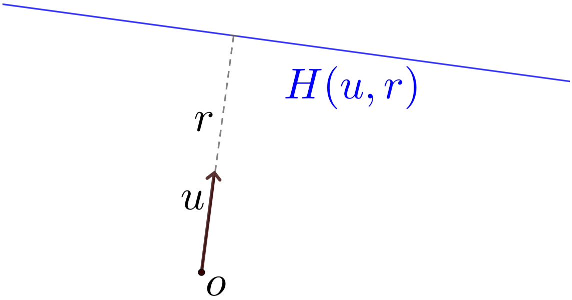
Poisson hyperplane process

- η stationary, isotropic Poisson hyperplane process in $\mathbb{R}^d$
- η has intensity measure $\gamma\mu_{d-1}$ with

$$\mu_{d-1}(\cdot) = 2\int_{S^{d-1}}\int_0^\infty 1\{H(u,r)\in\cdot\}\mathrm{d}r\sigma(\mathrm{d}u)$$

where

$$H(u,r) := \{x\in\mathbb{R}^d : \langle x,u\rangle = r\}.$$



Literature review

Textbook:

- Hug–Schneider 24. Poisson Hyperplane Tessellations

Some asymptotic results on Poisson hyperplanes:

- (Heinrich–Schmidt–Schmidt AAP 06): Central limit theorems
- (Hug–Reitzner–Schneider AP 04): Limit shape of the zero cell
- (Chenavier–Hemsley AdvAP 16): Poisson approximation for large cells. Restricted to
  - distributional convergence, no rates
  - $d = 2$
  - inradius

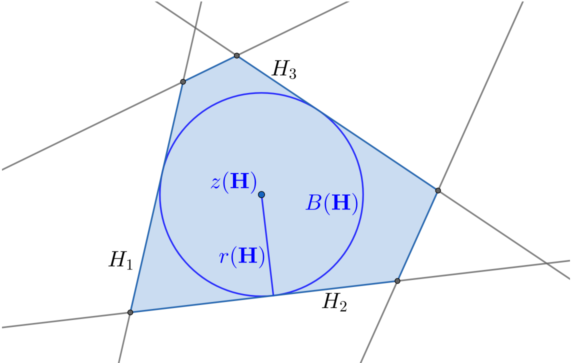
Our model

Consider the $\mathbb{R}^d \times \mathbb{R}$-valued point process

$$\zeta_n := \frac{1}{(d+1)!} \sum_{\mathbf{H} \in \eta_{\neq}^{d+1}} 1\{(\eta - \delta_{\mathbf{H}})(\mathbb{H}_{B(\mathbf{H})}) = \emptyset\} \delta_{(n^{-1/d} z(\mathbf{H}), 2\gamma r(\mathbf{H}) - \log n)}$$

where

$$\mathbb{H}_K := \{H \in \mathbb{H} : \ H \cap K \neq \emptyset\}.$$



Poisson approximation for cells with large inradius

Recall:

$$\zeta_n := \frac{1}{(d+1)!} \sum_{\mathbf{H} \in \eta^{d+1}_{\neq}} 1\{(\eta - \delta_{\mathbf{H}})(\mathbb{H}_{B(\mathbf{H})}) = \emptyset\} \delta_{(n^{-1/d} z(\mathbf{H}), 2\gamma r(\mathbf{H}) - \log n)}$$

Theorem (O. 23)

Let ν be a Poisson process on $\mathbb{R}^d \times \mathbb{R}$ with IM $\gamma^{(d)} \lambda_d \otimes \varphi$, where

- $\gamma^{(d)}$ cell intensity.
- φ measure on $\mathbb{R}$ given by $\varphi((c,\infty)) = e^{-c}$.

Then for all compact W and all $c \in \mathbb{R}$ there exists $C > 0$ s.t.

$$\mathbf{d_{KR}}(\zeta_n \cap W \times (c,\infty), \nu \cap W \times (c,\infty)) \leq C n^{-\delta} (\log n)^{2d},$$

where $\delta > 0$ solves $\delta = \frac{\omega_{d-1}}{\omega_d} \int_{\frac{2+\delta}{3+\delta}}^{1} (1 - x^2)^{\frac{d-3}{2}} \mathrm{d}x$.

