The shape of shortest paths in random spatial networks

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September 2024

Outline

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The model

1. We consider spatial networks on Poisson points in bounded Euclidean domains.
2. We assign edge weights = Euclidean length.
3. Consider the geodesics spanning two Poisson points x, y at Euclidean distance $|x - y|$. What do these look like?

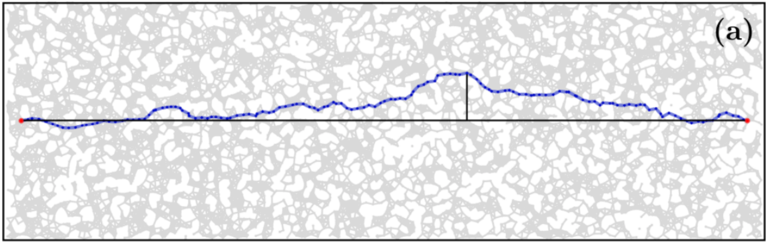


Figure: (a) A geodesic on the hrgg.

First passage percolation on spatial networks

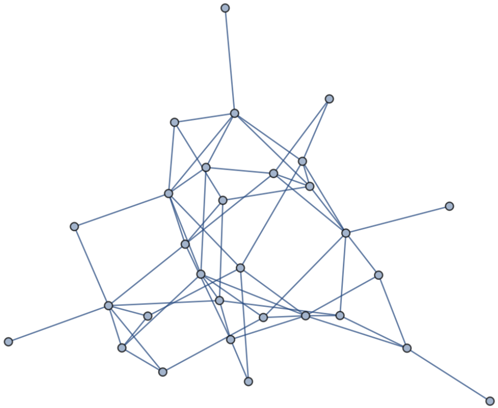


Figure: (a) An Erdos-Renyi random graph $G(30, 2/15)$.

First passage percolation on spatial networks

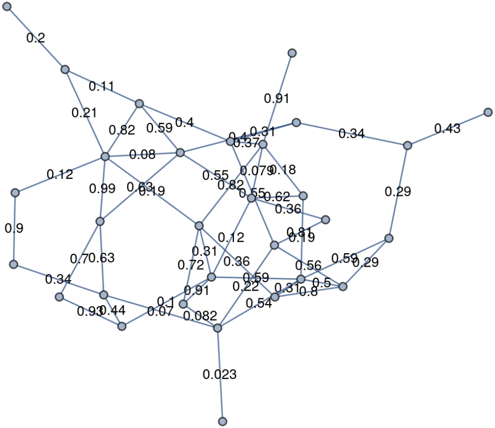


Figure: $G(30, 2/15)$ with added i.i.d. edge weights uniform on [0,1].

First passage percolation on spatial networks

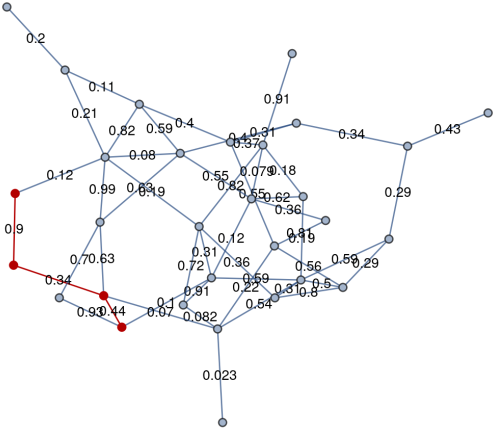


Figure: A geodesic highlighted.

First passage percolation on spatial networks

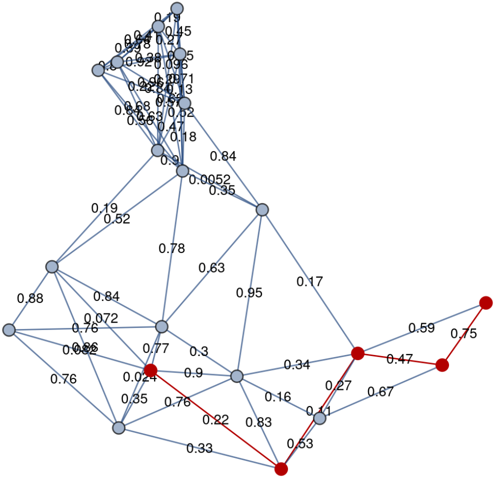


Figure: A geodesic highlighted.

