Crossing Number in a Projected Random Geometric Graph

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Workshop Stochastic Geometry in Action

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joint work with

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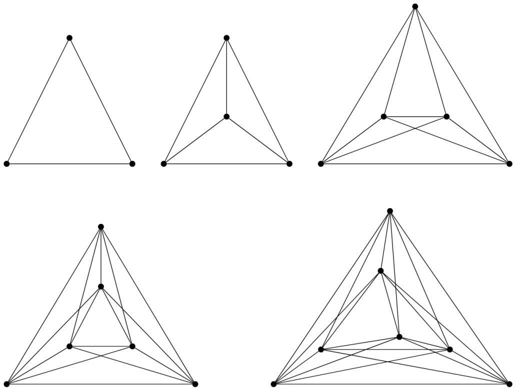
Crossing Number

Crossing number of the graph G
$=$ minimal number of edge crossings of a plane drawing of G

Example: Crossing Number of the complete graph $\mathrm{cr}(K_n)$



Picture from *Crossing Numbers of Graphs* by Schaefer

Harary-Hill Conjecture/ Guy's Conjecture 1960s

Conjecture

$$\mathrm{cr}(K_n) \overset{?}{=} \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Proven for $n \leq 10$ in [Guy 72]
and for $n \leq 12$ in [Pan and Richter 07]:

n	3	4	5	6	7	8	9	10	11	12
$\mathrm{cr}(K_n)$	0	0	1	3	9	18	36	60	100	150

and for some particular cases. Known

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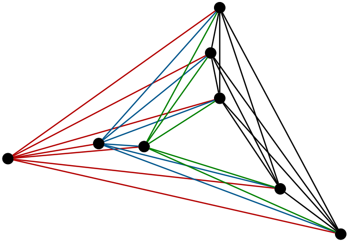
Question Is there always a drawing with edges as straight line segments and a minimal number of crossings?

Rectilinear Crossing Number

Rectilinear crossing number of the graph G
= minimal number of edge crossings of a plane drawing of G
with edges being line segments

Rectilinear Crossing Number
$\overline{\text{cr}}(G)$



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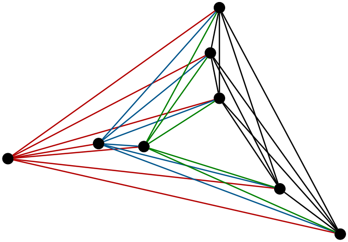

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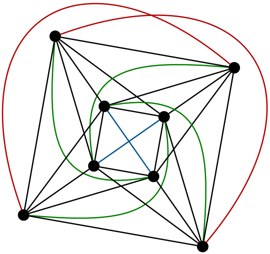

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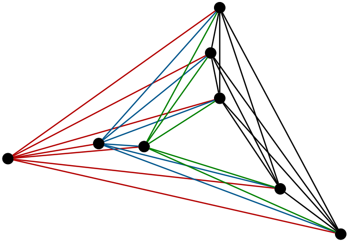

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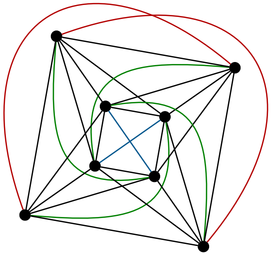

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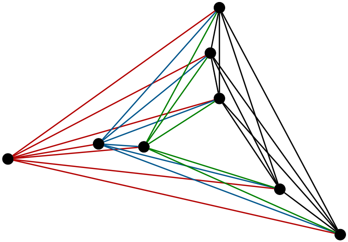

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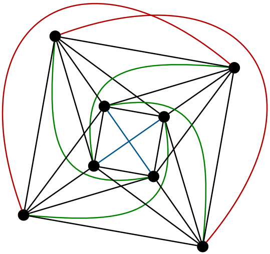

$\text{cr}(K_8) = 18$

$\rightarrow$ smallest complete graph with $\text{cr}(K_n) < \overline{\text{cr}}(K_n)$.