Poisson process approximation via Stein's method

Assume

- ζ finite point process with intensity measure $\mathbf{K}$
- For $\mathbf{K}$-a.a. $x \in \mathbb{R}^d$, let $\zeta_x \stackrel{d}{=} \zeta$ and let ζ^x be a reduced Palm version of ζ at x
- ν finite Poisson process with intensity measure $\mathbf{L}$

Theorem (Bobrowski–Schulte–Yogeshwaran 22)

We have

$$\mathbf{d_{KR}}(\zeta, \nu) \leq d_{TV}(\mathbf{K}, \mathbf{L}) + 2 \int \mathbb{E}\{(\zeta_x \Delta \zeta^x)(\mathbb{R}^d)\} \, \mathbf{K}(\mathrm{d}x).$$

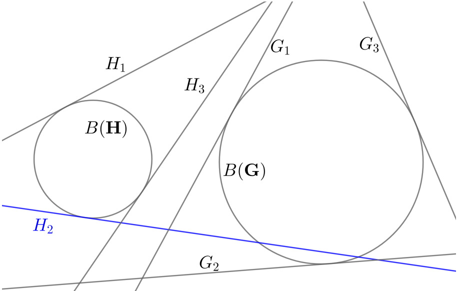
Proof ideas I

Chen-Stein method combined with a Palm coupling gives

$$\mathbf{d_{KR}}(\zeta_n \cap W \times (c,\infty), \nu \cap W \times (c,\infty)) \le E_1 + E_2 + E_3$$

where for $g_n(\mathbf{H}) := 1\{n^{-1/d} z(\mathbf{H}) \in W\} 1\{2\gamma r(\mathbf{H}) - \log n > c\}$,

$$E_1 = c_1 \int_{\mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1}} 1\{\{\mathbf{H}, \mathbf{G}\} \cap \mathbb{H}_{B(\mathbf{H})} \cap \mathbb{H}_{B(\mathbf{G})} \neq \emptyset\} g_n(\mathbf{H}) g_n(\mathbf{G})$$
$$\times\, e^{-2\gamma r(\mathbf{H})} e^{-2\gamma r(\mathbf{G})} \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{G}) \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{H}).$$

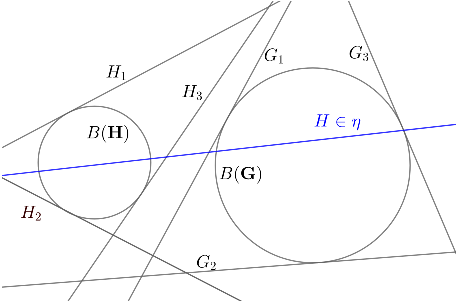


$H_2 \in \mathbb{H}_{B(\mathbf{G})}$

Proof ideas II

Recall that $g_n(\mathbf{H}) := 1\{n^{-1/d} z(\mathbf{H}) \in W\} 1\{2\gamma r(\mathbf{H}) - \log n > c\}$.

$$E_2 = c_2 \int_{\mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1}} 1\{r(\mathbf{H}) + r(\mathbf{G}) \le \|z(\mathbf{H}) - z(\mathbf{G})\|\} g_n(\mathbf{H}) g_n(\mathbf{G})$$
$$\times e^{-\mu_{d-1}(\mathbb{H}_{B(\mathbf{H})} \cup \mathbb{H}_{B(\mathbf{G})})} \mathbb{P}(\eta \cap \mathbb{H}_{B(\mathbf{H})} \cap \mathbb{H}_{B(\mathbf{G})} \neq \emptyset) \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{G}) \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{H}).$$



$H \in \mathbb{H}_{B(\mathbf{H})} \cap \mathbb{H}_{B(\mathbf{G})}$

Proof ideas III

To control E_2, we need to understand $\mu_{d-1}(\mathbb{H}_{B(z,r)} \cap \mathbb{H}_{B(w,s)})$.

