Boolean models in hyperbolic space

based on joint work with Günter Last (KIT) and Matthias Schulte (TUHH)

Daniel Hug | University of Bath, September 2024

Boolean Models in Euclidean Space: Applied Sciences

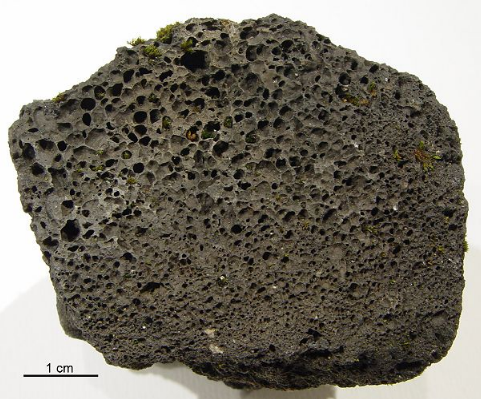


Figure: Lava, Puy de la Nugère, Volvic, France, https://www.kristallin.de/s2/f_diaman.htm, Matthias Bräunlich, Hamburg, CC-BY-SA 3.0,

Figure: Pores in aerated concrete created by the reaction of aluminum with the binding agents lime and cement used in production. https://www.bauen.de/a/porenbeton-fuer-den-hausbau.html

Figure: Oban, Scotland, 24/08/12

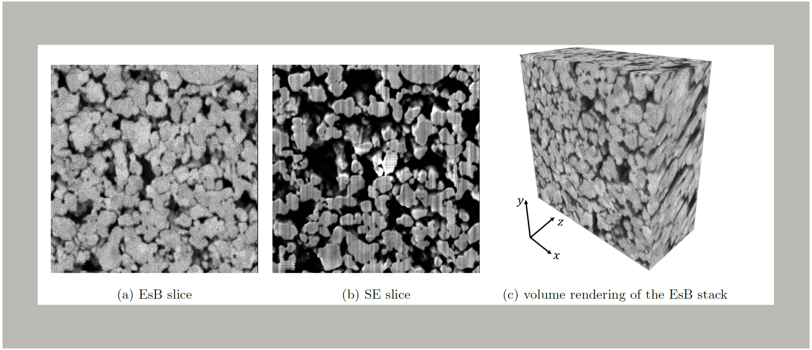


Figure: Porous structure of a zirconium dioxide filtration membrane, imaged by means of a Field Emission Scanning Electron Microscope (FE-SEM) & tomography package, Roldan, Redenbach, Schladitz, Klingele, Godehardt 2021

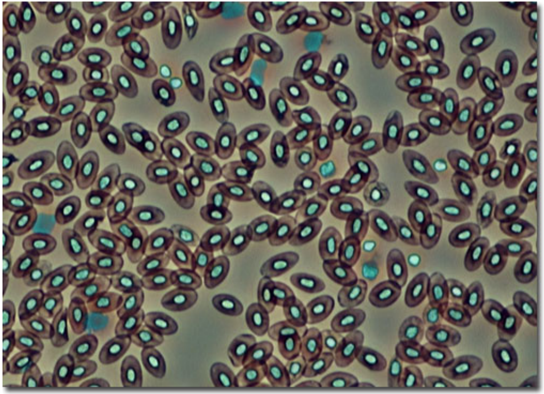
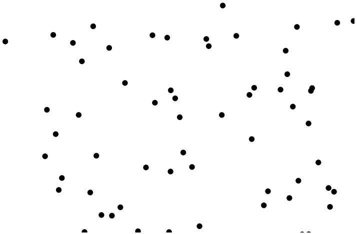


Figure: Blood cells of a frog, https://www.microscopyu.com/gallery-images/frog-blood-cells

Boolean Models in Euclidean Space: Construction



Poisson point process

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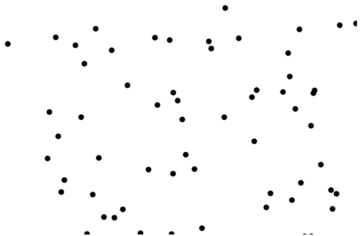 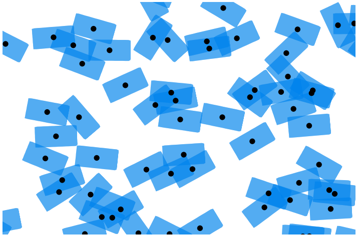

Poisson point process ⟶ Poisson particle process (indep. marking)

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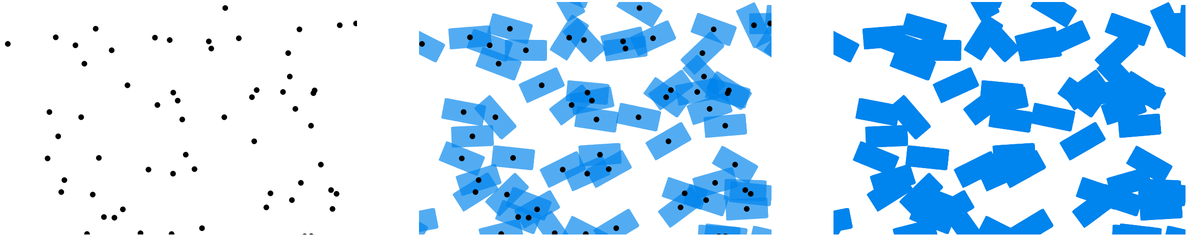

Poisson point process $\longrightarrow$ Poisson particle process (indep. marking) $\longrightarrow$ Boolean model

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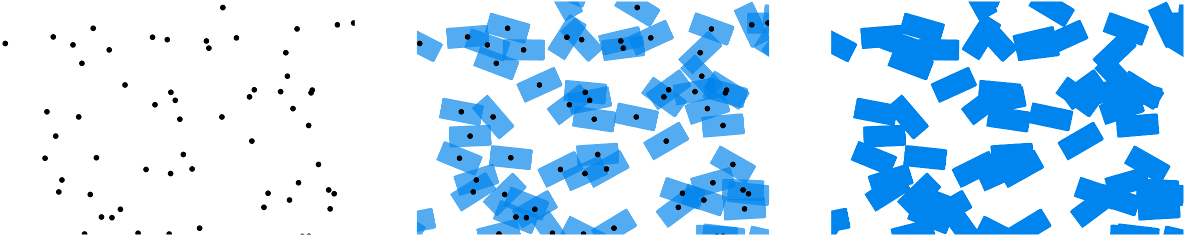

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Philosophy:

- Gain information about the **intensity** of the underlying Poisson process and the **shape distribution of the particles** via measurements on the Boolean model (union set) within an observation window W.