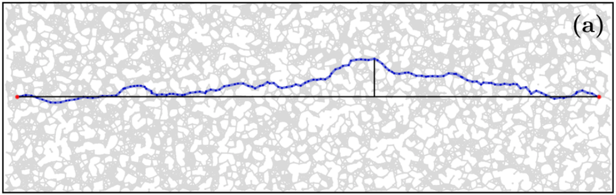
First passage percolation on spatial networks

It is known that in scale-invariant Euclidean geometry, we have power laws for various statistics of T: accordingly, the *fluctuation exponent* χ is defined by

$$\mathrm{Var}(T(x, y)) \sim |x - y|^{2\chi} \tag{1}$$

as $|x - y| \to \infty$. Similarly, the deviation $D(x, y)$ of the geodesic from the straight line from x to y is characterised by the wandering exponent ξ

$$\mathbb{E}(D(x, y)) \sim |x - y|^{\xi} \tag{2}$$



FPP review of results

Review of FPP on $\mathbb{Z}^2$, main results:

1. Cox-Durrett Shape theorem with $B(t) = \{z \in \mathbb{Z}^2 : T(0, z) \leq t\}$

$$\frac{1}{t}B(t) \to \mathcal{B} \subset \mathbb{R}^2 \tag{3}$$

   $\mathcal{B}$ is constant, symmetric, convex, non-empty, not all of $\mathbb{R}^2$, Cox-Durrett, Ann. Prob. 1981.

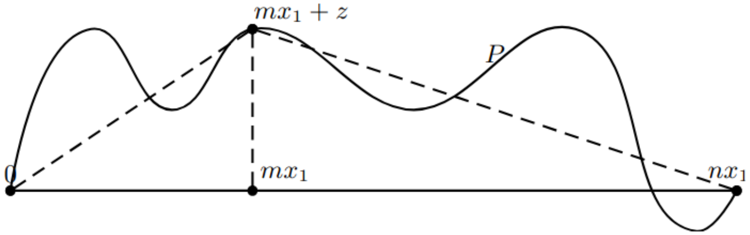
2. Almost nothing known until Newman and Piza in 1995 give bounds on wandering exponent.
3. In 2013, Chatterjee proves $\chi = 2\xi - 1$ with strong definition of exponents. Damron-Hanson also prove this without some assumptions on the edge-weight distribution.

First passage percolation

Sometimes we say τ is "time", ξ is "space", and χ is "fluctuation".

Time/space/fluctuation exponents of the Kadar-Parisi-Zhang (KPZ) class have $\chi = 1/3$, $\xi = 2/3$, and $\tau = 1$, with

$$\chi = 2\xi - 1 \tag{4}$$



from S. Chatterjee *The universal relation between scaling exponents in first-passage percolation*, Ann. Maths 2013.

Corner growth model

Example of 3:2:1 scaling:

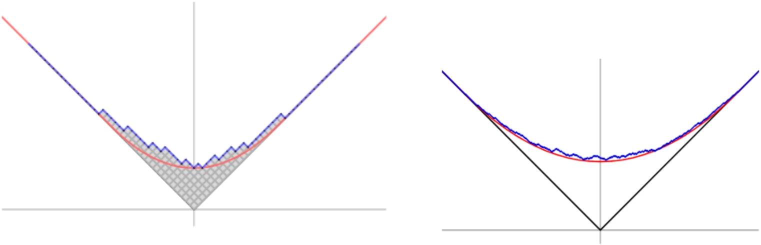


Figure: Simulation of the corner growth model (medium and long term). The blue curve has vertical fluctuations of order $t^{1/3}$ and decorrelates spatially on distances of order $t^{2/3}$ (*from I. Corwin, AMS Not. 2016*).

