Long-range connections and the giant component of a random geometric graph

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Based on joint work with Mathew Penrose

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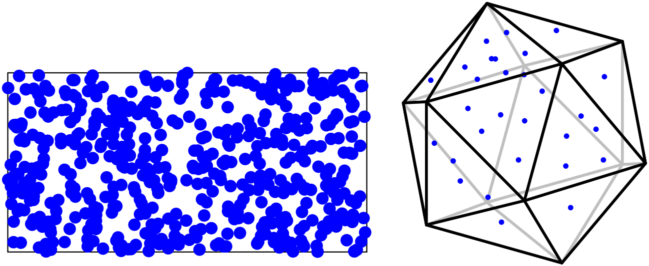
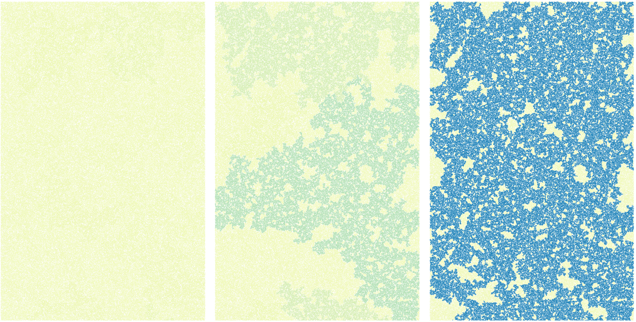
Stochastic Geometry *in Action*, Bath, 13th September 2024

Boolean model



In our setting: we use a homogeneous PPP of intensity n inside $A \subseteq \mathbb{R}^d$ and all the balls have the same radius r_n.

Continuum percolation



$A = \mathbb{R}^d$ or $A = \mathbb{H} := [0, \infty) \times \mathbb{R}^{d-1}$. Parameter $\lambda > 0$, homogeneous PPP $\mathcal{P}_\lambda$ on A and cluster $Z_\lambda := \bigcup_{x \in \mathcal{P}_\lambda} B(x, 1)$.
Known: non-trivial critical intensity $\lambda_c \in (0, \infty)$ (see Meester and Roy). Estimated $\lambda_c \approx 0.36$ in $\mathbb{R}^2$.
Tanemura 1993: same critical point for $\mathbb{R}^d$ and $\mathbb{H}$.
If $nr_n^d \to \lambda$ as $n \to \infty$, the Boolean model and continuum percolation are related.

An "application"

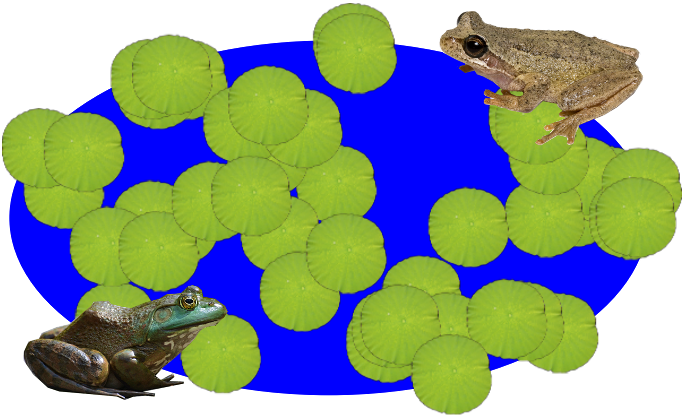


Figure: The frogs can't reach each other. Images by "Dana" and "DaPuglet" on Flikr and "Noodle Snacks" on Wikimedia