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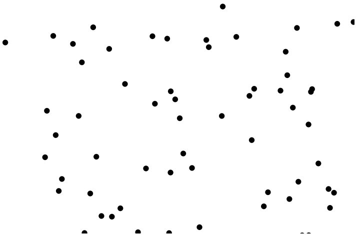

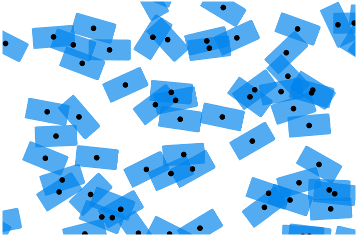

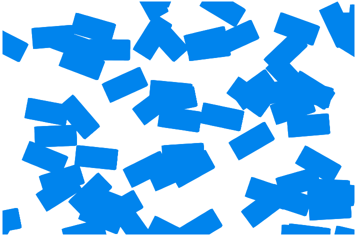

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- Gain information about the **intensity** of the underlying Poisson process and the **shape distribution of the particles** via measurements on the Boolean model (union set) within an observation window W.
- Approach: Derive **relations** between **local or asymptotic mean values** for various functionals of the typical particle and the Boolean model (state of the art up to, say, 2013).

Summary and outlook

- BMs in *d*-**dim. Euclidean space**: starting with **mean value relations** (Miles & Davy 1978), recently **variances** and **limit theorems** could be treated (Heinrich, Molchanov '99, Heinrich, Spiess '13, H., Last, Schulte 2016, H., Klatt, Last, Schulte 2017, Schulte, Yukich 2019, Betken, Schulte, Thäle 2022)).
- BMs arise by attaching to the points of a Poisson point process independently i.i.d. random shapes and taking the union set. Equivalently: union sets of Poisson particle processes.
- BM is a benchmark model in stochastic geometry and its applications.
- BMs in a much more general framework: spaces of constant curvature.
- Here we focus on hyperbolic space. Why?
  - Unexpected phenomena have been discovered in exploring Poisson processes of λ-geodesics / flats in hyperbolic space (Betken, Bühler, Herold, H., Rosen, Kabluchko, Thäle).
  - New effects and challenges occur, general trend to study random geometry in other spaces than Euclidean.
  - Hyperbolic structures are central in geometry, graph theory, ... and its fun!

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Classical Literature (Books)

- Matheron (Random Sets and Integral Geometry) 1975, Fontainebleau school (Geology)
- Serra (Image Analysis and Mathematical Morphology) 1982
- Hall (Coverage Processes) 1988
- Cressie (Spatial Statistics) 1992
- Molchanov (Statistics of random sets, the Boolean Model) 1995
- Meesters & Roy (Continuum Percolation) 1996
- (Chiu), Stoyan, Kendall, Mecke (Stochastic Geometry and Applications) 1987 (2013)
- Schneider & Weil (Integral and Stochastic Geometry) 2008

Basic hyperbolic geometry

- $\mathbb{H}^d$ d-dimensional hyperbolic space (simply connected RM with constant sectional curvature -1)

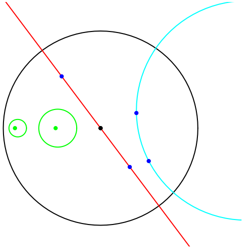

Figure: Circles and geodesics in the Poincaré disk model

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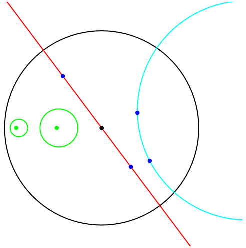

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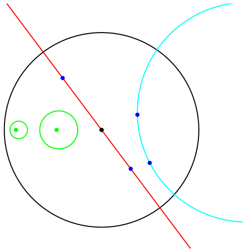

Figure: Circles and geodesics in the Poincaré disk model

- $\mathcal{K}^d$ compact convex subsets of $\mathbb{H}^d$
- $\mathcal{I}_d$ set of isometries, λ Haar measure on $\mathcal{I}_d$, $x \in \mathcal{I}_d$ fixed,

$$\mathcal{H}^d(\cdot) = \int_{\mathcal{I}_d} \mathbf{1}\{\varrho x \in \cdot\}\, \lambda(\mathrm{d}\varrho).$$

Balls in hyperbolic spaces

- p fixed point in $\mathbb{H}^d$
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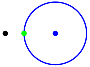
- Similar for $V_j(\mathbb{B}_R)/V_d(\mathbb{B}_R) \to d-1$ as $R \to \infty$ (with $j \in \{0, \dots, d-2\}$).

