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On the cluster density and the uniqueness of the infinite cluster in the stationary marked random connection model

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1. The random connection model

Setting

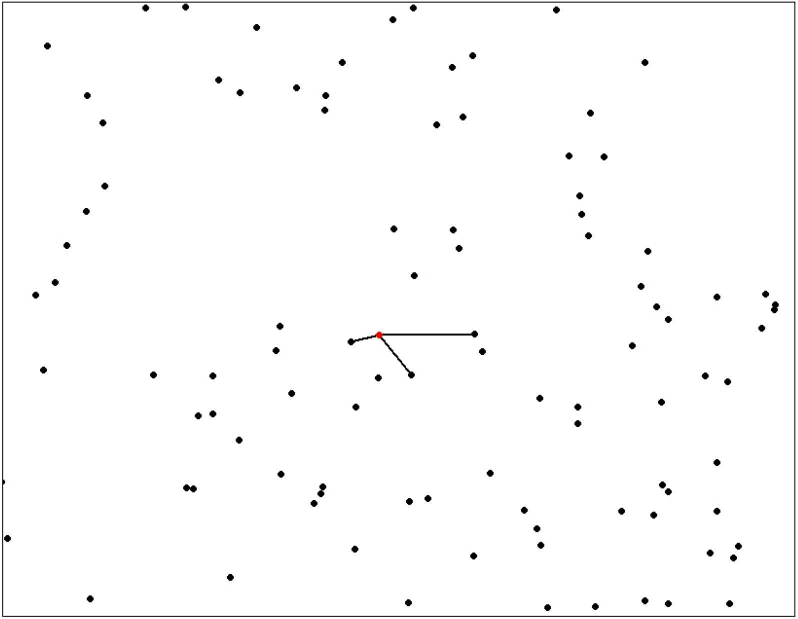
η is a Poisson process on a complete separable metric space (CSMS) $\mathbb{X}$ with intensity measure $t\lambda$, where λ is a σ-finite and diffuse measure on $\mathbb{X}$ and $t \geq 0$ is an intensity parameter.

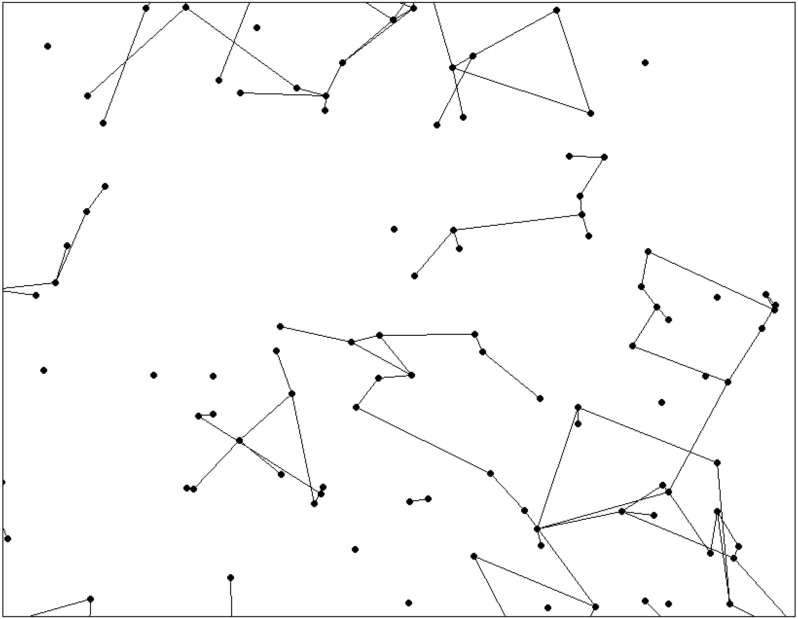
Definition

Let $\varphi\colon \mathbb{X} \times \mathbb{X} \to [0,1]$ be a measurable and symmetric connection function, satisfying

$$\int \varphi(x,y)\,\lambda(dy) < \infty, \quad \lambda\text{-a.e. } x.$$

Given η, connect any two distinct points $x, y \in \eta$ with probability $\varphi(x,y)$ independently of all other pairs. This gives the random connection model $\xi \equiv \Gamma_\varphi(\eta)$.





Definition

Write $\eta = \{X_n : n \leq \eta(\mathbb{X})\}$ and let $(U_{m,n})_{m,n \geq 1}$ be an iid-sequence of uniformly distributed random variables. Then the edges of the RCM $\Gamma_\varphi(\eta)$ are given by the point process

$$\{\{X_m, X_n\} : X_m < X_n, U_{m,n} \leq \varphi(X_m, X_n)\},$$

where $<$ on is a measurable partial ordering on $\mathbb{X}$, that orders the points of $\mathbb{X}$ λ-a.e. The RCM $\Gamma_\varphi(\eta')$ can be defined for any point process η' on $\mathbb{X}$.