The KPZ fixed point

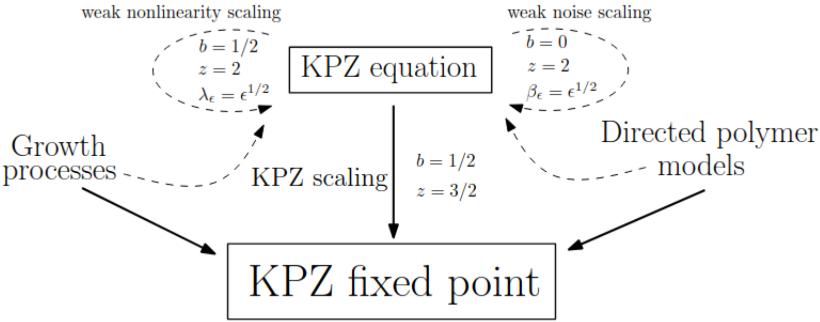


Figure: It is believed that these ideas extend to a variety of growth processes and directed polymer models (*from I. Corwin, Symp. App. Maths 2018*).

Outline

Four models of spatial networks

We simulate Euclidean FPP on four types of spatial network:

1. **Random geometric graph**
2. **k-Nearest Neighbour Graph** For this graph, we connect points of $\mathcal{X}$ to their $k \in \mathbb{N}$ nearest neighbours.
3. **Delaunay triangulation** The Delaunay triangulation of a set of points is the dual graph of their Voronoi tessellation.
4. **Beta skeleton** One adds edges between two points of a point set when a β-lens is empty of other spatial points.

Beta skeletons: definition

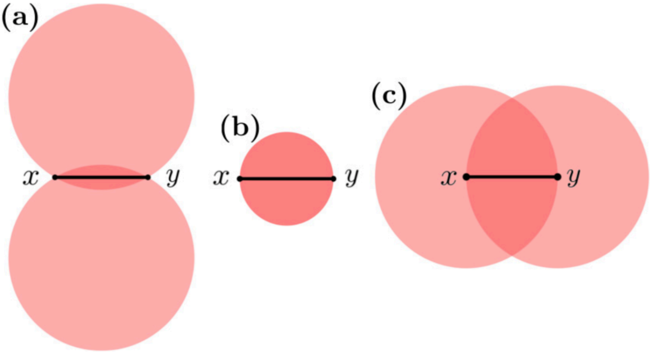
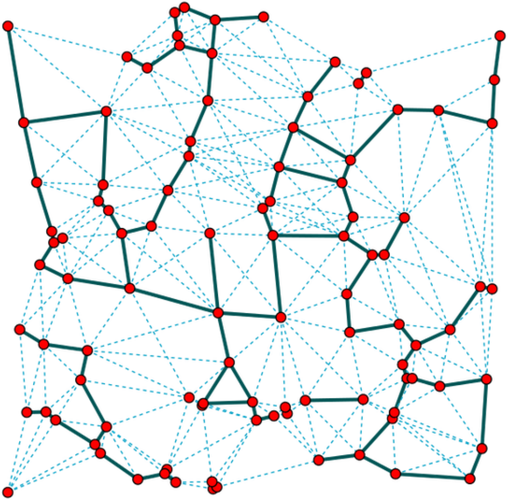


Figure: The geometry of the lune-based β-skeleton for (a) $\beta = 1/2$, (b) $\beta = 1$, and (c) $\beta = 2$.

Beta skeletons



Results

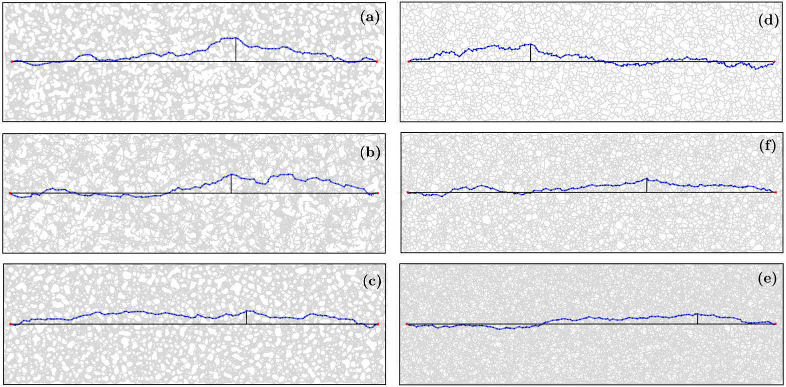


Figure: Geodesics of FPP on different spatial networks: (a) Hard (b) Soft (c) NN (d) RNG (e) Gabirel (f) DT.