Horoballs



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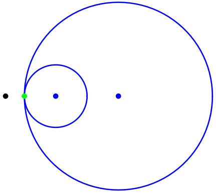

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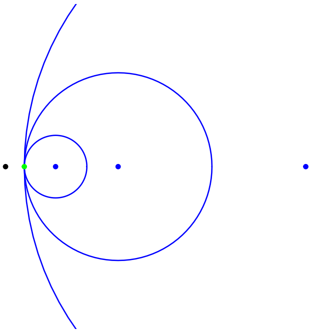

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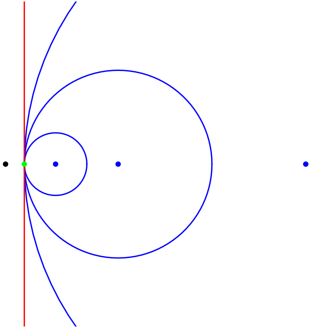

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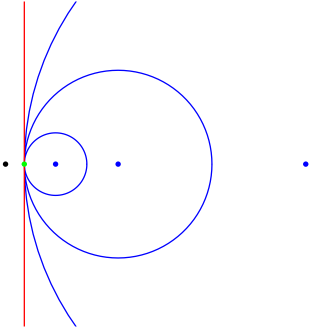

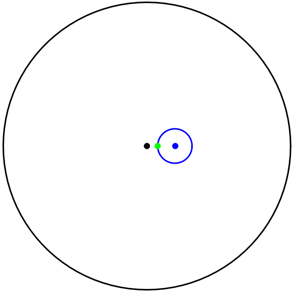

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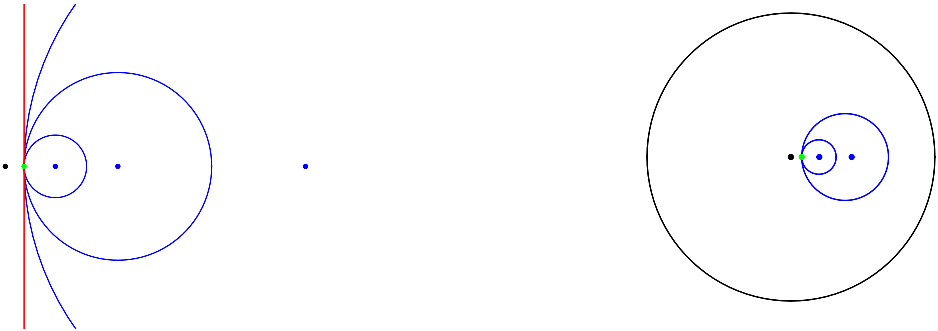

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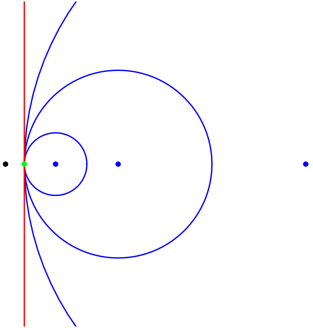

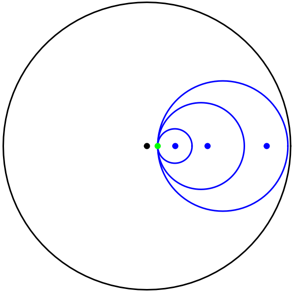

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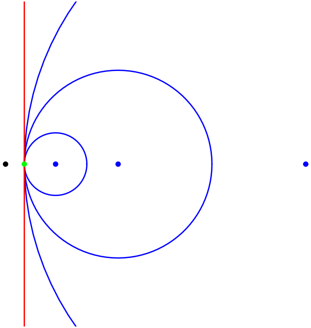

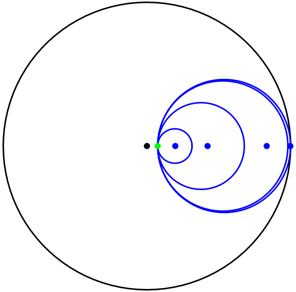

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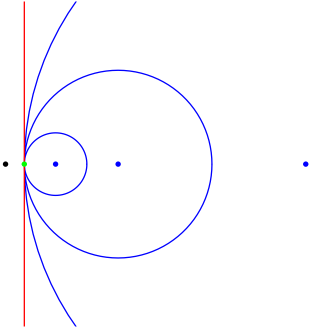

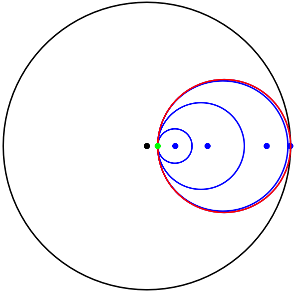

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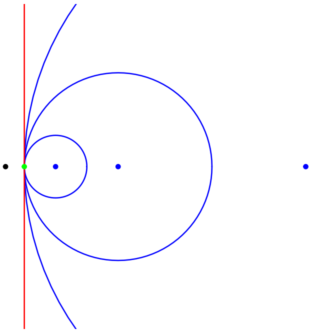

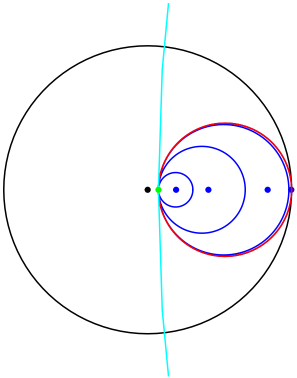

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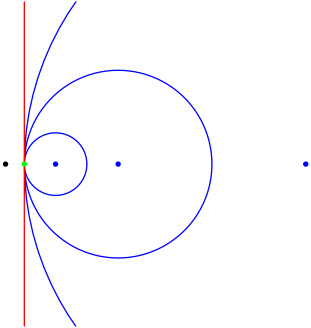
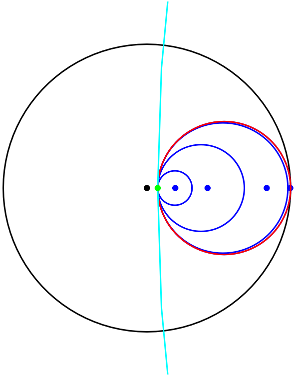
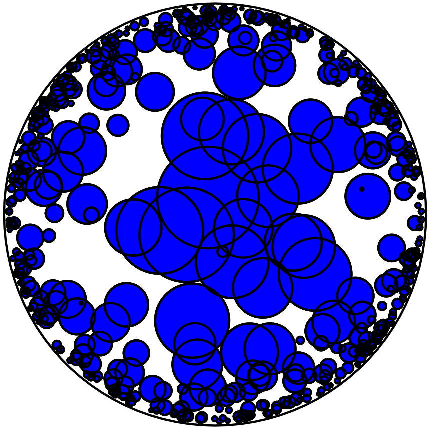




For $u \in \mathbb{S}_{\mathrm{p}}^{d-1}$ and $t \in \mathbb{R}$, let $\mathbb{B}_{u,t} = \lim_{R\to\infty} \mathbb{B}(\exp_{\mathrm{p}}((t+R)u), R)$. The limit $\mathbb{B}_{u,t}$ is a **horoball** (limit/border ball).