In fact, $\text{cr}(K_n) = \overline{\text{cr}}(K_n)$ for $n \leq 7$ and $n = 9$ only!

Crossing Numbers

Rectilinear crossing number of the graph G

Number n of Vertices	lower bound	min. crossings (so far)
3	0	0
4	0	0
5	1	1
6	3	3
7	9	9
8	19	19
9	36	36
10	62	62
11	102	102
12	153	153
13	229	229
14	324	324
15	447	447
16	603	603
17	798	798
18	1029	1029
19	1318	1318
20	1657	1657
21	2055	2055
22	2528	2528
23	3077	3077
24	3699	3699
25	4430	4430
26	5250	5250
27	6180	6180
28	7233	7234
29	8421	8423
30	9726	9726
31	11207	11213
32	12830	12836
33	14626	14634
34	16613	16620
35	18796	18808
36	21164	21175
37	23785	23803
38	26621	26635
39	29691	29715
40	33048	33071
41	36674	36700

see `http://www.ist.tugraz.at/staff/aichholzer/crossings.html;` 2015

Computer Scientists' View

The problem

Given a graph G, draw it in the plane with the minimal number of edge crossings.

is NP-complete.

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Interest from a computer science perspective

- chip design
- automatic graph drawing

Crossing Lemma

Upper Bounds for $\mathrm{cr}(G)$: constructions, heuristics,...
Lower Bounds for $\mathrm{cr}(G)$: much harder to argue!

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Upper Bounds for cr(G): constructions, heuristics,...
Lower Bounds for cr(G): much harder to argue!

Crossing Lemma. Consider a graph G on n vertices and m edges.

$$\exists c, d \geq 0 \text{ such that if } m \geq d \cdot n \text{ then } \operatorname{cr}(G) \geq c\frac{m^3}{n^2}.$$

[Ajtai et al. 82; Leighton 83]: $d = 4, c = 1/64$;
[de Klerk et al. 06]: $d = 7, c = 1/20$
or see the beautiful and short proof from THE BOOK in Aigner & Ziegler.

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Crossing Lemma for dense graphs, $m \sim n^2$:

- maximum no. of crossings: $\text{cr}(G) \leq \mathcal{O}(m^2)$
- Crossing Lemma: $\text{cr}(G) \geq c\frac{m^3}{n^2} \sim m^2$

$\Rightarrow$ Crossing Lemma is **optimal** for dense graphs.

Random Geometric Graphs

- Convex set $W \subset \mathbb{R}^d$ with $\mathrm{vol}_d(W) = 1$
- vertices: Poisson process of intensity t
- Consider radius δ_t dependent on t
  Draw an edge between u and v if $\|v - u\| \leq \delta_t$
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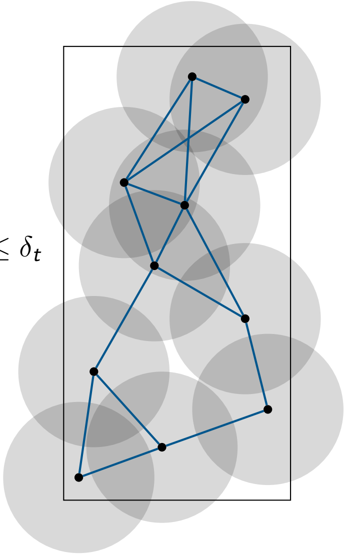
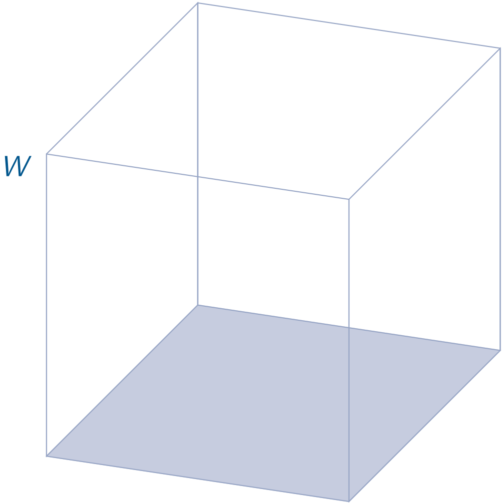
- typical degree of a vertex $\kappa_d t \delta_t^d$
- critical scaling
  for $t \to \infty$ and $\delta_t \to 0$
  with $\lim_{t\to\infty} t\delta_t^d = c \in (0, \infty)$.
  [Penrose 03; Reitzner, Schulte, Thäle 17]
- $L \subset \mathbb{R}^2$ a plane



