Dynamics in geometric scale-free networks

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based on joint work with *Arne Grauer*

Stochastic Geometry in Action

Bath, September 10-13, 2024

Structure of talk

The weight-dependent random connection model

The contact process on the weight-dependent random connection model

The dynamic weight-dependent random connection model

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- **Vertices**: a Poisson point process $\mathcal{X}$ of unit intensity on $\mathbb{R}^d \times [0,1]$ or $\mathbb{T}_n^d \times [0,1]$.

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- **Edges**: Given $\mathcal{X}$, edges are drawn independently of each other and there exists $\alpha, \kappa_1, \kappa_2 > 0$ such that, for every pair of vertices $\mathbf{x} = (x,t), \mathbf{y} = (y,s) \in \mathcal{X}$, it holds

Assumption (A1)

$$\alpha\left(1 \wedge \kappa_1 (t \wedge s)^{-\delta\gamma}|x-y|^{-\delta d}\right) \leq \mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \sim \mathbf{y}) \leq \kappa_2 (t \wedge s)^{-\delta\gamma}(t \vee s)^{\delta(\gamma-1)}|x-y|^{-\delta d},$$

where $\gamma \in (0,1)$, $\delta > 1$, $\alpha, \kappa_1, \kappa_2 > 0$.

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- **The age-dependent random connection model**:
  Let $\varphi : (0,\infty) \to [0,1]$ be a non-decreasing and satisfying $\varphi(r) \asymp r^{-\delta}$. Form edges between (x,t) and (y,s) with probability

$$\varphi\big(\beta^{-1}(t \wedge s)^{\gamma}(t \vee s)^{1-\gamma}|x-y|^{d}\big).$$

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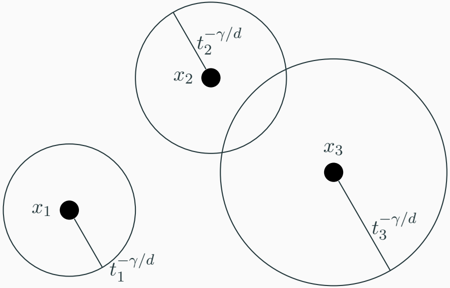
- **The soft Boolean model**:
  Let the radius distribution satisfy $\mathbb{P}(R_x > r) \asymp r^{-d/\gamma}$ as $r \to \infty$ and let $X(x,y)$ satisfy

$$\mathbb{P}(X > r) \asymp r^{-\delta d} \text{ as } r \to \infty.$$

  Connect to vertices when $\frac{|x-y|}{R_x+R_y} \leq X(x,y)$.

Examples of the weight-dependent random connection model

- **Boolean model:** $g^{\text{sum}}(s,t) = \left(s^{-\gamma/d} + t^{-\gamma/d}\right)^{-d}$



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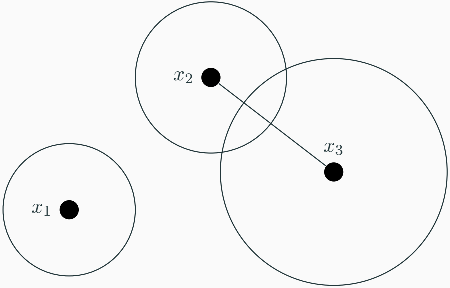

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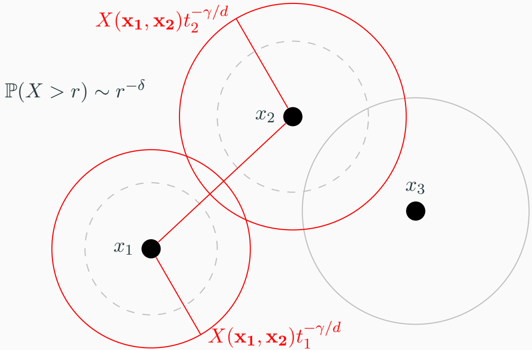

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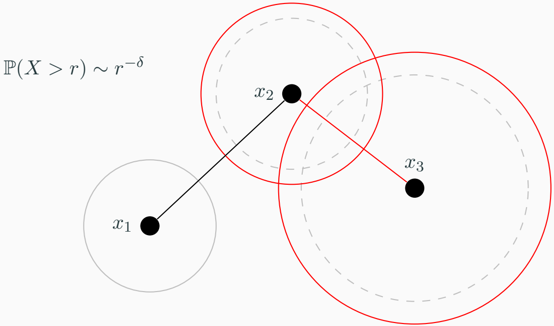

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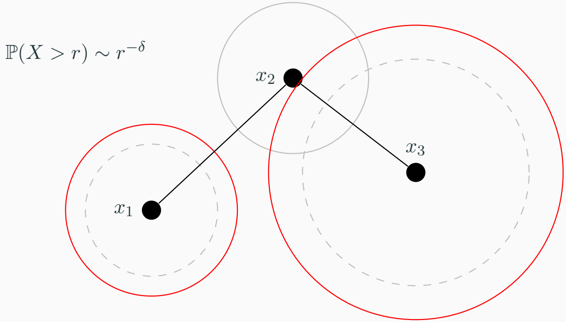

The contact process on the weight-dependent random connection model

The contact process

Let $G = (V, E)$ be a locally-finite graph.

