Connectivity in the 1-D random connection model

Ayalvadi Ganesh

University of Bristol

Joint work with Carl Dettmann and Michael Wilsher

Model and notation

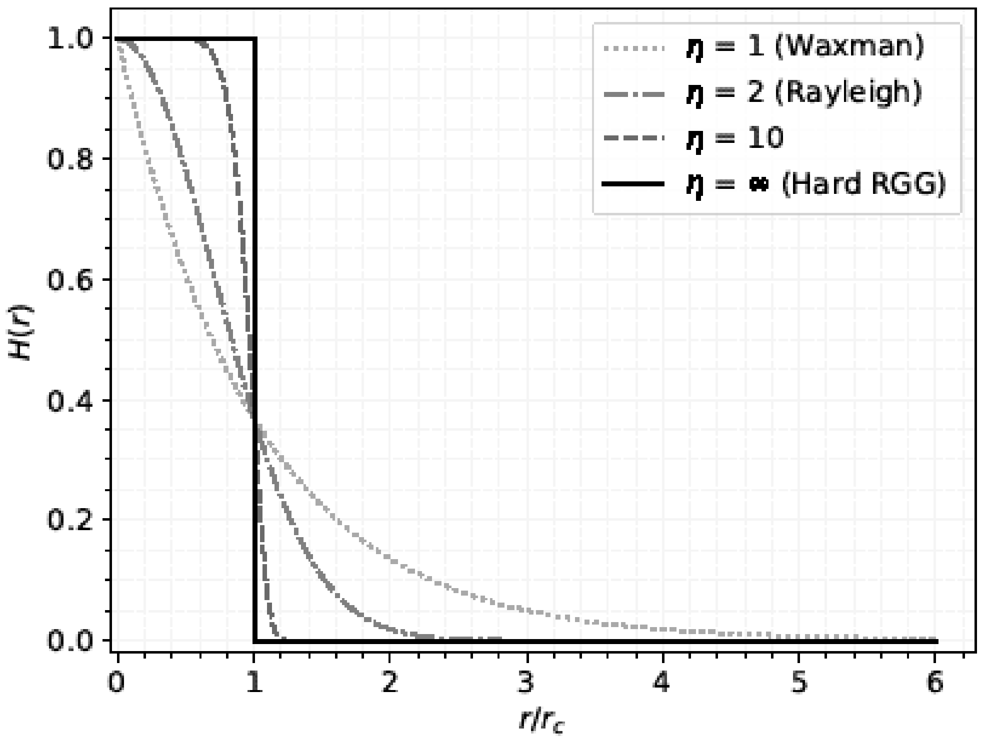
- Domain is interval $[0, L]$ with endpoints identified.
- Nodes form a Poisson process of unit intensity.
- Connection function $H(\cdot)$: edge between nodes a circular distance r apart present with probability $H(r)$, independent of other edges.
- Scaled connection functions: $H^L(r) = H(r/R_L)$
- Asymptotic regime: $L, R_L \to \infty, \quad R_L/L \to 0$

- Question: Is the resulting random graph connected?

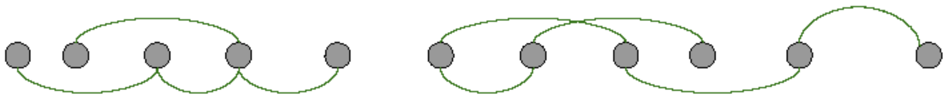
Remarks

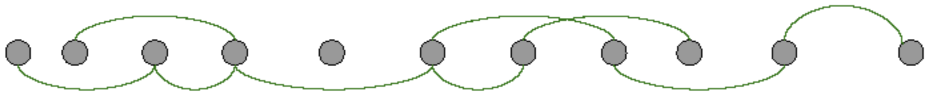
- Assumption: $\|H\|_1 := \int_0^\infty H(x)dx < \infty$.
- Mean number of nodes is L.
- Mean degree tends to $\|H\|_1$ (unscaled) or is asymptotic to $R_L\|H\|_1$ (scaled), as $L \to \infty$.
- Examples:
    - (Hard) Random geometric graph: $H(r) = 1(r < r_c)$
    - Waxman: $H(r) = \exp(-r/r_c)$
    - Rayleigh: $H(r) = \exp(-(r/r_c)^2)$