Projection Algorithm



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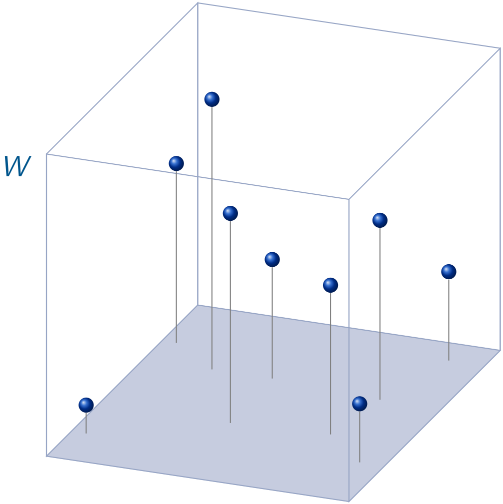

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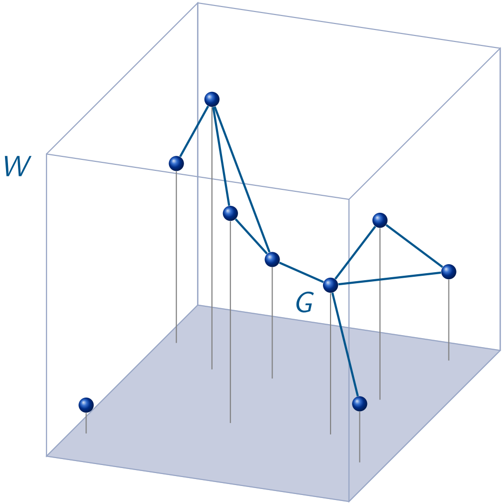

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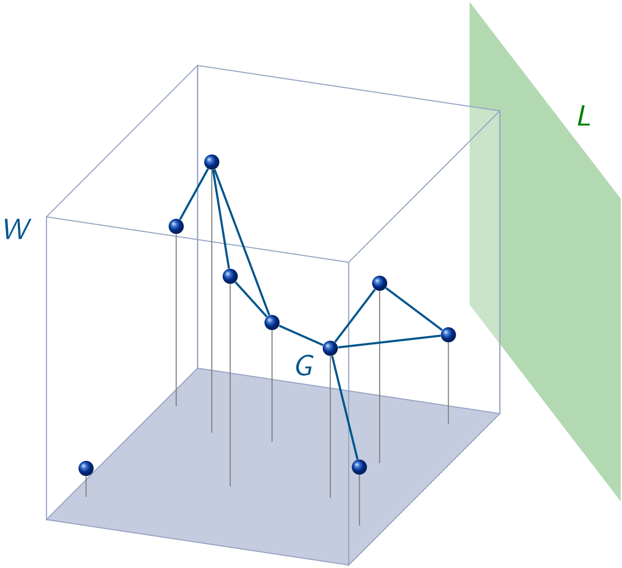