Proposition

We have for $z, w \in \mathbb{R}^d$ and $r, s > 0$,

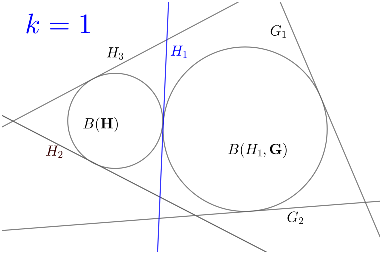
$$\mu_{d-1}(\mathbb{H}_{B(z,r)} \cap \mathbb{H}_{B(w,s)}) = \frac{2\omega_{d-1}}{\omega_d} \int_0^s \int_{\frac{t-r}{\|w-z\|} \vee -1}^{\frac{t+r}{\|w-z\|} \wedge 1} (1-x^2)^{\frac{d-3}{2}} \mathrm{d}x \mathrm{d}t.$$

Proof ideas IV

Recall that $g_n(\mathbf{H}) := 1\{n^{-1/d} z(\mathbf{H}) \in W\} 1\{2\gamma r(\mathbf{H}) - \log n > c\}$.

$$E_3 = c_3 \sum_{k=1}^{d} \int_{\mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1-k}} g_n(\mathbf{H}) g_n(\mathbf{H}_k, \mathbf{G}) e^{-\mu_{d-1}(\mathbb{H}_{B(\mathbf{H})} \cup \mathbb{H}_{B(\mathbf{H}_k, \mathbf{G})})}$$
$$\times 1\{r(\mathbf{H}) + r(\mathbf{H}_k, \mathbf{G}) \leq \|z(\mathbf{H}) - z(\mathbf{H}_k, \mathbf{G})\|\} \mu_{d-1}^{d+1-k}(\mathrm{d}\mathbf{G}) \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{H}),$$

where $\mathbf{H}_k := (H_1, \ldots, H_k)$.



Proof ideas IV

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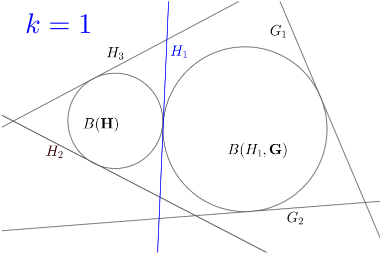

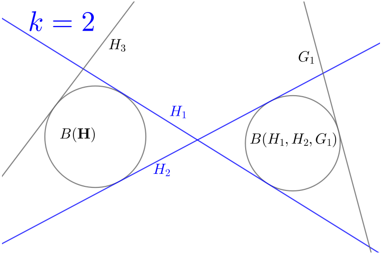

General size functionals

- $\Sigma : \mathcal{K}^d \to \mathbb{R}$ nice size functional, e.g. kth intrinsic volume
- Z typical cell in the mosaic generated by η
- F distribution function of $\Sigma(Z)$

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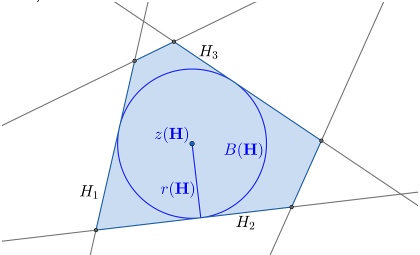
Let $G := \frac{1}{1-F}$. Then $G(\Sigma(Z)) \sim \text{Pareto}(1)$, i.e.

$$\mathbb{P}(G(\Sigma(Z)) > u) = u^{-1}, \quad u \geq 1.$$

Consider the $\mathbb{R}^d \times \mathbb{R}$-valued point process

$$\xi_n := \frac{1}{(d+1)!} \sum_{\mathbf{H} \in \eta_{\neq}^{d+1}} \mathbb{1}\{(\eta - \delta_{\mathbf{H}})(\mathbb{H}_{B(\mathbf{H})}) = \emptyset\} \delta_{(n^{-1/d} z(\mathbf{H}), n^{-1} G(\Sigma(C(\mathbf{H},\eta))))}$$



Poisson approximation for general size functionals

Recall:

$$\xi_n := \frac{1}{(d+1)!} \sum_{\mathbf{H} \in \eta_{\neq}^{d+1}} 1\{(\eta - \delta_{\mathbf{H}})(\mathbb{H}_{B(\mathbf{H})}) = \emptyset\} \delta_{(n^{-1/d} z(\mathbf{H}), n^{-1} G(\Sigma(C(\mathbf{H}, \eta))))}$$

Theorem (O. 23)

Assume that all extremal bodies of Σ are Euclidean balls. Let ν be a Poisson process on $\mathbb{R}^d \times \mathbb{R}$ with IM $\gamma^{(d)} \lambda_d \otimes \psi$, where

- $\gamma^{(d)}$ cell intensity,
- ψ measure on $(0, \infty)$ given by $\psi((a, \infty)) = a^{-1}$, $a > 0$.