The **contact process** on G is a continuous time Markov process $(\xi_t)_{t \geq 0}$ on $\{0, 1\}^V$.

For $\{x \in V : \xi_t(x) = 1\} \subset V$ we have the transition rates:

$$\begin{aligned} \xi_t \to \xi_t \backslash \{x\} \quad & \text{for } x \in \xi_t \text{ at rate } 1 \text{, and} \\ \xi_t \to \xi_t \cup \{x\} \quad & \text{for } x \notin \xi_t \text{ at rate } \lambda \cdot \left| \{y \in \xi_t : x \sim y\} \right| . \end{aligned}$$

Key questions

The process has the single absorbing state equal to $\emptyset$.

Consequently, the natural questions concern the **extinction time**

$$\tau_G := \inf\{t > 0 : \xi_t = \emptyset\}.$$

- Can τ_G be infinite?
- If yes, with what probability?
- On a graph with n vertices, how does τ_G change with n?
- How does λ affect the answer?

Non-extinction probability

Let the contact process with parameter λ on $G_{(0,T_0)}$ start in the origin $(0, T_0)$. Define:

$$\Gamma(\lambda) = \mathbb{P}_{(0,T_0)}\big(\xi_t^{(0,T_0)} \neq \emptyset \; \forall \, t \geq 0\big).$$

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Theorem (G. and Grauer, '24)

Let G be a general geometric random graph which satisfies Assumption (A1) for $\gamma > \frac{\delta}{\delta+1}$. Then, as $\lambda \to 0$,

$$\Gamma(\lambda) \asymp \frac{\lambda^{2/\gamma-1}}{\log(1/\lambda)^{(1-\gamma)/\gamma}}.$$

Proof idea

Some observations:

- The contact process can survive a constant amount of time on a “star” of size at least $\sim \lambda^{-2}$. (Mountford, Valesin, and Yao, 2013)
- On a chain of disjoint stars of size at least $\sim \lambda^{-2} \log(1/\lambda)$, separated by at most bounded many steps, the contact process survives with probability at least p for some constant $p > 0$. (Linker, Mitsche, Schapira, and Valesin, 2021)

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Lower bound: A chain of star$\rightarrow$connector$\rightarrow$bigger star$\rightarrow \ldots$ exists *a.s.*

Upper bound: Truncated first moment bound counting argument, based on the counting argument by *G., Grauer and Mörters, 2022*.

Exponential extinction time on finite restrictions

Let G_n be the restriction of G to $\left[-\frac{n^{1/d}}{2}, \frac{n^{1/d}}{2}\right]^d$.

We denote by $\tau_n := \inf\{t > 0 : \xi_t^{G_n} = \emptyset\}$ the extinction time of the contact process on G_n.

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Theorem (G. and Grauer, '24)

Let $(G_n)_{n\in\mathbb{N}}$ be the restricted finite graph sequence of a general geometric random graph which satisfies Assumption (A1) for $\gamma > \frac{\delta}{\delta+1}$, and let the graph start fully infected. For any $\lambda > 0$, there exists $c > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\{\tau_n \geq e^{cn}\} = 1.$$

Proof sketch

Proposition (G. and Grauer, ’24)

Let $S > 0$ *be given and* $(G_n)_{n\in\mathbb{N}}$ *the restricted finite graph sequence of a general geometric random graph which satisfies Assumption (A1) for* $\gamma > \frac{\delta}{\delta+1}$*. Then, there exists* $b > 0$ *and* $\varepsilon > 0$ *such that, for* n *sufficiently large, the probability that* G_n *has a connected subgraph containing* $b \cdot n$ *disjoint stars of at least* S *vertices each is larger than* $1 - \exp(-n^{\varepsilon})$*.*