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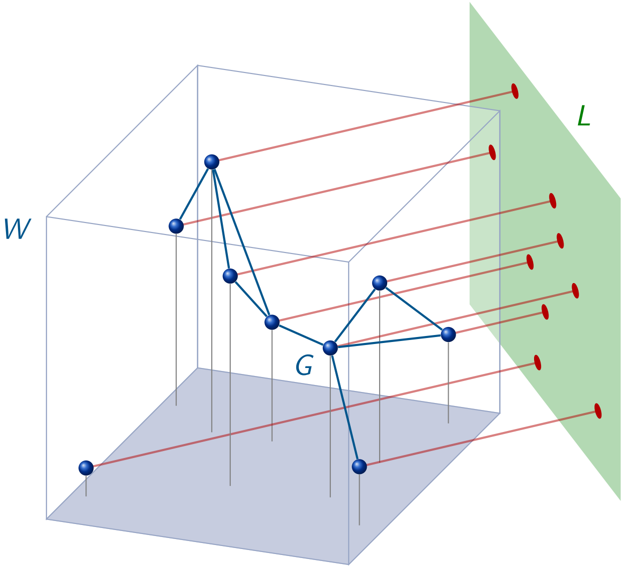

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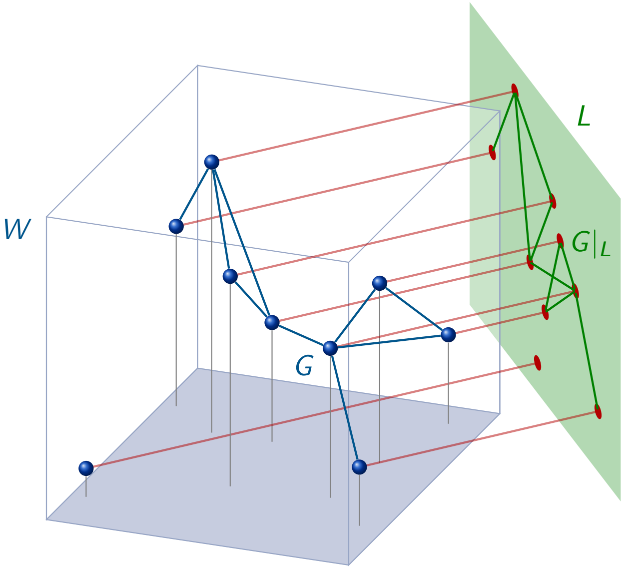

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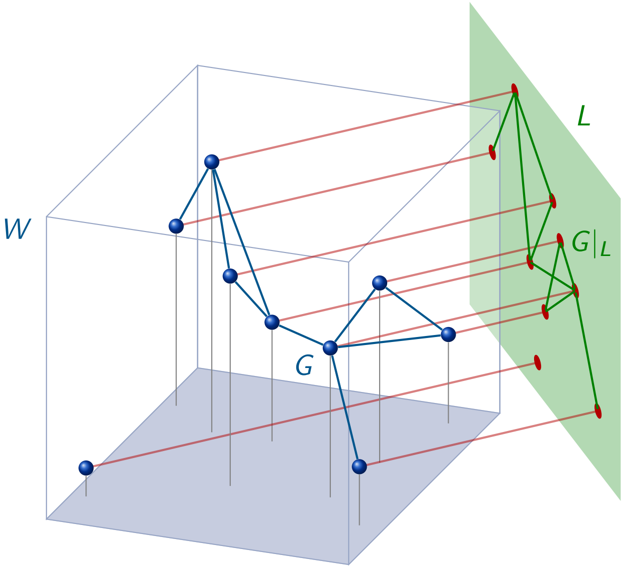

G_0 = abstract graph of G: $\mathrm{cr}(G_0) \le \overline{\mathrm{cr}}(G_0) \le \overline{\mathrm{cr}}(G|_L)$

Crossing number

Number of crossings in G after projecting onto L

line segment after projection on L

$$\overline{\text{cr}}(G|_L) = \frac{1}{8} \sum_{(v_1,v_2,v_3,v_4)\in V^4_{\neq}} \mathbb{1}\big([v_1, v_2]|_L \cap [v_3, v_4]|_L \neq \emptyset, \|v_1 - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t\big)$$

is **U-statistic of order** 4

Mean crossing number

$$\mathbb{E}_V \, \overline{\mathrm{cr}}(G_L)$$

$$= \frac{1}{8}\mathbb{E}_V \sum_{(v_1,v_2,v_3,v_4)\in V^4_{\neq}} \mathbb{1}\big([v_1, v_2]|_L \cap [v_3, v_4]|_L \neq \emptyset,\ \|v_1 - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t\big)$$