2. Percolation

Definition

- Given $x \in \eta$ the cluster $C(x, \xi)$ is the connected component of x in ξ.
- Given $x \in \mathbb{X}$ let ξ^x denote the RCM arising from ξ by adding the point x along with independently drawn connections between x and the points from η.
- The cluster $C^x \equiv C^x(x, \xi^x)$ is the connected component of x in ξ^x.

Definition

If the RCM ξ has an infinite cluster, then it *percolates*. The position dependent percolation probability is defined by

$$\theta^x(t) := \mathbb{P}_t(|C^x| = \infty), \quad x \in \mathbb{X}.$$

Theorem

The RCM percolates with positive probability iff

$$\lambda(\{x : \theta^x(t) > 0\}) > 0.$$

3. The stationary marked RCM

Setting

- $\mathbb{M}$ is a CSMS equipped with a probability measure $\mathbb{Q}$, the mark distribution.
- $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$ and $\lambda = t\lambda_d \otimes \mathbb{Q}$, where λ_d denotes Lebesgue measure on $\mathbb{R}^d$.
- The symmetric connection function satisfies

$$\varphi((x, p), (y, q)) = \varphi((0, p), (y - x, q))$$

for all $x, y \in \mathbb{R}^d$ and all $p, q \in \mathbb{M}$.

Remark

The stationary marked RCM ξ is stationary and ergodic w.r.t. shifts of the spatial coordinate.

Assumption

The expected degree of the typical vertex is denoted by

$$d_\varphi := \iint \varphi(x,p,q)\,dx\,\mathbb{Q}^2(d(p,q)),$$

where $\varphi(x,p,q) := \varphi((0,p),(x,q))$. We assume that

$$d_\varphi < \infty.$$

Definition

Define

$$\theta(t) := \int \mathbb{P}_t\big(|C^{(0,p)}| = \infty\big)\, \mathbb{Q}(dp), \quad t \geq 0,$$

as the probability that the cluster of a typical vertex has infinite size.

Theorem

Let $t > 0$. Then $\theta(t) > 0$ iff ξ percolates $\mathbb{P}_t$-almost surely.

Definition

The critical intensity is defined by

$$t_c := \inf\{t \geq 0 : \theta(t) > 0\}.$$

Remark

In the unmarked case it was shown in the seminal paper Penrose '91 that $0 < t_c < \infty$ (under the minimal assumption $0 < \int \varphi(x)\, dx < \infty$).

Example

Suppose that $\mathbb{M}$ equals the space $\mathcal{C}^d$ of all non-empty compact subsets of $\mathbb{R}^d$, equipped with the Hausdorff metric. Assume that the connection function is given by

$$\varphi((x,K),(y,L)) = \mathbf{1}\{(K+x)\cap(L+y)\neq\emptyset\}.$$

The random closed set

$$Z := \bigcup_{(x,K)\in\eta} K+x.$$

is known as the Boolean model. Percolation in ξ is equivalent to percolation in Z.

Example

Assume that $\mathbb{M} = (0,1)$ equipped with Lebesgue measure $\mathbb{Q}$. Assume also that

$$\varphi((x,p),(y,q)) = \rho(g(p,q)\|x-y\|^d),$$

for a decreasing function $\rho\colon [0,\infty) \to [0,1]$ and a function $g\colon (0,1)\times(0,1) \to [0,\infty)$ which is increasing in both arguments. We assume that $m_\rho := \int \rho(\|x\|^d)\,dx$ is positive and finite. This model is sometimes called weight-dependent random connection model.

4. Deletion stability

Setting

ξ is a (Poisson driven) stationary marked RCM.

Definition

For $(x, p) \in \mathbb{R}^d \times \mathbb{M}$ let $N^\infty(x, p)$ denote the number of infinite clusters in $C^{(x,p)} - \delta_{(x,p)}$.

Theorem (Chebunin and L. '24)

The infinite clusters of a stationary marked RCM are deletion stable, that is

$$\mathbb{P}(N^\infty(x, p) \geq 2) = 0, \quad \lambda\text{-a.e. } (x, p).$$

5. Proof of deletion stability

Definition

The cluster density is defined by

$$\kappa(t) := \int \mathbb{E}_t\big[|C^{(0,p)}|^{-1}\big]\,\mathbb{Q}(dp), \quad t \in \mathbb{R}_+.$$

Remark

$t\kappa(t)$ is the intensity of finite clusters.

Definition

Let $(x,p) \in \eta$.