Results

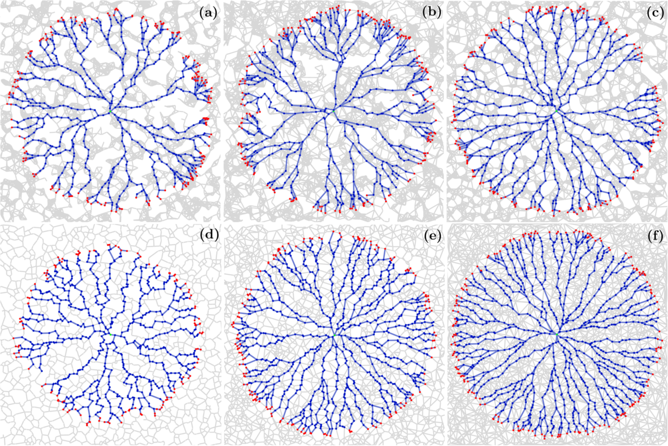


Figure: FPP on different spatial networks (a) Hard (b) Soft (c) NN (d) RNG (e) Gabirel (f) DT.

Outline

Results

What is known:

For the **KPZ class**, we have transversal deviation $\sim |x - y|^{2/3}$, and standard deviation of travel time $\sim |x - y|^{1/3}$, abbreviated as 3:2:1 time:space:fluctuation scaling.

What is new:

New universality class (1) gives 5:3:1 scaling. This includes the soft and hard RGG, based on neighbour connectivity.

New universality class (2) gives 10:7:4 scaling. Networks based on skeleton formation, such as RNG, and Delaunay triangulation.

Two new universality classes:

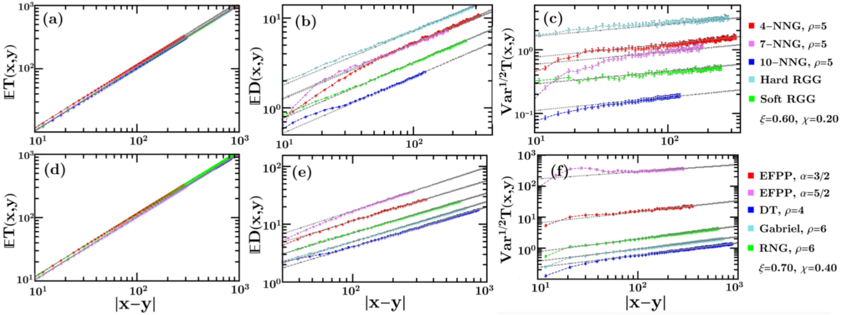


Figure: (Top) New class 1 with **5:3:1** scaling, containing the RGGs and the *k*-NN graph. (Below) New class 2 with **10:7:4** scaling, containing the proximity graphs, triangulations, and Euclidean FPP.

Results

TABLE I. Exponents ξ and χ and passage-time distribution for the various networks considered.

Network	ξ	χ	Distribution of T
	Proximity graphs		
Hard RGG	$3/5$	$1/5$	Normal (Conj.)
Soft RGG with Rayleigh fading	$3/5$	$1/5$	Normal (Conj.)
k-NNG	$3/5$	$1/5$	Normal
	Excluded region graphs		
DT	$7/10$	$2/5$	Normal
GG	$7/10$	$2/5$	Normal
β skeletons	$7/10$	$2/5$	Normal
RNG	$7/10$	$2/5$	Normal
	Euclidean FPP		
With $\alpha = 3/2$	$7/10$	$2/5$	Normal
With $\alpha = 5/2$	$7/10$	$2/5$	Normal

Conclusions and Open Questions

Interesting universality in spatial networks is present in their geodesics.

- **Open question 1**: Can you prove anything rigourously about these exponents?
- **Open question 2**: Can you find another model with these exponents? What is the most straightforward random growth process that displays them?
- **Open question 3**: Can you observe Tracy-Widom for any sort of probabilistic fluctuations in a spatial network model?

Lots of comparison with actual complex networks is also interesting, with connections to scale invariant spatial networks.

Thank you

Joint work with Carl Dettmann at Bristol University, and Marc Barthelemy at IPhT in Paris.

Thank you.