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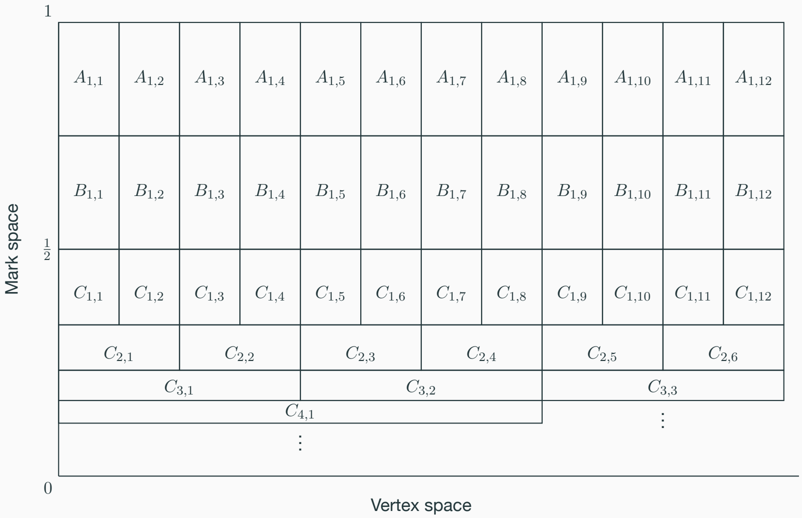
The proof by *Mountford, Mourrat, Valesin, and Yao, 2016* for the configuration model than yields the exponential extinction time claim.

Proof sketch (Subgraph construction)



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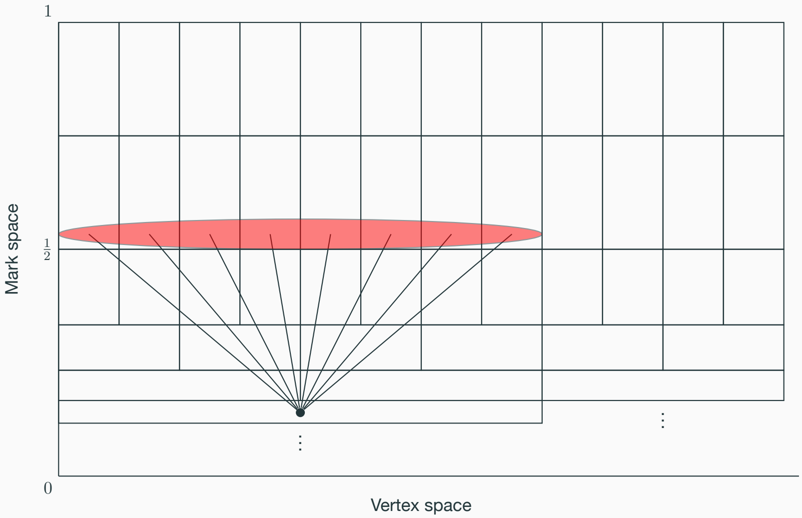

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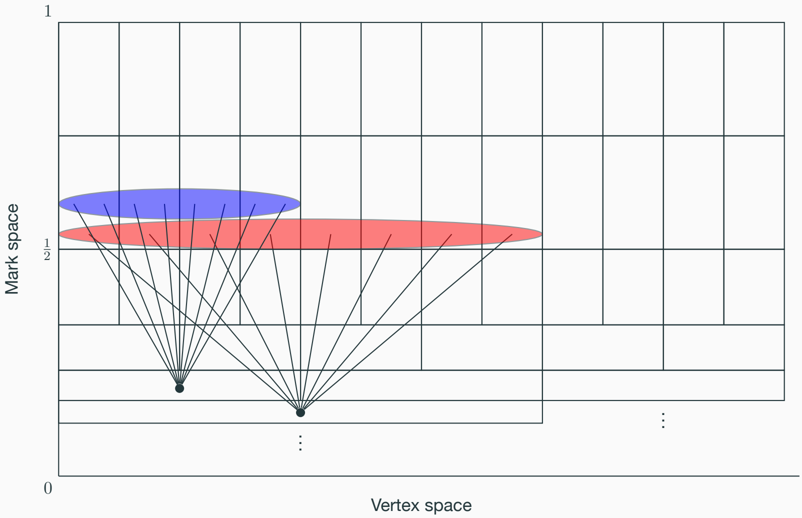

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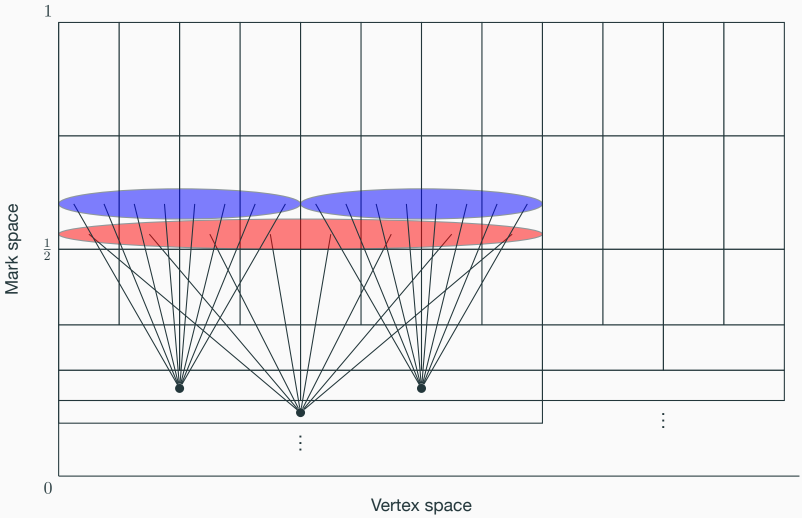