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by Multivariate Slivnyak-Mecke

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By Fubini and substitution, the indicator equals 1 if

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- v_4 lies in a ball of radius δ_t around v_3: $\sim \delta_t^d$

$$\sim \delta_t^{2d+2}$$

Mean crossing number ...more precisely

$$\mathbb{E}_V \,\overline{\mathsf{cr}}(G_L) = \frac{1}{8} c_d \, t^4 \delta_t^{2d+2} \int\limits_{W|_L} \lambda_{d-2}((v + L^\perp) \cap W)^2 \, dv + o(\delta_t^{2d+2} t^4),$$

where $c_d = 8\pi\kappa_{d-2}^2 \,\mathbf{B}\big(3, \frac{d}{2}\big)^2$

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expected number of vertices $\mathbb{E}_V n = t$
expected number of edges $\mathbb{E}_V m = \frac{\kappa_d}{2} t^2 \delta_t^d + \mathcal{O}(t^2 \delta_t^{d+1} \mathrm{surf(W)})$

For G_0 the abstract graph of G, we heuristically have

Crossing Lemma — density

$$c \cdot \frac{m^3}{n^2} \leq \mathrm{cr}(G_0) \leq \overline{\mathrm{cr}}(G_0) \leq \mathbb{E}_V \, \overline{\mathrm{cr}}(G|_L) \leq C \cdot \frac{m^3}{n^2} \cdot \left(\frac{m}{n^2} \right)^{\frac{2-d}{d}}$$

Mean crossing number and LLN

Corollaries (Chimani, HD, Reitzner, 2018)

- A random geometric graph G in $\mathbb{R}^2$ **is** an **expected constant-factor** approximation for $\mathrm{cr}(G_0)$ and $\overline{\mathrm{cr}}(G_0)$.

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- Let d and density m/n^2 fixed. Picking **any** projection plane L for a random geometric graph in $\mathbb{R}^d$ yields an **expected constant-factor** approximation for $\mathrm{cr}(G_0)$ and $\overline{\mathrm{cr}}(G_0)$.

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Corollary (Chimani, HD, Reitzner, 2018) **law of large numbers**:
For given L, the normalized random crossing number converges in prob. (with resp. to the PPP V) as $t \to \infty$,

$$\frac{\overline{\mathrm{cr}}(G_L)}{t^4 \delta_t^{2d+2}} \to \frac{1}{8} c_d \lambda_{d-2}((v + L^\perp) \cap W).$$

Crossing point process

- The *crossing point process* is the random measure ξ_t defined for Borel sets $A \subset L$ by

$$\xi_t(A) = \frac{1}{8} \sum_{(v_1,v_2,v_3,v_4)\in V^4_{\neq}} \mathbb{1}([v_1, v_2]|_L \cap [v_3, v_4]|_L \cap A \neq \emptyset) \cdot \mathbb{1}(\|v_1 - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t).$$

- scaling: $t^2\delta_t^{d+1} \overset{t\to\infty}{\longrightarrow} c > 0$ part of sparse regime

Theorem (HD, de Jonge, 2024+)
Let $t^2\delta_t^{d+1} \to c > 0$. Then there exists a Poisson point process ζ on L with finite intensity measure such that

$$d_{KR}(\xi_t, \zeta) = \mathcal{O}(\delta_t) + \mathcal{O}(c^2 - t^4\delta_t^{2d+2}).$$

Convergence in distribution of ξ_t to ζ follows.

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with Kantorovich–Rubinstein distance

$$\begin{aligned} d_{KR}(\xi, \zeta) &= \inf_{\mathbf{C} \text{ a coupling of } \xi \text{ and } \zeta} \int d_{TV}(\omega_1, \omega_2)\mathbf{C}d(\omega_1, \omega_2) \\ &\geq d_{TV}(\xi, \zeta) \end{aligned}$$

Proof idea

Intensity measure of ξ_t: $\mathbf{M}_t(A) := \mathbb{E}\xi_t(A)$.