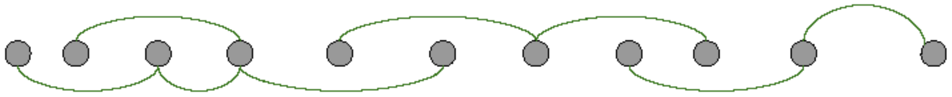
Examples of connection functions



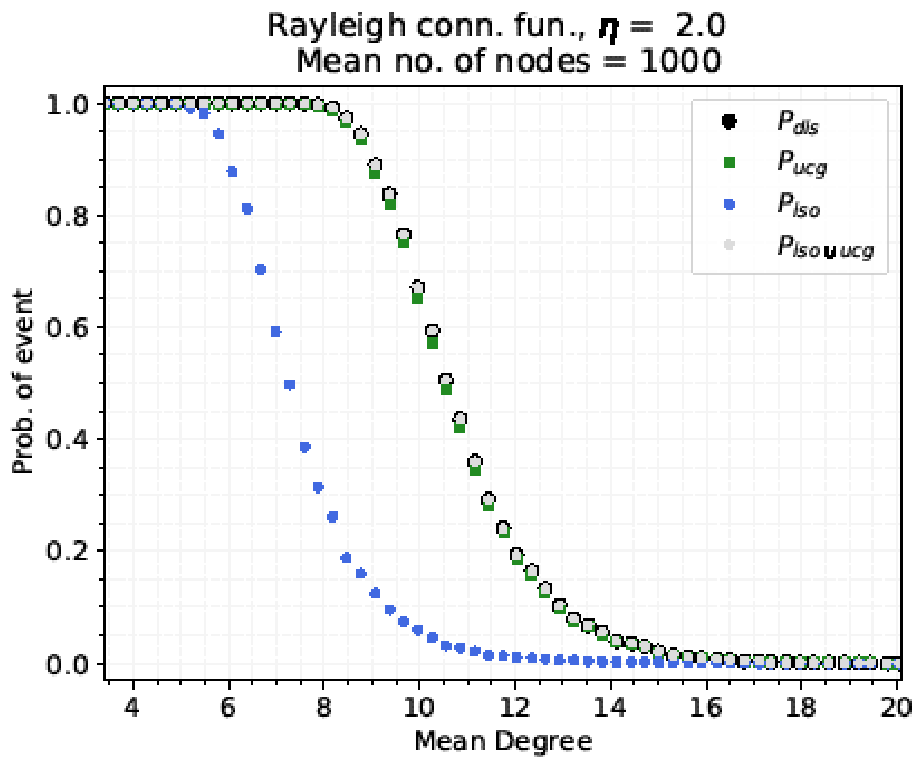
Possible causes of disconnection

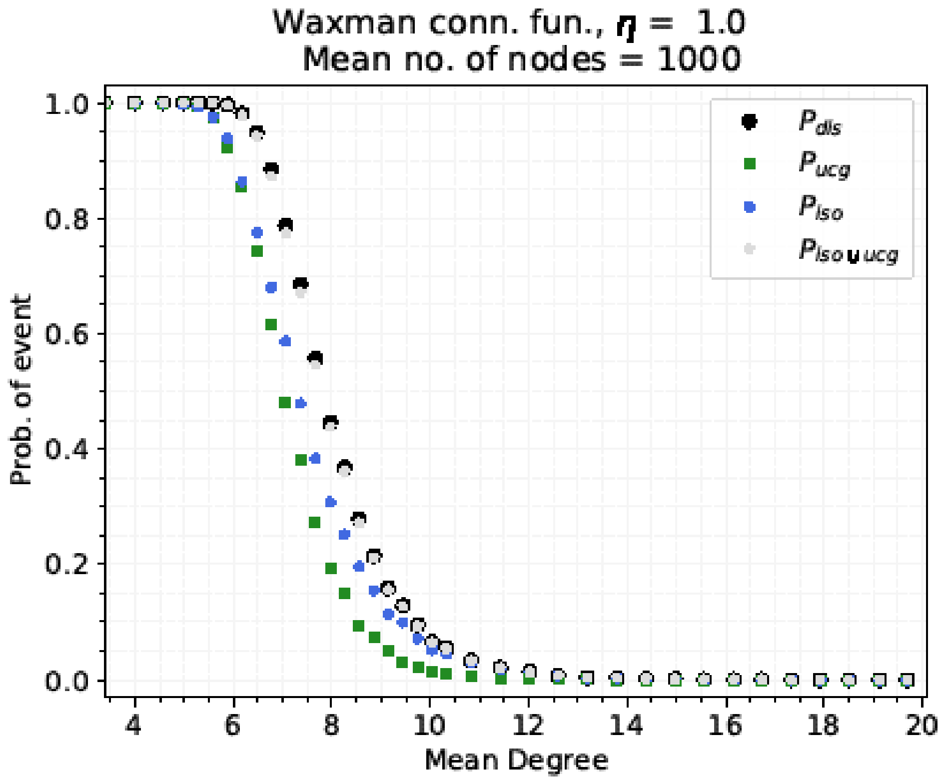