- $N^0(x,p)$ denotes the number of clusters in $C^{(x,p)} - \delta_{(x,p)}$.
- $N^+(x,p)$ is defined similarly, except that at most one infinite cluster is counted, i.e.

$$N^+(x,p) := N^0(x,p) - \mathbf{1}\{N^\infty(x,p) \geq 1\}(N^\infty(x,p) - 1).$$

Goal

Show that

$$N^+(x,p) = N^0(x,p), \quad \lambda\text{-a.e. } (x,p) \in \mathbb{R}^d \times \mathbb{M}.$$

- Use a Margulis–Russo type formula and analytic tools to prove that $\kappa(t)$ is differentiable!
- Let $(B_n)_{n\in\mathbb{N}}$ be an increasing sequence of convex and compact sets with union $\mathbb{R}^d$.
- Define finite volume counterparts $N_n^0(x,p)$ and $N_n^+(x,p)$ of $N^0(x,p)$ resp. $N^+(x,p)$ based on empty resp. wired boundary conditions.
- Use Margulis–Russo to show for $\star \in \{0,+\}$ that

$$\frac{d}{dt}\mathbb{E}_t M_{n,\star} = \lambda_d(B_n) - \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^\star(x,p)\, dx\, \mathbb{Q}(dp),$$

where $M_{n,*}$ is the number of clusters in B_n with boundary condition $*$.

- We have

$$\lim_{n\to\infty}(\lambda_d(B_n))^{-1}\mathbb{E}_t M_{n,\star} = t\kappa(t)$$

- For each $n \in \mathbb{N}$ the function $t \mapsto \mathbb{E}_t M_{n,\star} + \lambda_d(B_n) d_\varphi t^2/2$ is convex.
- Conclude that $t \mapsto t\kappa(t) + d_\varphi t^2/2$ is convex. Hence its derivative is the limit of the derivatives of

$$\lambda_d(B_n)^{-1}\mathbb{E}_t M_{n,\star} + d_\varphi t^2/2.$$

Definition

Define

$$d_\varphi^{(1)}(p,q) \equiv d_\varphi(p,q) := \int \varphi(x,p,q)\,dx, \quad p,q \in \mathbb{M},$$

and inductively

$$d_\varphi^{(n+1)}(p,q) = \int d_\varphi^{(n)}(p,r) d_\varphi^{(1)}(r,q)\,\mathbb{Q}(dr),\ n \in \mathbb{N}.$$

Then $\int d_\varphi^{(n)}(p,q)\mathbf{1}\{q \in A\}\,\mathbb{Q}(dq)$ is the expected number of paths of length n from $(0,p)$ to some location with mark in a measurable set $A \subset \mathbb{M}$.

Theorem

We have that

$$\sup_{n\geq 1} d_\varphi^{(n)}(p,q) > 0, \quad \mathbb{Q}^2\text{-a.e. } (p,q) \in \mathbb{M}^2$$

iff ξ is irreducible, that is

$$\mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1,x_2}) > 0, \quad \lambda^2\text{-a.e. } (x_1,x_2) \in \mathbb{X}^2.$$

7. Uniqueness of the infinite cluster

Theorem (Chebunin and L. '24)

An irreducible stationary marked RCM has $\mathbb{P}$-a.s. at most one infinite cluster.

Idea of the proof

- We need to prove that any two points of η cannot belong to two different infinite clusters.
- By the bivariate Mecke equation we need to prove that for λ^2-a.e. (x_1, x_2) it is impossible that the clusters $C^{x_1}(\xi^{x_1,x_2})$ and $C^{x_2}(\xi^{x_1,x_2})$ are infinite and not connected.
- By irreducibility there is a positive probability that x_1 and x_2 are connected in ξ^{x_1,x_2}.
- By the multivariate Mecke equation we can find a measurable set $B \subset B$ with $0 < \lambda(B) < \infty$, an $n \in \mathbb{N}$ and independent r.v.'s $Y_1, \ldots, Y_n$ with distribution $\lambda(\cdot \cap B)/\lambda(B)^{-1}$ such that x_1 and x_2 are connected in the RCM based on $x_1, x_2, Y_1, \ldots, Y_n$ with positive probability.
- It can be shown that the augmented RCM $\xi^{x_1,x_2,Y_1,\ldots,Y_n}$ is still deletion stable.

Theorem (Chebunin and L. ’24)

Assume that ξ has almost surely at most one infinite cluster. Then the infinite cluster of ξ is deletion stable.

8. Analytic properties of the cluster density

Theorem (Chebunin and L. ’24)

The function $t \mapsto t\kappa(t) + d_\varphi t^2/2$ is continuously differentiable on $(0, \infty)$, convex on $\mathbb{R}_+$ and right differentiable at zero. (Recall that d_φ is expected degree of a typical vertex.)