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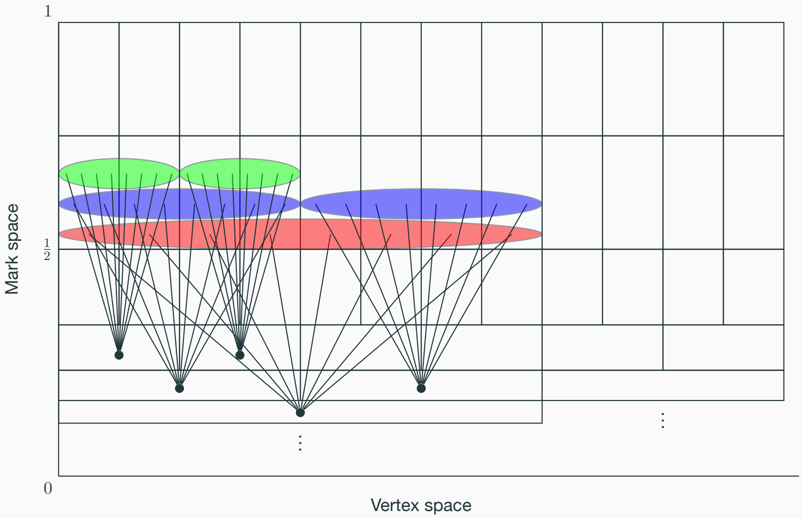

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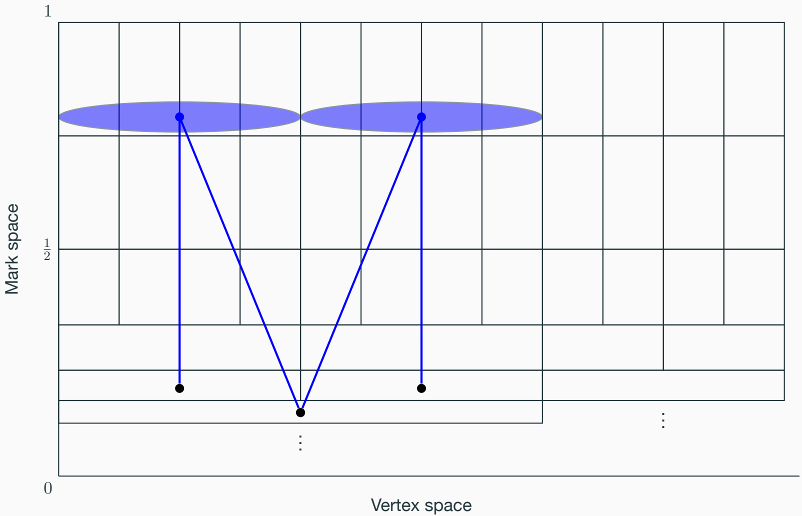

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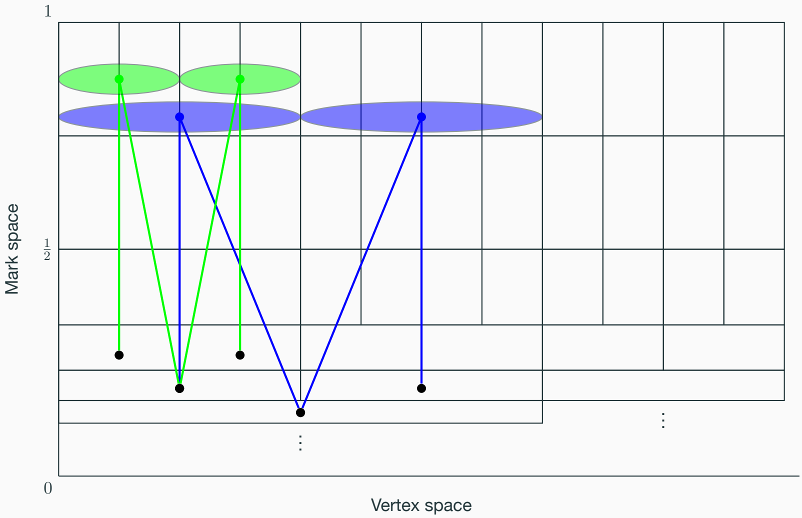