What determines connectivity?





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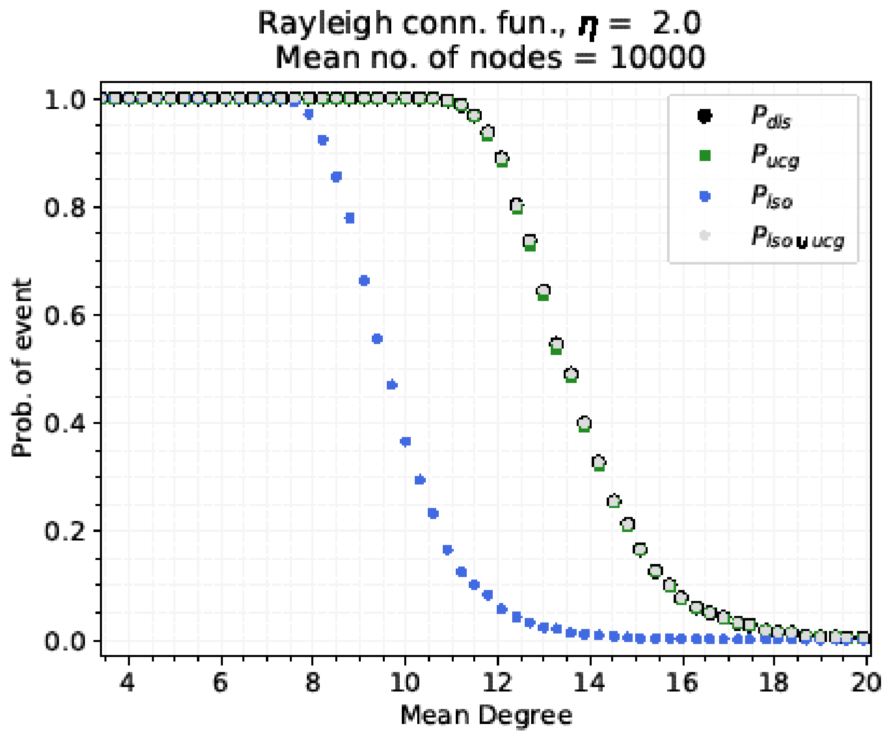
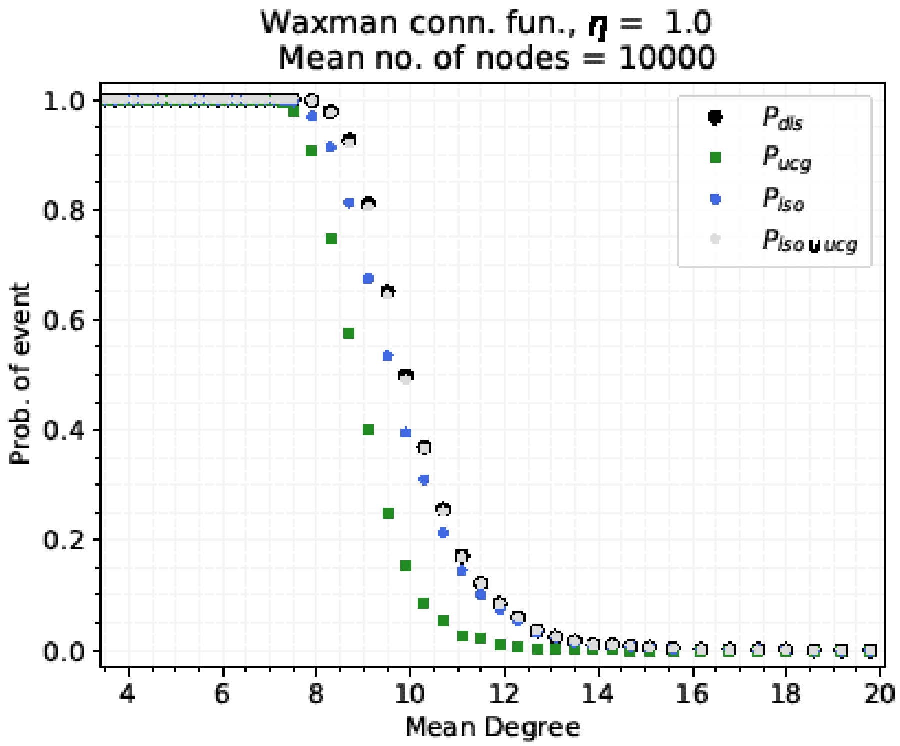