Boolean model in hyperbolic space



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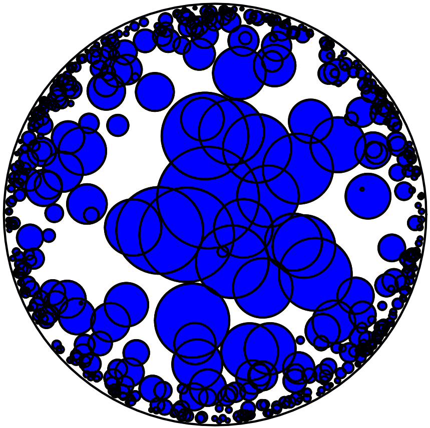

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$$\varrho_\sharp \Lambda = \Lambda \quad \text{for } \varrho \in \mathcal{I}_d.$$

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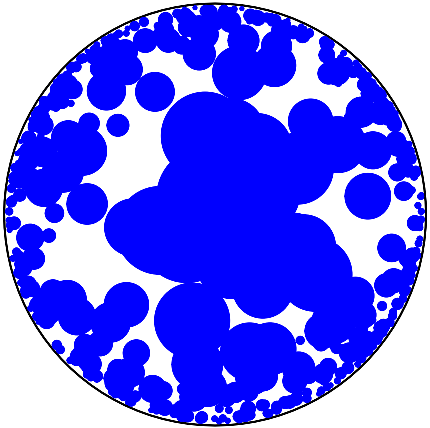

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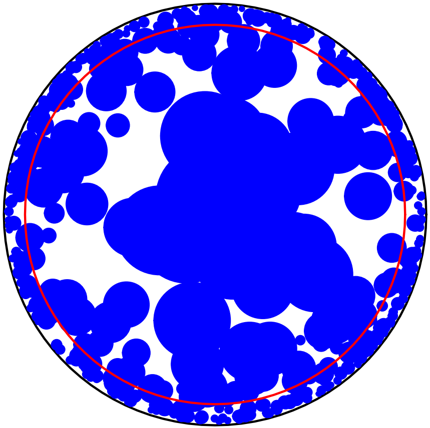

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- Consider $Z \cap W$ and $Z \cap \mathbb{B}_R$ as $R \to \infty$

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Suppose that Λ is locally finite, i.e.,

$$\Lambda(\{K \in \mathcal{K}^d : K \cap C \neq \varnothing\}) < \infty \quad \text{for all compact} \quad C \subset \mathbb{H}^d.$$

Let $c_h : \mathcal{K}^d \to \mathbb{H}^d$ be an isometry covariant **centre function** and $\gamma := \int \mathbf{1}\{c_h(K) \in \mathbb{B}\}\,\Lambda(\mathrm{d}K) < \infty$.

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Theorem (H., Last, Schulte '24+)

If $f : \mathcal{K}^d \to [0, \infty)$, then

$$\int_{\mathcal{K}^d} f(K)\,\Lambda(\mathrm{d}K) = \gamma \int_{\mathcal{K}^d} \int_{\mathcal{I}_d} f(\varrho G)\,\lambda(\mathrm{d}\varrho)\,\mathbf{Q}(\mathrm{d}G)$$

with $\gamma \in [0, \infty)$ and a probability measure $\mathbf{Q}$ invariant under all $\varrho \in \mathcal{I}_d$ with $\varrho(\mathrm{p}) = \mathrm{p}$ and concentrated on $\mathcal{K}^d_{\mathrm{p}} := \{K \in \mathcal{K}^d : c_h(K) = p\}$.

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Compare with the Euclidean situation.

A basic functional: volume

Define $v_d = \mathbf{E}\,\mathrm{Vol}(\mathrm{G})$ and the **mean covariogram function** of the typical grain

$$C(x,z) = \mathbf{E}\lambda(\{\varrho \in \mathcal{I}_d : \varrho x, \varrho z \in \mathrm{G}\}) = \mathbf{E}C_{\mathrm{G}}(x,z) \qquad , \qquad x,z \in \mathbb{H}^d.$$

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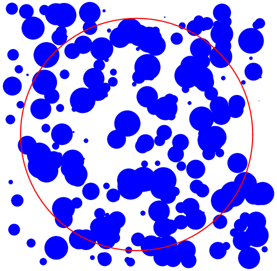
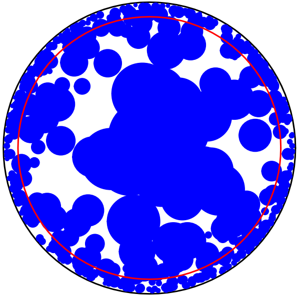
If $\mathbb{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$, then with independent $U \sim \mathrm{Uniform}(\mathbb{S}^{d-1}_{\mathrm{p}})$ and $-T \sim \mathrm{Exp}(d-1)$.