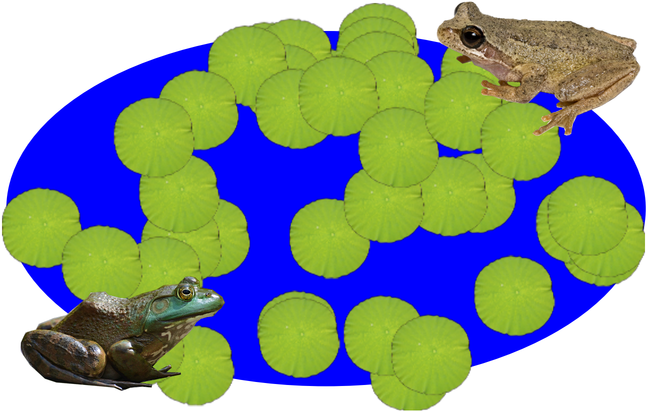
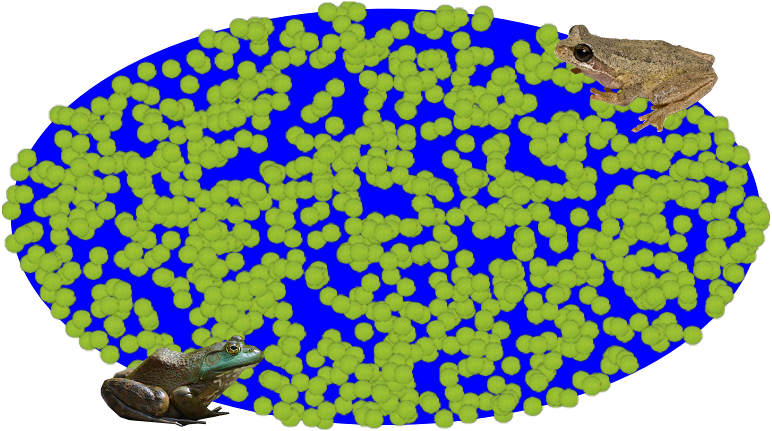
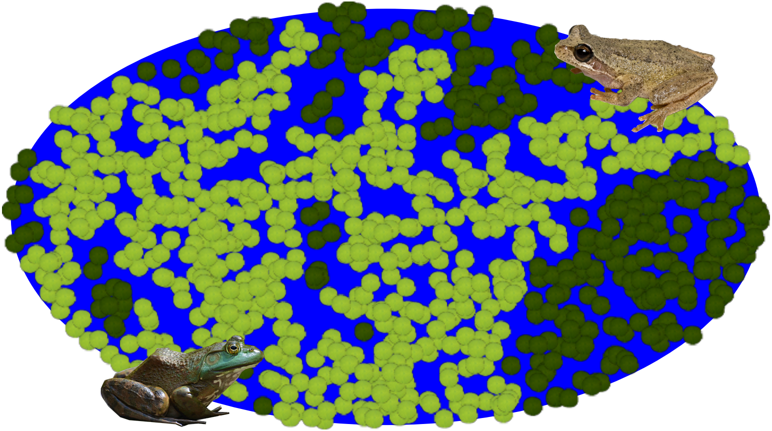


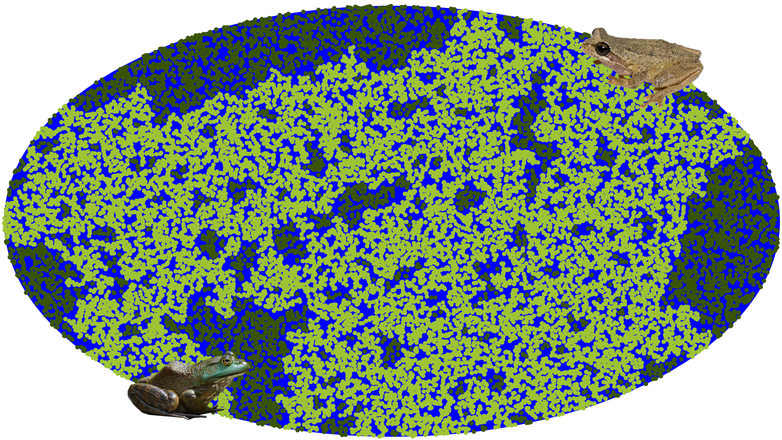
Figure: This time they *can* meet



How to understand the “communication” event?



As a *percolation* event, via a giant component



When the pond is huge or the lilypads are tiny, does the geometry of the boundary matter?

Let $\theta_A(\lambda) := \mathbb{P}[0 \text{ is in an unbounded component of } Z_\lambda]$.

Proposition ((A corollary of) Penrose, 2022)

Suppose $\lambda \neq \lambda_c(\mathbb{R}^2)$. Let $A = [-1/2, 1/2]^2$. Let V be uniformly distributed on $[-1/2, 1/2]^2$, then

$$\lim_{\substack{n\to\infty \\ nr_n^2=\lambda}} \mathbb{P}[0 \leftrightarrow V \textit{ via the Boolean model}] = \theta_{\mathbb{R}^d}(\lambda)^2.$$

(This was for the soft RGG / random connection model.)

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0 and V don't really know that they aren't in $\mathbb{R}^2$.

What about boundary effects?