Results: Isolated nodes

- Theorem (Wilsher, Dettmann, G, 2020):

Take $R_L = \gamma \log L$, so that the mean degree is $2\gamma \|H\|_1 \log L$. Then,

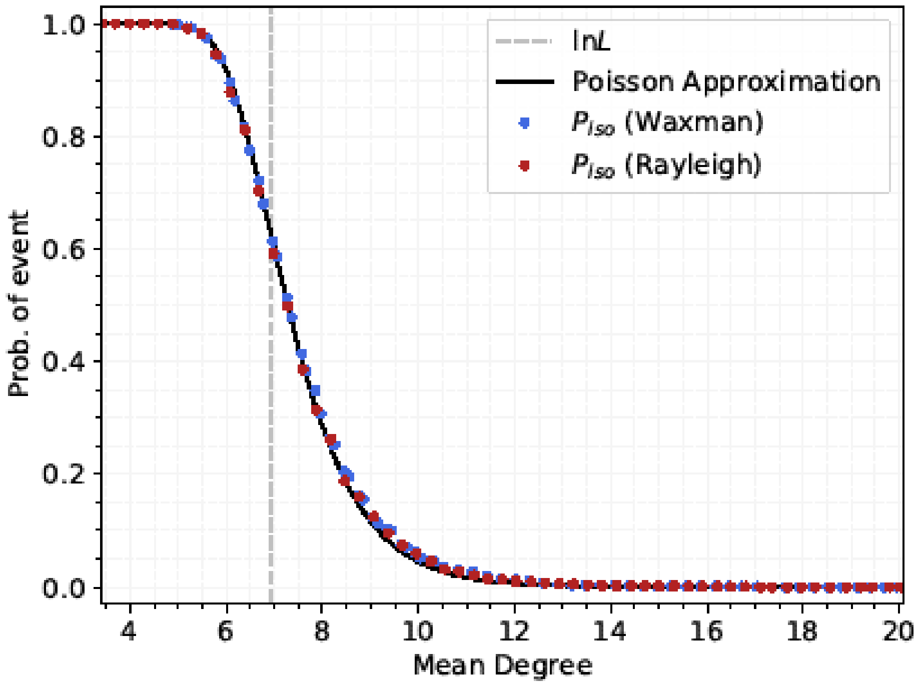
$$P(N_{iso} = 0) \to \begin{cases} 0, & 2\gamma \|H\|_1 < 1, \\ 1, & 2\gamma \|H\|_1 > 1. \end{cases}$$