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,\mathrm{Vol}(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} = e^{-2\gamma v_d}\int_{\mathbb{H}^d} \big(e^{\gamma C(\mathrm{p},z)} - 1\big)\mathbf{P}(z \in \mathbb{B}_{U,T})\,\mathcal{H}^d(\mathrm{d}z).$$

Comparison with Euclidean case





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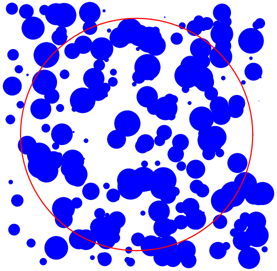

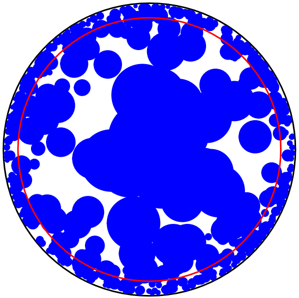

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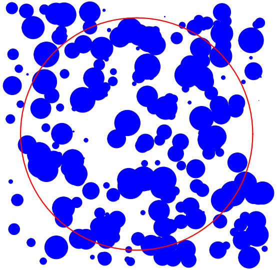

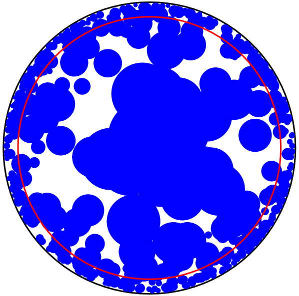

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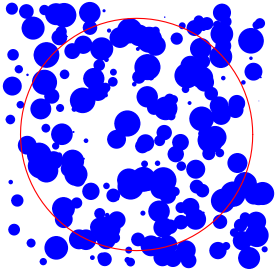

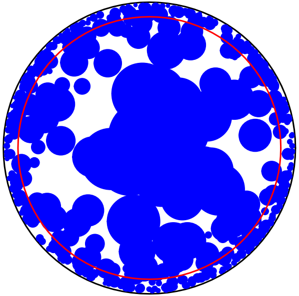

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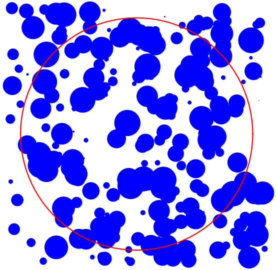

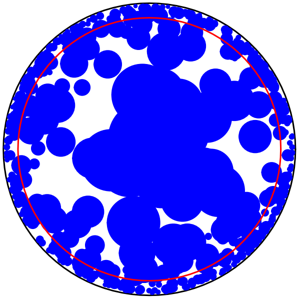

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Asymptotic expectations

On the set of horoballs $\mathbb{B}_h^d$, we define the (σ-finite) measure

$$\mu_{\mathrm{hb}}(\cdot) = \frac{d-1}{\omega_d} \int_{\mathbb{S}_{\mathrm{p}}^{d-1}} \int_{\mathbb{R}} \mathbf{1}\{\mathbb{B}_{u,t} \in \cdot\} e^{(d-1)t} \, \mathrm{d}t \, \mathcal{H}_{\mathrm{p}}^{d-1}(\mathrm{d}u).$$

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Compare to Euclidean counterpart! (not in the literature, no continuity required) $m_{\phi,Z_{\mathrm{Euc}}} = \mathbf{E}\phi(Z_{\mathrm{Euc}} \cap [0,1)^d)$

Example of an explicit mean value formula

Corollary (H., Last, Schulte '24+)

If $d = 2$ and $W \in \mathcal{K}^2$, then $\chi = \frac{1}{2\pi}(V_0 - V_2)$

$$\mathbf{E}\chi(Z \cap W) = (1 - e^{-\gamma v_2}) + V_1(W)e^{-\gamma v_2}\frac{\gamma v_1}{2\pi} + V_2(W)e^{-\gamma v_2}\left(\gamma + \frac{\gamma v_2}{2\pi} - \frac{(\gamma v_1)^2}{4\pi}\right), \tag{1}$$

where $v_i := \mathbf{E} V_i(\mathrm{G})$. The asymptotic density of the Euler characteristic is given by

$$m_{\chi,Z} = \gamma e^{-\gamma v_2} - \gamma^2 e^{-\gamma v_2}\frac{v_1^2}{4\pi} + \gamma e^{-\gamma v_2}\frac{1}{2\pi}(v_1 + v_2).$$

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$$\mathbf{E}\chi(Z \cap W) = (1 - e^{-\gamma v_2}) + V_1(W)e^{-\gamma v_2}\frac{\gamma v_1}{2\pi} + V_2(W)e^{-\gamma v_2}\left(\gamma + \frac{\gamma v_2}{2\pi} - \frac{(\gamma v_1)^2}{4\pi}\right), \qquad (1)$$

where $v_i := \mathbf{E}V_i(\mathrm{G})$. The asymptotic density of the Euler characteristic is given by

$$m_{\chi,Z} = \gamma e^{-\gamma v_2} - \gamma^2 e^{-\gamma v_2}\frac{v_1^2}{4\pi} + \gamma e^{-\gamma v_2}\frac{1}{2\pi}(v_1 + v_2).$$

In the Euclidean case, $\chi = V_0$ and (classical formula by Miles '76)

$$m_{\chi,Z_{\mathrm{Euc}}} = \gamma e^{-\gamma v_2} - \gamma^2 e^{-\gamma v_2}\frac{\mathbf{E}\mathcal{H}^1(\partial\,\mathrm{G})^2}{4\pi}.$$

Asymptotic variances

Theorem (H., Last, Schulte '24+)

Assume that $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$. If $\phi : \mathcal{R}^d \to \mathbb{R}$ is a geometric functional that is continuous on $\mathcal{K}^d$, then

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,\phi(Z\cap\mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{B}_h^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \phi^*(G\cap K_2\cap\ldots\cap K_n\cap B)^2$$
$$\times\, \Lambda^{n-1}(\mathrm{d}\,(K_2,\ldots,K_n))\,\mathbf{Q}(\mathrm{d}G)\,\mu_{\mathrm{hb}}(\mathrm{d}B)$$

with $\phi^*(\cdot) = \mathbf{E}\phi(Z\cap\cdot) - \phi(\cdot)$.