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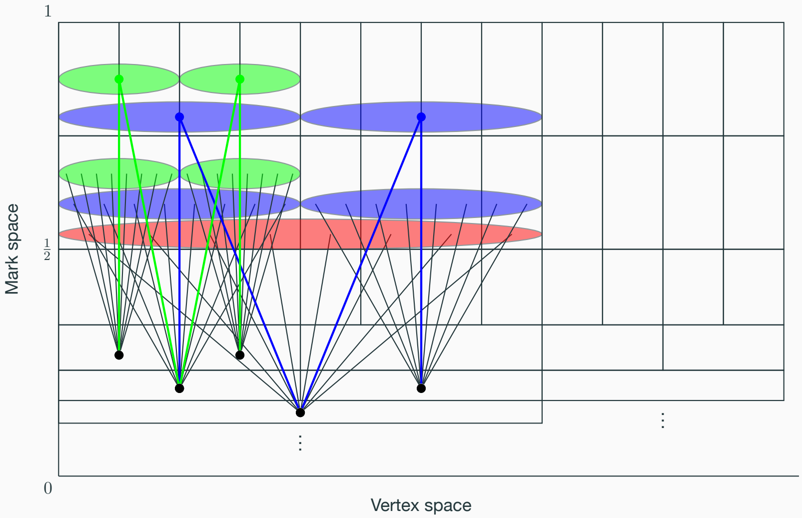

Beyond the weight-dependent random connection graph satisfying (A1)

The stated results also hold for the weight-dependent random connection model with the factor kernel, i.e.

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \sim \mathbf{y}) \asymp t^{-\delta\gamma} s^{-\delta\gamma} |x - y|^{-\delta d}$$

for $\delta > 1$ and $\gamma > \frac{1}{2}$ (Linker, Mitsche, Schapira, and Valesin, 2021).

Key difference: stars connect directly.

The dynamic weight-dependent random connection model

Setup

- **Vertices**: a Poisson point process $\mathcal{X}$ of intensity λ on $\mathbb{R}^d \times [0,1]$ or $\mathbb{T}_n^d \times [0,1]$ at time $t = 0$.

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- **Marks**: Marks do not update*.

Setup

- **Edges**: Given $\mathcal{X}_0$, an edge is drawn between every pair of vertices $\mathbf{x} = (x, t), \mathbf{y} = (y, s) \in \mathcal{X}$ if $U_{x,y} < \mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \sim \mathbf{y})$, where $\mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \sim \mathbf{y})$ satisfies (A1), where $U_{x,y} \sim \mathrm{Unif}(0, 1)$, sampled independently for every pair $\{\mathbf{x}, \mathbf{y}\}$.

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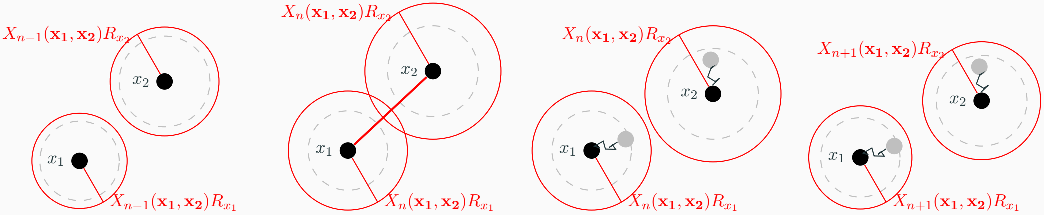
Individual edge/vertex neighbourhood updating at i.i.d. exponentially distributed random times makes the proofs more difficult, but does not change the results.

Edge updating (Example)



***(a)** Just before time n.* ***(b)** At time n.* ***(c)** Just before $n+1$.* ***(d)** At time $n+1$.*

Summary

The randomness in the model comes from:

- The **starting locations** and **marks** of the vertices.
- The **motion** of the vertices.
- Given the above, the **random occurrence of the edges**.

Percolation time

Consider the dynamic scale-free geometric random graph on $\mathbb{R}^d \times (0,1)$.

We define the *percolation time* as

$$T_{\text{perc}} := \inf\{t \geq 0 : \exists \mathbf{x} \in \mathcal{C}_\infty^t \textbf{ s.t. } \mathbf{x} \sim \mathbf{0}\},$$

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Theorem (G. and Grauer, '24[1])

Let $\gamma > \frac{\delta}{\delta+1}$. Then there exists a constant $c > 0$ such that on the $\mathbb{R}^d$ with $d \geq 1$, the percolation time $T_{\textsf{perc}}$ satisfies

$$\mathbb{P}(T_{\textsf{perc}} > t) \leq \exp\{-ct^{1/c}\},$$

for any vertex intensity $\lambda > 0$ and any $t > 0$ sufficiently large.

[1] Also holds for factor kernel in "robust" phase, i.e. $\gamma > \frac{1}{2}$.

Broadcast time

Let $\mathbb{T}_n$ be the d-dimensional torus of volume n and consider the dynamic scale-free geometric random graph on $\mathbb{T}_n \times (0, 1)$.