Then there exists $b > 0$ such that for all compact W and all $c > 0$ there exists $C > 0$ s.t.

$$\mathbf{d_{KR}}(\xi_n \cap W \times (c, \infty), \nu \cap W \times (c, \infty)) \leq C n^{-b}.$$

Asymptotic shape of the typical cell

- Let Σ be a k-homogeneous size functional.

Asymptotic shape of the typical cell

- Let Σ be a *k-homogeneous* size functional.
- We call $K \in \mathcal{K}^d$ an extremal body if it minimizes $\frac{\mu_{d-1}(\mathbb{H}_K)}{\Sigma(K)^{1/k}}$.

Asymptotic shape of the typical cell

- Let Σ be a k-homogeneous size functional.
- We call $K \in \mathcal{K}^d$ an extremal body if it minimizes $\dfrac{\mu_{d-1}(\mathbb{H}_K)}{\Sigma(K)^{1/k}}$.
- Define the deviation functional

$$\vartheta(K) := \min\left\{\frac{R-r}{R+r} : \, rB^d + z \subset K \subset RB^d + z, \, r, R > 0, \, z \in \mathbb{R}^d\right\}.$$

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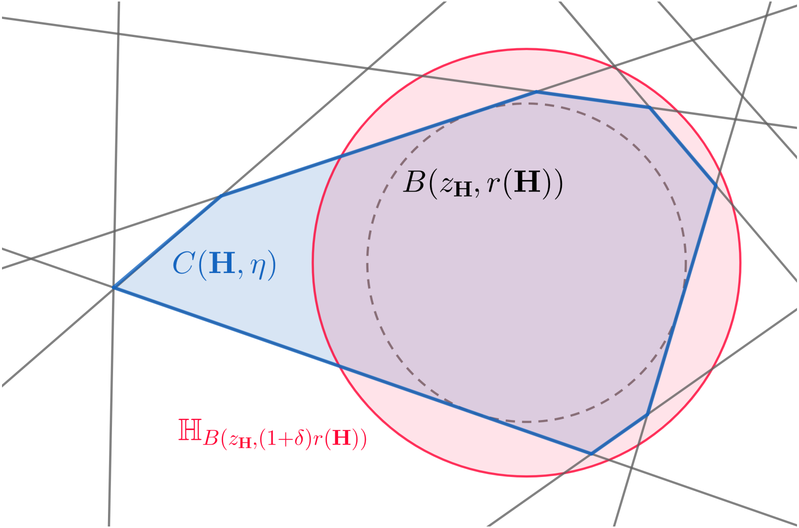
$$\vartheta(K) := \min\left\{\frac{R-r}{R+r} : \, rB^d+z \subset K \subset RB^d+z, \ r, R > 0, \ z \in \mathbb{R}^d\right\}.$$

- There is a function s with $s(\varepsilon) > 0$ for $\varepsilon > 0$ and $s(0) = 0$ s.t.

$$\mathbb{P}(\vartheta(Z) > \varepsilon \mid \Sigma(Z) > u) \leq C \exp(-s(\varepsilon)u^{1/k}\gamma),$$

where C does not depend on u and ε.