9. Sharp phase transition

Definition

Define the critical intensities

$$t_T := \inf\left\{t \ge 0 : \mathbb{E}_t \int |C^{(0,p)}|\, \mathbb{Q}(dp) = \infty\right\},$$
$$t_T^\infty := \inf\left\{t \ge 0 : \operatorname*{ess\,sup}_{p\in\mathbb{M}} \mathbb{E}_t |C^{(0,p)}| = \infty\right\}$$

and note that $t_T^\infty \le t_T \le t_c$.

Definition

Define

$$d_\varphi^* := \operatorname*{ess\,sup}_{p\in\mathbb{M}} \int d(p,q)\, \mathbb{Q}(dq) = \operatorname*{ess\,sup}_{p\in\mathbb{M}} \int \varphi(x,p,q)\, dx\, \mathbb{Q}(dq).$$

Theorem (Chebunin and L. '24)

Assume that $d^*_\varphi \in (0,\infty)$. *Then* $t_T^{(\infty)} \ge 1/d^*_\varphi$. *Moreover, for each* $t < t_T^{(\infty)}$ *there exists* $\delta = \delta(t) > 0$ *such that*

$$\operatorname*{ess\,sup}_{p\in\mathbb{M}} \mathbb{E}_t e^{\delta|C^{(0,p)}|} < \infty$$

Theorem (Chebunin and L. '24)

Assume that

$$0 < \operatorname*{ess\,sup}_{p\in\mathbb{M}} \int d(p,q)^2\,\mathbb{Q}(dq) < \infty.$$

Then $t_c = t_T^\infty$ *and*

$$\mathbb{E}_{t_c} \int |C^{(0,p)}|\,\mathbb{Q}(dp) = \infty.$$

Remarks

- In the unmarked case the equality $t_c = t_T$ was proved in Meester '95 (under additional assumptions on the connection function).
- The preceding theorems generalize some of the results in Ziesche '18, Dickson and Heydenreich '22, Caicedo and Dickson '23 and Küpper and Penrose '24.
- For the spherical Boolean model our assumption require the random radii to be deterministically bounded. This was significantly relaxed in Duminil-Copin, Raoufi and Tassion '20.

Example

Consider the unmarked stationary RCM ξ. It can be proved that ξ is irreducible. Hence there is at most one infinite cluster. This generalizes an earlier result (see Meester and Roy '96), where it is assumed that $\varphi(x) = \tilde{\varphi}(\|x\|)$, $x \in \mathbb{R}^d$, for a decreasing function $\tilde{\varphi}\colon [0, \infty) \to [0, 1]$. The proof there is based on the Burton–Keane '89 approach.

Example

Consider the Boolean model ξ. Assume that the grains contain almost surely an open neighborhood of the origin. Then ξ is irreducible and there exists at most one infinite cluster. For the spherical Boolean model this result can be found in Meester and Roy '96.

Example

The weighted RCM is irreducible and hence has at most one infinite cluster; see Gracar, Lüchtrath and Mörters '21.

11. References

- Aizenman, M., Kesten, H. and Newman, C.M. (1987). Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Commun. Math. Phys.* **111**, 505–531.
- Burton, R.M. and Keane, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121**, 501–505.
- Chebunin, M. and Last, G. (2024). On the uniqueness of the infinite cluster and the cluster density in the Poisson driven random connection model. arXiv:2403.17762
- Caicedo, A. and Dickson, M. (2023). Critical exponents for marked random connection models. arXiv:2305.07398.
- Dickson, M. and Heydenreich, M. (2022). The triangle condition for the marked random connection model. arXiv:2210.07727.

- Gracar, P., Lüchtrath, L. and Mörters, P. (2021). Percolation phase transition in weight-dependent random connection models. *Adv. Appl. Probab.* **53** (4), 1090–1114.
- Küpper, N. and M.D. Penrose. M.D. (2024). Largest component and sharpness in subcritical continuum percolation. arXiv:2407.10715
- Last, G., Nestmann, F. and Schulte, M. (2021). The random connection model and functions of edge-marked Poisson processes: second order properties and normal approximation. *Ann. Appl. Probab.* **31**, 128–168.
- Last, G. and Penrose, M.D. (2017). *Lectures on the Poisson Process.* Cambridge University Press.
- Meester, R., Penrose, M.D. and Sarkar, A. (1997). The random connection model in high dimensions. *Stat. Probab. Lett.* **35** 145-153.

- Meester, R. and Roy, R. (1996). *Continuum Percolation.* Cambridge University Press, Cambridge.
- Penrose, M.D. (1991). On a continuum percolation model. *Adv. Appl. Probab.* **23** (3), 536–556.
- Penrose, M. (2003). *Random Geometric Graphs*. Oxford University Press, Oxford.
- Penrose, M.D. (2018). Inhomogeneous random graphs, isolated vertices, and Poisson approximation. *J. Appl. Probab.* **55**, 112-136.

Thank You!