At time $t = 0$ an arbitrary vertex starts broadcasting information to all vertices in its connected component. Every vertex that receives this broadcast begins broadcasting it further.

The *broadcast time* T_{bc} is defined as the smallest time at which every vertex of the network has received the broadcast.

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Theorem (G. and Grauer, ’24[2])

On the d-dimensional torus of volume n with $d \geq 1$ with $\gamma > \frac{\delta}{\delta+1}$, the broadcast time T_{bc} is with high probability $O(\log n (\log\log n)^{\epsilon})$ for any $\epsilon > 0$ and any $\lambda > 0$.

[2] Also holds for factor kernel in “robust” phase, i.e. $\gamma > \frac{1}{2}$.

A few remarks

- Dynamic geometric random graphs are strongly correlated spatially and temporally.
- Since the vertex marks are not being updated, exceptionally “unlucky” vertices are possible.
- It is not *a priori* clear how the largest connected component evolves over time.

Static tools: Good cubes

Let $a \in (0, \frac{1}{\log 2})$ and $\Theta \in (\frac{\log 2}{\gamma+\gamma/\delta}, \log 2)$ be two constants.

Definition (t-α-dense cubes)

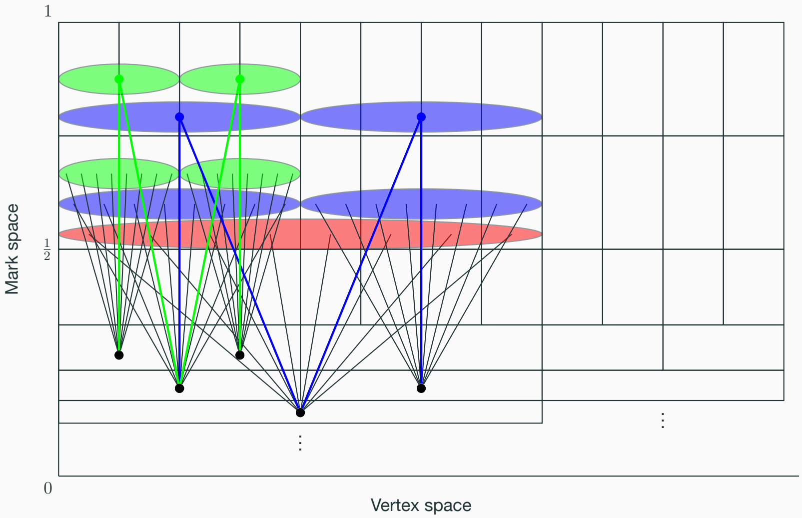
Let $t > 0$ and $I_k := (\frac{1}{2}e^{-(k+1)\Theta d}, \frac{1}{2}e^{-k\Theta d})$, $k \in \{0, \dots, \lfloor (a \log t)/d \rfloor\}$ and $I_{-1} := (\frac{1}{2}, 1)$. We say a cube $Q \subset \mathbb{R}^d$ is *t-α-dense*, if for every $k \in \{-1, 0, \dots, \lfloor (a \log t)/d \rfloor\}$, the locations of the vertices in $Q \times I_k$ contain as a subset an independent Poisson point process of intensity $\lambda(1-\alpha)|I_k|$ on $\mathbb{R}^d$, with marks in I_k.

Static tools: Evenly spread subgraphs

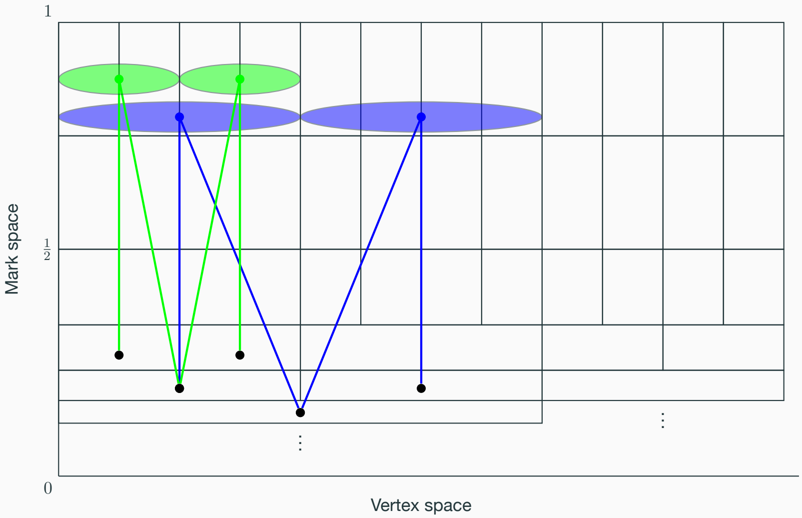
Definition (Evenly spread subgraphs)

We call a finite connected subgraph of $\mathcal{G}$ contained inside Q_K an *evenly spread subgraph of* $\mathcal{G}$ *inside* Q_K, if it contains at least $b \cdot K$ vertices for some constant $b > 0$ and if every subcube of Q_K of the form $\times_{i=1}^{d}(2^{k_p}v_i, (2^{k_p}+1)v_i)$, $v_i \in \mathbb{Z}$, $k_p = \lfloor a \log K \rfloor$ contains a vertex with mark smaller than $\frac{1}{2}e^{-k_p \Theta d}$ belonging to the evenly spread component. We call these vertices the *bottom vertices* of the evenly spread component.