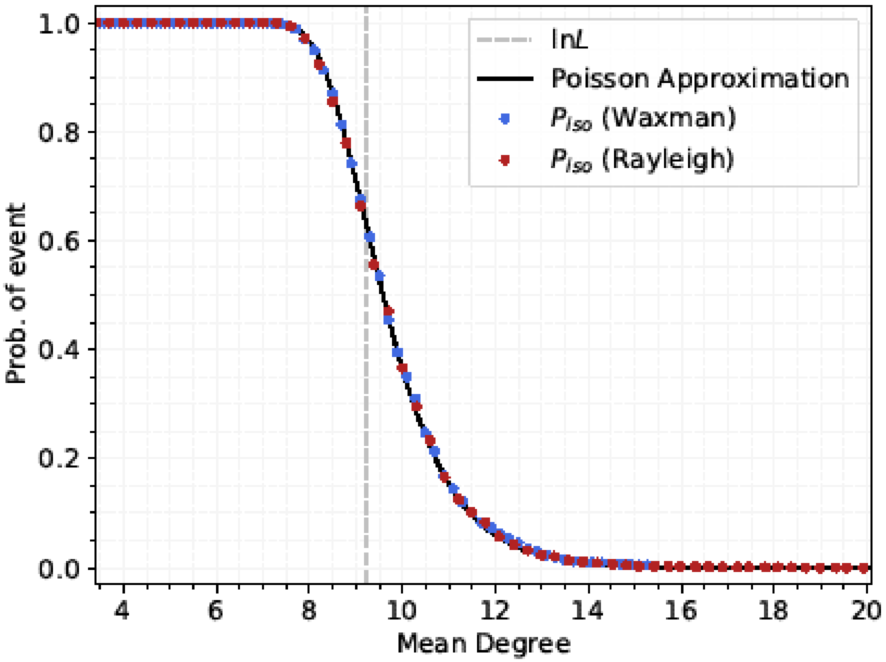
- Theorem (WDG, 2022):

If the mean degree minus $\log L$ tends to a constant, c, then

$$P(N_{iso} = 0) \to \exp(-\exp(-c))$$

Simulation vs. theory





Results: Uncrossed gaps

- Theorem (WDG, 2020): Suppose, further, that H is monotone decreasing and has unbounded support. Take $R_L = \gamma \log L$. Then, for any $\gamma > 0$,

$$P(N_{ucg} = 0) \to 1$$

- Remarks: The theorem says that, at the threshold at which isolated nodes appear, there are no uncrossed gaps, whp.
- Hence, we conjecture that the probability of disconnection is asymptotically the same as the probability of isolated nodes being present.

Threshold for uncrossed gaps

- Consider the generalized Rayleigh connection function,

$$H(r) = \exp(-(r/r_c)^{\eta})$$

- Then, the threshold for the emergence of uncrossed gaps is

$$R_L = {}^{\log L}\!\big/_{C\,(\log\log L)^{\theta}} \ ,$$

- where C and θ depend on η.

References

- Wilsher, Dettmann, Ganesh, Connectivity in one-dimensional soft random geometric graphs, *Phys. Rev. E*, 2020.
- Wilsher, Dettmann, Ganesh, The distribution of the number of isolated nodes in the 1-Dimensional soft random geometric graph, *Stat. and Prob. Letters*, 2022.