Proof ideas I



Idea: Distinguish by the deviation of large cells from a ball

Proof ideas II

- Let $g_n(\mathbf{H}, \omega) := 1\{n^{-1} G(\Sigma(C(\mathbf{H}, \omega))) > c\} 1\{\omega \cap \mathbb{H}_{B(\mathbf{H})} = \emptyset\}$.
- Assume that $\omega \mapsto \mathcal{S}(\mathbf{H}, \omega)$ is a stopping set s.t.

$$g_n(\mathbf{H}, \omega) = g_n(\mathbf{H}, \omega \cap S), \quad S \supset \mathcal{S}(\mathbf{H}, \omega).$$

Proof ideas II

- Let $g_n(\mathbf{H},\omega) := 1\{n^{-1}G(\Sigma(C(\mathbf{H},\omega))) > c\}1\{\omega \cap \mathbb{H}_{B(\mathbf{H})} = \emptyset\}$.
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$$g_n(\mathbf{H},\omega) = g_n(\mathbf{H},\omega \cap S), \quad S \supset \mathcal{S}(\mathbf{H},\omega).$$

- We have

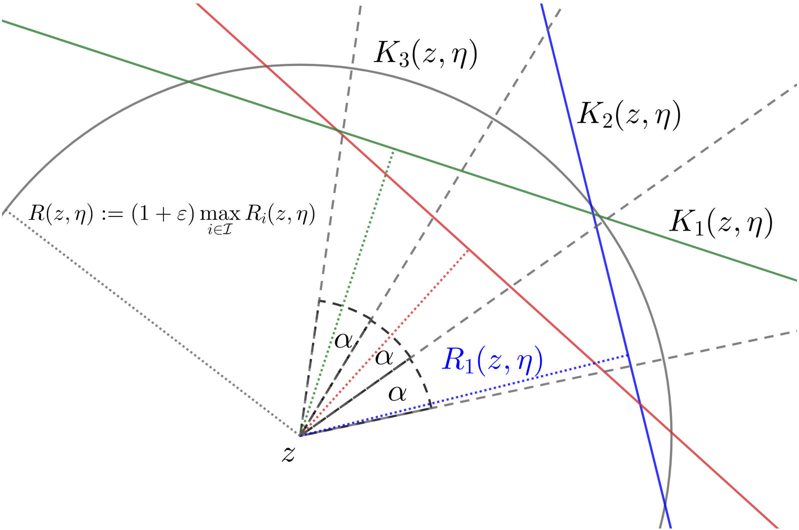
$$\mathbf{d_{KR}}(\xi_n \cap W \times (c,\infty), \nu \cap W \times (c,\infty)) \leq E_0 + E_1 + E_2 + E_3,$$

where for $S_{\mathbf{H}} := \mathbb{H}_{B(z_{\mathbf{H}},(1+\delta)r(\mathbf{H}))}$,

$$E_0 = c_0 \int_{\mathbb{H}^{d+1}} 1\{n^{-\frac{1}{d}} z_{\mathbf{H}} \in W\} \mathbb{E}\Big\{g_n(\mathbf{H},\eta_{\mathbf{H}}) 1\{\mathcal{S}(\mathbf{H},\eta) \not\subseteq S_{\mathbf{H}}\}\Big\} \mu_{d-1}^{d+1}(\mathrm{d}\mathbf{H})$$

and E_1, E_2, E_3 only use $g_n(\mathbf{H},\eta_{\mathbf{H}} \cap S_{\mathbf{H}})$!

Stopping sets



Stopping sets

- Let $\alpha \in (0, \pi/6)$
- Let $\mathbf{H} = (H_1, \ldots, H_{d+1})$ in general position with $z = z(\mathbf{H})$
- Let $K_i(z)$, $i \in \mathcal{I}$, be infinite open cones with apex z, angular radius $\alpha/2$ and union $\mathbb{R}^d$
- For $i \in \mathcal{I}$ let

  $$R_i(z, \omega) := \inf\{r > 0 : \exists u \in S^{d-1} \cap K_i(z) \text{ s.t. } H(u, \langle u, z\rangle + r) \in \omega\}$$

- Let

  $$R(z, \omega) := (\cos\alpha)^{-1} \max_{i \in \mathcal{I}} R_i(z, \omega).$$

Lemma

The mapping

$$\omega \mapsto \mathbb{H}_{B(z(\mathbf{H}), R(z(\mathbf{H}), \omega))} =: \mathcal{S}(\mathbf{H}, \omega)$$

is a stopping set and g_n is localized to $\mathcal{S}$.

Thank you!