Intensity measure of ζ: $\mathbf{M}(A) := \frac{1}{8} c_d c^2 \int_A \lambda_{d-2}((v + L^\perp) \cap W)^2 dv$.

- Intensity measure converges:

$$d_{TV}(\mathbf{M}_t, \mathbf{M}) = \mathcal{O}(\delta_t) + \mathcal{O}(c^2 - t^4 \delta_t^{2d+2})$$

- Difference of variance and expectation converges to zero:

$$\mathbb{V}\xi_t(L) - \mathbb{E}\xi_t(L) = \mathcal{O}(\delta_t)$$

- Apply [Decreusefond, Schulte, Thäle '16]:

$$\begin{aligned} d_{KR}(\xi_t, \zeta) &\leq d_{TV}(\mathbf{M}_t, \mathbf{M}) + 2(\mathbb{V}\xi_t(L) - \mathbb{E}\xi_t(L)) \\ &= \mathcal{O}(\delta_t) + \mathcal{O}(c^2 - t^4 \delta_t^{2d+2}). \end{aligned}$$

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$$\text{stress}(G) := \frac{1}{2} \sum_{(v_1,v_2)\in V^2_{\neq}} \underbrace{w(v_1,v_2)}_{\text{often } \frac{1}{d_0(v_1,v_2)^2}} \cdot \Big(\underbrace{d_0(v_1,v_2)}_{\text{desired (graph-theoretic?) distance}} - \underbrace{d_1(v_1,v_2)}_{\text{distance in drawing}} \Big)^2$$

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$$\text{stress}(G) := \frac{1}{2} \sum_{(v_1,v_2)\in V^2_{\neq}} \underbrace{w(v_1,v_2)}_{\text{often } \frac{1}{d_0(v_1,v_2)^2}} \cdot \Big(\underbrace{d_0(v_1,v_2)}_{\text{desired (graph-theoretic?) distance}} - \underbrace{d_1(v_1,v_2)}_{\text{distance in drawing}} \Big)^2$$

Find low-stress drawings via **Multidimensional Scaling (MDS)**:

1. Embed graph in high dimensional space, satisfying the distances
2. Seek a projection to minimize stress ← good algorithms!

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If stress and crossing number positively correlated

$\Rightarrow$ MDS yields crossing number approximations?!

Not really (graph-theoretic $\neq$ our geometric distances), but close.

Stress

- stress of the projected random geometric graph

$$\begin{aligned}\mathrm{stress}(G) &:= \frac{1}{2} \sum_{(v_1,v_2)\in V^2_{\neq}} \frac{1}{d_0(v_1,v_2)^2} \cdot \big(d_0(v_1,v_2) - d_1(v_1,v_2)\big)^2 \\ &= \frac{1}{2} \sum_{(v_1,v_2)\in V^2_{\neq}} \left(1 - \frac{d_1(v_1,v_2)}{d_0(v_1,v_2)}\right)^2 \\ &= \frac{1}{2} \sum_{(v_1,v_2)\in V^2_{\neq}} \left(1 - \frac{d_E(v_1|_L, v_2|_L)}{d_E(v_1,v_2)}\right)^2\end{aligned}$$

Positive Correlation ... for fixed L

- $\overline{\mathrm{cr}}$ and stress are both U-statistics and increasing, i.e. $D_v\big(\overline{\mathrm{cr}}(G)\big) \geq 0$ and $D_v\big(\mathrm{stress}(G)\big) \geq 0$
- Harris-FKG inequality [Fortuin–Kasteleyn–Ginibre (1971)]:

$$\mathbb{E}f(\eta)g(\eta) \geq \mathbb{E}f(\eta) \cdot \mathbb{E}g(\eta),$$

  if $f, g \in L^2(\mathbb{P}_\eta)$ are increasing.