Evenly spread



Evenly spread



Static tools: Evenly spread subgraphs

Proposition (G. and Grauer, '24)

Fix K large enough and consider the cube $Q_K \subset \mathbb{R}^d$. Assume that Q_K is K-α-dense for some $\alpha > 0$ and let $\gamma > \frac{\delta}{\delta+1}$. Then, there exists $\epsilon > 0$ such that for any $\lambda > 0$ and K large enough, there exists an evenly spread subgraph of $\mathcal{G}$ inside Q_K with probability at least

$$1 - \exp\{-K^{\epsilon}\}.$$

Static tools: Evenly spread subgraphs

Lemma (G. and Grauer, ’24)

Assume that Q_K is K-α-dense for some $\alpha > 0$ and let $\gamma > \frac{\delta}{\delta+1}$ and $\lambda > 0$. Let furthermore G be an evenly spread subgraph of $\mathcal{G}^{1-\varepsilon}$ inside Q_K. Then, a given vertex $\mathbf{x}$ with a mark in $(0,1)$ and arbitrary location within Q_K belongs to the same connected component of $\mathcal{G}$ as G with probability bounded away from 0.

Static tools: Summary

- If a cube Q_K of volume t^d is t-α-dense, an evenly spread subgraph of size K exists $wep(K)$.
- An arbitrary vertex anywhere in the cube Q_K has probability bounded away from 0 of belonging to the same connected component as the evenly spread subgraph.

Key question

How often is the cube Q_K t-α-dense during the time interval $[0, t]$?

Dynamics: Decoupling of particles

Proposition (Peres, Sinclair, Sousi, and Stauffer, 2011[3])

Fix $K > \ell > 0$ *and consider the cube* $Q_{K^d} \subset \mathbb{R}^d$ *tessellated into subcubes of side length* ℓ*. Let* Π_0 *be an arbitrary point process at time* 0 *that contains at least* $\beta\ell^d$ *vertices in each subcube of the tessellation for some* $\beta > 0$*. Let* Π_Δ *be the point process obtained at time* Δ *from* Π_0 *after the vertices have moved for time* Δ*. Fix* $\epsilon \in (0,1)$ *and let* Ψ *be an independent Poisson point process of intensity* $(1-\epsilon)\beta$ *on* Q_K*. Then there exists a coupling of* Ψ *and* Π_Δ *and constants* c_1, c_2, c_3 *that depend on* d *only, such that if* $\Delta \geq \frac{c_1\ell^2}{\epsilon^2}$ *and* $K' \leq K - c_2\sqrt{\Delta \log \epsilon^{-1}} > 0$*, the vertices of* Ψ *are a subset of the vertices* Π_Δ *inside the cube* $Q_{K'}$ *with probability at least*

$$1 - \frac{K^d}{\ell^d} \exp\{-c_3\epsilon^2\beta\Delta^{d/2}\}.$$

[3] Generalisations of this result exist also for (uniformly elliptic) lattices and fractal lattices.

Dynamics: Cubes are good/dense most of the time

Proposition (G. and Grauer, '24)

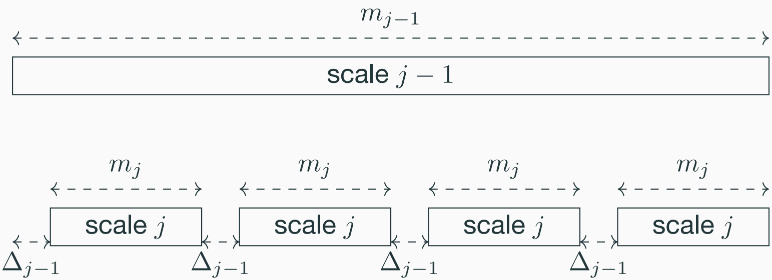
Let $t > 0$ be a sufficiently large integer and $\xi, \epsilon \in (0,1)$ two constants. Consider the cube Q_{L^d}, for $L = t$. Define for $i = 0, \ldots, t$ the events

$$A_i = \{\textit{at time } i \textit{ the cube } Q_{L^d} \textit{ is } t\text{-}\xi\textit{-dense}\}.$$