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Local formulas, limsup is finite (for mb, add., loc. bounded functionals); covariances, . . .

Local covariances (example)

For local covariances of mb., add., loc. bounded functionals in W, we have series expansions. For special functionals such as V_{d-1}, V_d, these can be simplified. Assume $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$. Define measures

$$M_{i,j} := \mathbf{E} \int_{\mathbb{H}^d} \int_{\mathbb{H}^d} \mathbf{1}\{(x,y) \in \cdot\}\, C_i(\mathrm{G}, \mathrm{d}x)\ C_j(\mathrm{G}, \mathrm{d}y) \quad \text{for } i,j \in \{d-1, d\} \text{ on } (\mathbb{H}^d)^2.$$

Theorem (H., Last, Schulte '24+)

Assume that $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$. Let $v_i = \mathbf{E}\, V_i(\mathrm{G})$ for $i \in \{0, \dots, d\}$. If $W \in \mathcal{K}^d$, then

$$\begin{aligned}\mathrm{Cov}(V_{d-1}(Z \cap W), V_d(Z \cap W)) &= -e^{-2\gamma v_d}\gamma v_{d-1} \int \left(e^{\gamma C(\mathrm{p},z)} - 1\right) C_W(\mathrm{p}, z)\, \mathcal{H}^d(dz) \\ &\quad + e^{-2\gamma v_d}\gamma \int e^{\gamma C(y,z)} C_W(y,z)\, M_{d-1,d}(\mathrm{d}(y,z)) \\ &\quad + e^{-2\gamma v_d} \iint (e^{\gamma C(y,z)} - 1)\mathbf{1}\{z \in W\}\, C_{d-1}(W, \mathrm{d}y)\, \mathcal{H}^d(\mathrm{d}z).\end{aligned}$$

Asymptotic covariances (example)

For asymptotic covariances of continuous geometric functionals we have series expansions. For special functionals, these can be simplified. Assume that $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$. Recall that

$$M_{i,j} := \mathbf{E} \int_{\mathbb{H}^d} \int_{\mathbb{H}^d} \mathbf{1}\{(x,y) \in \cdot\}\, C_i(\mathrm{G}, \mathrm{d}x)\ C_j(\mathrm{G}, \mathrm{d}y) \quad \text{for } i,j \in \{d-1, d\}.$$

Theorem (H., Last, Schulte '24+)

Assume that $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$. If $u \in \mathbb{S}_{\mathrm{p}}^{d-1}$ is (arbitrarily) fixed, then

$$\begin{aligned}
\sigma_{d-1,d} = &-e^{-2\gamma v_d}\gamma v_{d-1} \iint \left(e^{\gamma C(\mathrm{p},z)} - 1\right)\mathbf{1}\{\mathrm{p}, z \in B\}\, \mathcal{H}^d(\mathrm{d}z)\, \mu_{\mathrm{hb}}(\mathrm{d}B) \\
&+ e^{-2\gamma v_d}\gamma \iint e^{\gamma C(y,z)}\mathbf{1}\{y, z \in B\}\, M_{d-1,d}(\mathrm{d}(y,z))\, \mu_{\mathrm{hb}}(\mathrm{d}B) \\
&+ (d-1)\, e^{-2\gamma v_d} \int \left(e^{\gamma C(\mathrm{p},z)} - 1\right)\mathbf{1}\{z \in \mathbb{B}_{u,0}\}\, \mathcal{H}^d(\mathrm{d}z).
\end{aligned}$$

Central limit theorems

Theorem (H., Last, Schulte '24+)

Let N be a standard Gaussian random variable. Let $\phi : \mathcal{R}^d \to \mathbb{R}$ be a geometric functional such that

$$\liminf_{R\to\infty} \frac{\mathrm{Var}\,\phi(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} > 0.$$

a) If $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$, then

$$S_R := \frac{\phi(Z \cap \mathbb{B}_R) - \mathbb{E}\phi(Z \cap \mathbb{B}_R)}{\sqrt{\mathrm{Var}(\phi(Z \cap \mathbb{B}_R))}} \xrightarrow{d} N \quad \text{as} \quad R \to \infty.$$

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b) If $\mathbf{E} \operatorname{Vol}(\mathrm{G}^{(1)})^4 < \infty$, there exists a constant $C \in (0, \infty)$ such that

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}(S_R \le t) - \mathbf{P}(N \le t) \right| \le \frac{C}{\sqrt{\operatorname{Vol}(\mathbb{B}_R)}}, \quad R \ge 1.$$

Lower bounds for variances

Theorem (H., Last, Schulte '24+)

Assume that $\mathbf{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$ and let $\phi : \mathcal{R}^d \to \mathbb{R}$ be a geometric functional. If there exists some $m \in \mathbb{N}_0$ such that

$$\int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^m} \mathbf{1}\{\phi(G \cap K_1 \cap \ldots \cap K_m) \neq 0\}\, \Lambda^m(\mathrm{d}\,(K_1, \ldots, K_m))\, \mathbf{Q}(\mathrm{d}G) > 0,$$

then

$$\liminf_{R\to\infty} \frac{\mathrm{Var}\,\phi(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} > 0.$$

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then

$$\liminf_{R \to \infty} \frac{\mathrm{Var}\,\phi(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} > 0.$$

For $m = 0$, the hypothesis means that $\mathbf{P}(\phi(G) \neq 0) > 0$.