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Theorem (Chimani, HD, Reitzner, 2018)
Let $G|_L$ be the projection of an RGG in $\mathbb{R}^d$, $d \geq 3$, onto a two-dimensional plane L. Assume that $\text{stress}(G) \in L^2$. Then

$$\mathbb{E}_V\, \overline{\text{cr}}(G|_L)\text{stress}(G) \geq \mathbb{E}_V\, \overline{\text{cr}}(G|_L)\, \mathbb{E}_V\text{stress}(G)$$

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- Thus the correlation of $\overline{\text{cr}}$ and stress is positive. It can be calculated explicitly. *similar result for random L for W rotational inv.*

Multivariate CLT

Theorem (HD, de Jonge, 2024+) For the covariance matrix Σ,

$$\left(\frac{\overline{\mathrm{cr}}(G|_L) - \mathbb{E}\,\overline{\mathrm{cr}}(G|_L)}{t^{7/2}\delta_t^{2d+2}}, \frac{\mathrm{stress}(G, G_L) - \mathbb{E}\mathrm{stress}(G, G_L)}{t^{3/2}} \right) \xrightarrow{d} N \sim \mathcal{N}(0, \Sigma)$$

as $t \to \infty$ in the thermodynamic regime $t\delta_t^d \to c$.

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Proof: Apply Malliavin-Stein method, in particular [Schulte, Yukich '19]:
$\mathcal{H}_m^{(3)}$ = class of all C^3-functions $h : \mathbb{R}^m \to \mathbb{R}$ such that the absolute values of the 2nd and 3rd partial derivatives are bounded by 1.
Let $F = (F_1, F_2)$ be a vector of Poisson functionals with $EF_i = 0$ and

$$\begin{aligned} d_3(F, Z) &:= \sup_{h \in \mathcal{H}_m^{(3)}} |\mathbb{E}h(F) - \mathbb{E}h(Z)| \\ &\leq \sum_{i,j=1}^{2} |\sigma_{ij} - \text{Cov}(F_i, F_j)| + 2\gamma_1 + \gamma_2 + \gamma^3 \end{aligned}$$

for Z a 2-dim. centered Gaussian random vector with cov. matrix $(\sigma_{ij})_{i,j}$.

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$$\gamma_1 = t^3 \left(\sum_{i,j=1}^{2} \int_{W^3} \sqrt{\mathbb{E}(D^2_{x_1,x_3} F_i)^2 (D^2_{x_2,x_3} F_i)^2} \cdot \sqrt{(\mathbb{E}(D_{x_1} F_j)^2 (D_{x_2} F_j)^2} \lambda_d^3(d(x_1, x_2, x_3)) \right)^{1/2},$$

$$\gamma_2 = t^3 \left(\sum_{i,j=1}^{2} \int_{W^3} \sqrt{\mathbb{E}(D^2_{x_1,x_3} F_i)^2 (D^2_{x_2,x_3} F_i)^2} \cdot \sqrt{\mathbb{E}(D^2_{x_1,x_3} F_j)^2 (D^2_{x_2,x_3} F_j)^2} \lambda_d^3(d(x_1, x_2, x_3)) \right)^{1/2},$$

$$\gamma_3 = t \sum_{i=1}^{2} \int_W \mathbb{E}|D_x F_i|^3 \lambda(dx) \qquad \text{with } D_x F(V) := F(V \cup \{x\}) - F(V).$$

$F_1 = \frac{\overline{\mathrm{cr}}(G|_L) - \mathbb{E}\,\overline{\mathrm{cr}}(G|_L)}{t^{7/2}\delta_t^{2d+2}}$ and $F_2 = \frac{\mathrm{stress}(G,G_L) - \mathbb{E}\mathrm{stress}(G,G_L)}{t^{3/2}}$:

$2\gamma_1 + \gamma_2 + \gamma^3 = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$

Wrapping up...

Summary (Markus Chimani, H. D., Matthias Reitzner 2018)
For a **random geometric graph** with $\lim_{t\to\infty} t\delta_t^d = c, \ldots$

1. . . . a trivial projection yields an **expected** crossing number approximation **with high probability**.
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Thank you for your attention!