Then, there exist two positive constants c_1, c_2 such that

$$\mathbb{P}\left(\sum_{i=0}^{t-1} \mathbb{1}_{A_i} \geq (1-\epsilon)t\right) \geq 1 - \exp\{-c_1 t^{c_2}\}.$$

Dynamics: Proof of proposition (sketch)

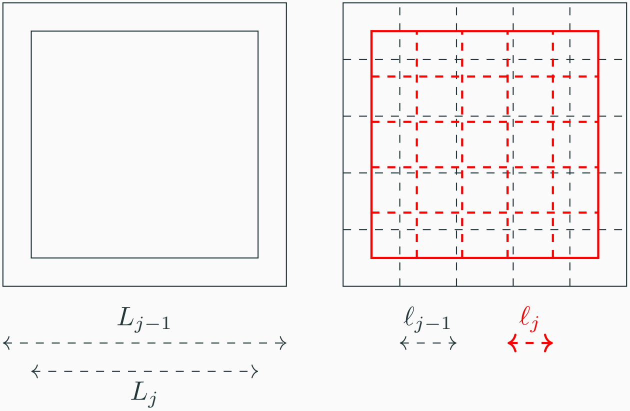


The temporal multi-scale recursion

The values are chosen as:

$$m_1 = t, \quad m_j = \frac{m_{j-1} - 4\Delta_{j-1}}{4}, \quad \text{and} \quad \frac{\Delta_j}{m_j} = \frac{\epsilon}{8\kappa}.$$

Dynamics: Proof of proposition (sketch)



The spatial multi-scale recursion

The values are chosen as: $L_1 = L^2$, $L_\kappa = L$, $\ell_1 = c\sqrt{t}$, $\ell_{j+1} = \ell_j \sqrt{\frac{1}{4} - \frac{\epsilon}{8\kappa}}$, with $\kappa = O(\log t)$.

Dynamics: Proof of proposition (sketch)

Let D_j be the event that all subcubes of side-length ℓ_j inside the cube of side length L_j are *good* for a fraction of at least $(1 - \frac{\epsilon_j}{2})$ time intervals of scale j.

1. $\mathbb{P}(D_1) \geq 1 - \exp\{-c_1 t^{c_2}\}$
2. Set
$$E = \{\text{at time } b' \text{ all subcubes are good for the scale } j-1\},$$
with $b' = b - \Delta_{j-1}$. Then
$$\mathbb{P}([b, b+m_j) \text{ is good}, E|F) \geq 1 - \exp\{-c_1 t^{-a\Theta} \ell_j^d / \kappa^2\},$$
with F measurable w.r.t. events that occurred up to time b'.
3. $\mathbb{P}(D_j^c \cap D_{j-1}) \leq \exp\{-\frac{ct^{c_4}}{\kappa^6}\}$.

Dynamics: Proof of proposition (sketch)

Consequently,

$$\mathbb{P}(D_\kappa^c) \leq \mathbb{P}(D_\kappa^c \cap D_{\kappa-1}) + \mathbb{P}(D_{\kappa-1}^c),$$

which gives

$$\mathbb{P}(D_\kappa^c) \leq \sum_{j=2}^{\kappa} \mathbb{P}(D_j^c \cap D_{j-1}) + \mathbb{P}(D_1^c).$$

Broadcast time

Consider a torus of volume n and let $t = C \log n(\log\log n)^{\epsilon}$.

- Tessellate the torus into cubes of side length t.
- Each such cube is dense at least $(1-\epsilon)t$ amount of time $\Rightarrow$ Contains $wep(t)$ evenly spread subgraph throughout this time.
- At each time, the second largest component on the torus (of size n) is $wep(t)$ of size $o(t)$ by *Jorritsma, Komjáthy, and Mitsche, 2024*.

Consequently, when an evenly spread subgraph exists, it is in the largest connected component of the torus.

Broadcast time

- Each vertex (including the "origin vertex") has positive (bdd. away from 0) probability of belonging to the same component as the "local" evenly spread subgraph.
  $\Rightarrow$ Infection enters the large component during $[0, t/2)$ with probability greater than $1-(1-p)^{t/2}$.
- During $[t/2, t]$, a vertex belongs to the large component at least once with probability greater than $1-(1-p)^{t/2}$.
- Since there are only $\Theta(n)$ vertices on the torus $whp(n)$, every vertex gets the information with probability at least

$$1-(1+\delta)n(1-p)^{(1-\epsilon)t/2}.$$

  Setting C large concludes the proof.

Open problems and future work

- Tightness of broadcast time bound.
- **Non-robust regimes.**
- Different edge updating mechanisms.
- **Contact process on dynamic graph.**

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Thank you for listening![4]

[4] and/or being physically present.