Proof strategies

- Approaches for Euclidean Boolean models: H., Last & Schulte (2016), Schulte & Yukich (2019) and Betken, Schulte & Thäle (2022) (Euclidean Poisson cylinder processes)
- Careful analysis of boundary effects for asymptotic formulas for expectation and variance
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- Lower variance bound by Schulte & Trapp (2024+): If a functional F of η with $\mathbf{E}F^2 < \infty$ satisfies

$$\mathbf{E}\int_{(\mathcal{K}^d)^2} (D^2_{K_1,K_2}F)^2\,\Lambda^2(\mathrm{d}\,(K_1,K_2)) \leq \alpha\,\mathbf{E}\int_{\mathcal{K}^d}(D_K F)^2\,\Lambda(\mathrm{d}K) < \infty, \quad \text{for some } \alpha \in (0,\infty),$$

then

$$\operatorname{Var} F \geq \frac{4}{(\alpha+2)^2}\mathbf{E}\int_{\mathcal{K}^d}(D_K F)^2\,\Lambda(\mathrm{d}K).$$

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- Various results from integral geometry:

Results from integral geometry I

Lemma (H., Last, Schulte '24+)

Let $\varphi : \mathcal{K}^d \cup \mathbb{B}_h^d \to \mathbb{R}$ be a bounded, measurable functional. Assume that

$$\lim_{R\to\infty} \varphi(\mathbb{B}(\exp_{\mathrm{p}}((t+R)u), R)) = \varphi(\mathbb{B}_{u,t}) \tag{2}$$

for $\mathcal{H}_{\mathrm{p}}^{d-1} \otimes \mathcal{H}^1$ almost all $(u, t) \in \mathbb{S}_{\mathrm{p}}^{d-1} \times \mathbb{R}$ and that there exists some $r_0 \in (0, \infty)$ such that

$$\varphi(K) = 0 \quad \text{for all } K \in \mathcal{K}^d \text{ with } K \cap \mathbb{B}_{r_0} = \varnothing. \tag{3}$$

Then

$$\lim_{R\to\infty} \frac{1}{V_d(\mathbb{B}_R)} \int_{\mathbb{H}^d} \varphi(\mathbb{B}(x, R))\, \mathcal{H}^d(\mathrm{d}x) = \int_{\mathbb{B}_h^d} \varphi(B)\, \mu_{\mathrm{hb}}(\mathrm{d}B).$$

Results from integral geometry II

Lemma (H., Last, Schulte '24+)

If $L \in \mathcal{K}^d$ and $\psi : \mathcal{K}^d \to \mathbb{R}$ is a continuous functional such that $\psi(\varnothing) = 0$, then the map $\varphi\colon \mathcal{K}^d \cup \mathbb{B}_h^d \to \mathbb{R}$, $K \mapsto \psi(L \cap K)$, satisfies the hypothesis of the preceding lemma and

$$\lim_{R\to\infty} \frac{1}{V_d(\mathbb{B}_R)} \int_{\mathbb{H}^d} \psi(L \cap \mathbb{B}(x,R))\, \mathcal{H}^d(\mathrm{d}x) = \int_{\mathbb{B}_h^d} \psi(L \cap B)\, \mu_{\mathrm{hb}}(\mathrm{d}B).$$

In particular, if $L \in \mathcal{K}^d$ and $y, z \in \mathbb{H}^d$, then

$$\mathrm{Vol}(L) = \int_{\mathbb{B}_h^d} \mathrm{Vol}(L \cap B)\, \mu_{\mathrm{hb}}(\mathrm{d}B)$$

and

$$\lim_{R\to\infty} \frac{\mathcal{H}^d\left(\mathbb{B}(y,R) \cap \mathbb{B}(z,R)\right)}{V_d(\mathbb{B}_R)} = \int_{\mathbb{B}_h^d} \mathbf{1}\{y, z \in B\}\, \mu_{\mathrm{hb}}(\mathrm{d}B).$$

Results from integral geometry III

Lemma (H., Last, Schulte '24+)

(a) Let $A \in \mathcal{K}^d$ and $u \in \mathbb{S}_{\mathrm{p}}^{d-1}$ be fixed.

$$\int_{\mathbb{B}_h^d} V_d(A \cap B) V_{d-1}(A \cap B)\, \mu_{\mathrm{hb}}(\mathrm{d}B)$$
$$= \int_{\mathbb{B}_h^d} V_d(A \cap B) C_{d-1}(A, B)\, \mu_{\mathrm{hb}}(\mathrm{d}B) + (d-1) \int_{\mathcal{I}_d} \mathbf{1}\{\mathrm{p} \in \varrho A\} V_d(\varrho A \cap \mathbb{B}_{u,0})\, \lambda(\mathrm{d}\varrho),$$

(b)

$$\int_{\mathbb{B}_h^d} V_{d-1}(A \cap B)^2\, \mu_{\mathrm{hb}}(\mathrm{d}B) = \int_{\mathbb{B}_h^d} C_{d-1}(A, B)^2\, \mu_{\mathrm{hb}}(\mathrm{d}B)$$
$$+ (d-1) \int_{\mathcal{I}_d} \mathbf{1}\{\mathrm{p} \in \varrho A\} \big(2 C_{d-1}(\varrho A, \mathbb{B}_{u,0}) + C_{d-1}(\mathbb{B}_{u,0}, \varrho A)\big)\, \lambda(\mathrm{d}\varrho).$$

Sketch of proof

$$\int_{\mathbb{B}_h^d} V_d(A\cap B)V_{d-1}(A\cap B)\,\mu_{\mathrm{hb}}(\mathrm{d}B) = \int_{\mathbb{B}_h^d} V_d(A\cap B)C_{d-1}(A,B)\,\mu_{\mathrm{hb}}(\mathrm{d}B)+(d-1)\int_{\mathcal{I}_d} \mathbf{1}\{\mathrm{p}\in\varrho A\}\,V_d(\varrho A\cap\mathbb{B}_{u,0})\,\lambda(\mathrm{d}\varrho).$$