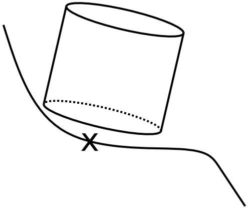
Theorem (H. and Penrose, 2024+)

Let $A \subseteq \mathbb{R}^d$ be bounded, $A = \overline{A^\circ}$, with a C^2 boundary. Suppose $\lambda \neq \lambda_c(\mathbb{R}^d)$. Fix distinct $x, y \in \partial A$, then

$$\lim_{\substack{n\to\infty \\ nr_n^d=\lambda}} \mathbb{P}[x \leftrightarrow y \textit{ via the Boolean model}] = \theta_{\mathbb{H}}(\lambda)^2.$$

Boundary effects

Our theorem says: once we get away from the boundary it's very easy to connect.
Several techniques to deal with the boundary: osculating spheres (good for local events) and "fitting a cylinder."



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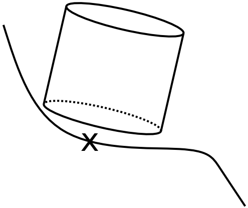

To connect with high probability: renormalisation.

Why $\lambda \neq \lambda_c$?

It is conjectured (known in $d = 2$ and $d \geq 11$ for $\mathbb{Z}^d$) that $\theta_{\mathbb{R}^d}(\lambda_c) = 0$.

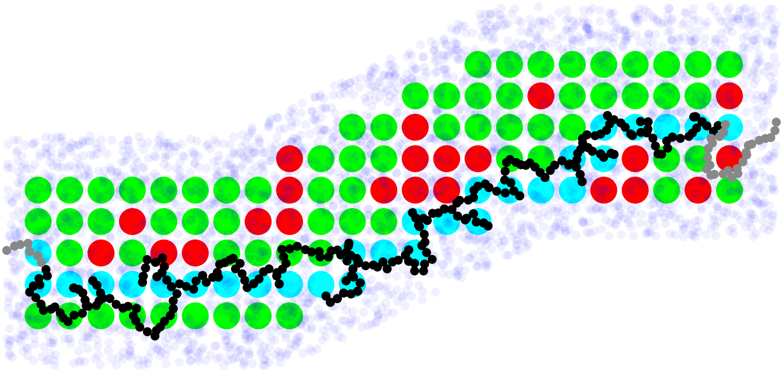

Figure: To connect C_x and C_y we build a grid of balls, and connect them via a path of "good" sites.

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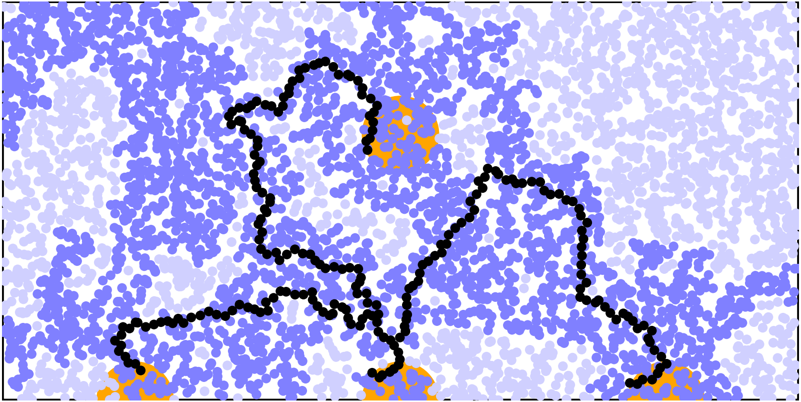

Figure: There must be a unique large component in the box, so the paths between balls are in the same component.

Why $\lambda \neq \lambda_c$?

We rely on a very useful result:

Theorem (Penrose and Pisztora, 1996)

Let $\lambda > \lambda_c(\mathbb{R}^d)$ and suppose $\phi : \mathbb{N} \to \mathbb{R}$ satisfies $\phi(n)/\log n \to \infty$ as $n \to \infty$ and $\phi(n) \leq n$ for all n. Then with high probability there exists a unique connected component of $Z_\lambda \cap [0, n]^d$ of diameter at least $\phi(n)$ if n is large.

Extended to the soft RGG for $d = 2$ by Lichev, Lodewijks, Mitsche, and Schapira (2023).

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If we could replace "$\lambda > \lambda_c$" with $\theta(\lambda) > 0$, then we'd be able to prove our result at $\lambda = \lambda_c$. But that would also solve the major open problem of determining $\theta(\lambda_c)$.

Thank you all for a great workshop!