From the lemma,

$$\int_{\mathbb{B}_h^d} V_d(A\cap B)V_{d-1}(A\cap B)\,\mu_{\mathrm{hb}}(\mathrm{d}B) = \lim_{R\to\infty}\frac{\int_{\mathbb{H}^d} V_d(A\cap\mathbb{B}(x,R))V_{d-1}(A\cap\mathbb{B}(x,R))\,\mathcal{H}^d(\mathrm{d}x)}{V_d(\mathbb{B}_R)}.$$

The integral in the numerator on the rhs is the sum of

$$I_1(R) := \int_{\mathbb{H}^d} V_d(A\cap\mathbb{B}(x,R))C_{d-1}(A,\mathbb{B}(x,R))\,\mathcal{H}^d(\mathrm{d}x),$$

$$I_2(R) := \int_{\mathbb{H}^d} V_d(A\cap\mathbb{B}(x,R))C_{d-1}(\mathbb{B}(x,R),A)\,\mathcal{H}^d(\mathrm{d}x),$$

The limit wrt I_1 can then be treated with the first integral geometric lemma.

Starting point for the proof of CLTs

Write

$$\sigma_R := \sqrt{\operatorname{Var} \phi(Z \cap \mathbb{B}_R)} \qquad \text{and} \qquad F_R := \frac{\phi(Z \cap \mathbb{B}_R) - \mathbf{E}\phi(Z \cap \mathbb{B}_R)}{\sigma_R}.$$

It follows from Last, Peccati, Schulte '16 that

$$\mathbf{d}_{\mathrm{Wass}}(F_R, N) \leq T_1 + T_2 + T_3$$

with

$$T_1 = 2\left(\int_{(\mathcal{K}^d)^3} \sqrt{\mathbf{E}(D_{K_1}F_R)^2(D_{K_2}F_R)^2}\sqrt{\mathbf{E}(D^2_{K_1,K_3}F_R)^2(D^2_{K_2,K_3}F_R)^2}\,\Lambda^3(\mathrm{d}(K_1,K_2,K_3))\right)^{1/2},$$

$$T_2 = \left(\int_{(\mathcal{K}^d)^3} \mathbf{E}(D^2_{K_1,K_3}F_R)^2(D^2_{K_2,K_3}F_R)^2\,\Lambda^3(\mathrm{d}(K_1,K_2,K_3))\right)^{1/2},$$

$$T_3 = \int_{\mathcal{K}^d} \mathbf{E}|D_K F_R|^3\,\Lambda(\mathrm{d}K).$$

Lemma (H., Last, Schulte ’24+)

Let $\phi : \mathcal{R}^d \to \mathbb{R}$ be measurable, additive, and locally bounded. Let $m \in \mathbb{N}$. Then there exists a constant $C_m \in (0, \infty)$, depending only on m, d, γ, and $\mathbb{Q}$, such that

(a)
$$\mathbf{E}|\phi(Z \cap W)|^m \leq c_{1,d}^m M(\phi)^m \mathbf{E} 2^{m\eta([\mathbb{B}_{1/2}])} \overline{V_d}(W)^m$$
for all $W \in \mathcal{K}^d$,

(b)
$$\mathbf{E}|D_K \phi(Z \cap W)|^m \leq C_m M(\phi)^m \overline{V_d}(K \cap W)^m$$
for all $K, W \in \mathcal{K}^d$,

(c)
$$\mathbf{E}|D^2_{K_1,K_2} \phi(Z \cap W)|^m \leq C_m M(\phi)^m \overline{V_d}(K_1 \cap K_2 \cap W)^m$$
for all $K_1, K_2, W \in \mathcal{K}^d$.

Continuation of the proof of CLTs

From the Cauchy–Schwarz inequality and the preceding Lemma (b), (c) it follows that

$$T_1^2 \leq \frac{4C_4 M(\phi)^4}{\sigma_R^4} \int_{(\mathcal{K}^d)^3} \overline{V_d}(K_1 \cap \mathbb{B}_R)\overline{V_d}(K_2 \cap \mathbb{B}_R)\overline{V_d}(K_1 \cap K_3 \cap \mathbb{B}_R) \times \overline{V_d}(K_2 \cap K_3 \cap \mathbb{B}_R)\,\Lambda^3(\mathrm{d}(K_1, K_2, K_3)). \tag{4}$$

The integral on the right-hand side can be rewritten as

$$J_1 := \gamma^3 \mathbf{E} \int_{\mathcal{I}_d^3} \overline{V_d}(\varrho_1\, \mathrm{G}_1 \cap \mathbb{B}_R)\overline{V_d}(\varrho_2\, \mathrm{G}_2 \cap \mathbb{B}_R)\overline{V_d}(\varrho_1\, \mathrm{G}_1 \cap \varrho_3\, \mathrm{G}_3 \cap \mathbb{B}_R)\overline{V_d}(\varrho_2\, \mathrm{G}_2 \cap \varrho_3\, \mathrm{G}_3 \cap \mathbb{B}_R) \times \lambda^3(\mathrm{d}(\varrho_1, \varrho_2, \varrho_3))$$

with independent copies G_1, G_2 and G_3 of the typical particle.

Economic covering of hyperbolic space

Almost disjoint decomposition of $\mathbb{R}^d$ by (half-open) cubes is a key tool in dealing with additive functionals in stochastic geometry.

The next lemma is used to bound additive, locally bounded functionals and their iterated difference operators in terms of the volume functional.

Lemma (H., Last, Schulte '24+)

There exist a countable set $\mathcal{M} \subset \mathbb{H}^d$ and a constant $c_d \in \mathbb{N}$ such that

$$\bigcup_{x \in \mathcal{M}} \mathbb{B}(x, 1/2) = \mathbb{H}^d$$

and

$$|\{x \in \mathcal{M} \colon y \in \mathbb{B}(x, 1/2)\}| \leq c_d \quad \text{for all } y \in \mathbb{H}^d.$$

Advertisement: Book on Poisson Hyperplane Tessellations