Path distances in the Manhattan Poisson Line Cox Process

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Outline

1. Poisson point and line processes and Cox processes
2. Spatial model
3. Path distance to a typical intersection
4. Path distance to a typical point
5. k nearest neighbour path distance to a typical intersection

Motivation

Suppose we want to answer the question:

How far to the nearest facility in a road network?

- We could consider Euclidean ("crow flies") distance but it is more natural to measure path distance, along the roads.
- The answer will inform urban planning (the road network) as well as the facilities, which could be static (eg charging points) or mobile (eg taxis). There are also applications in industrial contexts.
- Other relevant research includes stochastic geometry, spatial networks, communications.

Poisson point processes

In the following $A \subset \mathbb{R}^d$ is a Borel set.

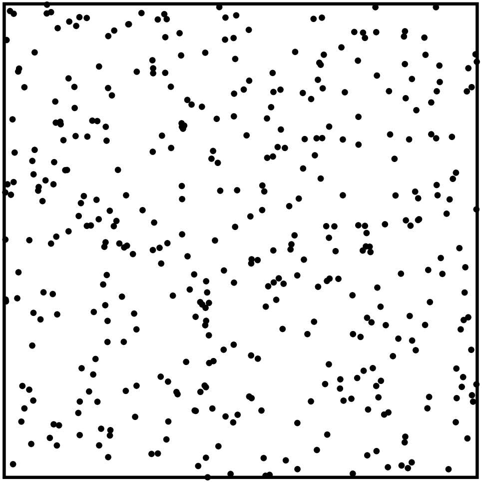
A **point process** Φ refers either to a random point set or to the random measure for which $\Phi(A)$ gives the number of points in A. $\Lambda = \mathbb{E}(\Phi)$ is called the **intensity measure** of the point process.

A **Poisson point process** (PPP) Φ with intensity measure Λ is a point process satisfying

1. $\mathbb{P}(\Phi(A) = k) = \frac{e^{-\Lambda(A)}\Lambda(A)^k}{k!}$
2. $\{\Phi(A_i)\}_{1 \leq i \leq n}$ are independent if $\{A_i\}$ are disjoint.

We assume that Λ is σ-finite and diffuse, which implies that the above two statements are equivalent.

PPP: Illustration



Conditioning Poisson point processes

A stochastic process is **stationary** if the distribution is invariant under translation.

A stationary PPP has intensity measure $\Lambda = \lambda \nu_d$ where $\lambda \in \mathbb{R}$ is the intensity and ν_d is d-dimensional Lebesgue measure.

Slivnyak's theorem: For a stationary PPP, conditioning on a point at the origin is equivalent to adding a point at the origin.

To simulate a PPP numerically: Choose a bounded region A, defining the probability measure Λ_A as

$$\Lambda_A(B) = \frac{\Lambda(A \cap B)}{\Lambda(A)}$$

The **binomial point process** (BPP) is a fixed number n points iid with respect to Λ_A. For the PPP, choose $n = \Phi(A) \sim \mathrm{Poi}(\Lambda(A))$.

Poisson line processes

Now, consider a PPP on the space

$$\rho \in \mathbb{R}, \qquad \theta \in [0, \pi)$$

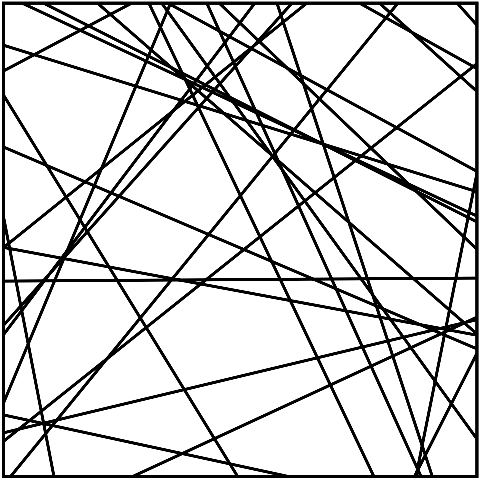
where each point (ρ, θ) is mapped to the line in $\mathbb{R}^2$ with closest point to the origin given by (ρ, θ) in polar coordinates, that is,

$$\frac{y - \rho \sin \theta}{x - \rho \cos \theta} = -\cot \theta$$

It can be shown that if the PPP is stationary, then so is the corresponding **Poisson line process** (PLP).

Not considered here, but related: For $\rho \in [-R, R]$ and fixed number of lines: The **binomial line process**

Poisson line process: Illustration



Manhattan Poisson line processes

Simpler (and still realistic): Consider a PPP on the space

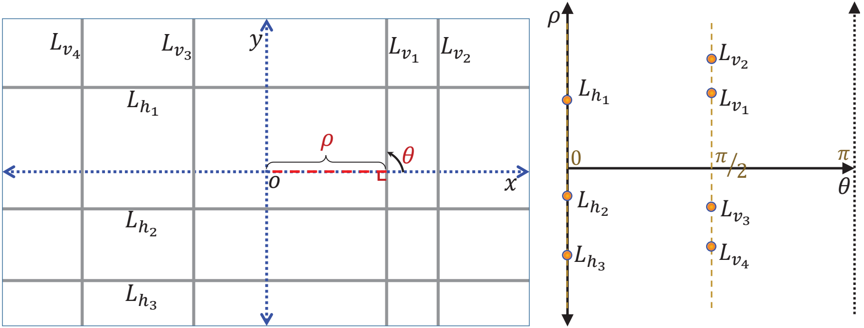
$$\rho \in \mathbb{R}, \qquad \theta \in \{0, \frac{\pi}{2}\}$$

with the same mapping as the PLP.

If the PPP has a uniform intensity λ_l on each of the two lines, the result is equivalent to the union of horizontal and vertical lines, with coordinates each given by a one dimensional PPP: The **Manhattan Poisson line process** (MPLP).

The line density μ_l is the mean line length per unit area and is given by $\mu_l = 2\lambda_l$.

MPLP: Illustration



Manhattan Poisson line Cox process

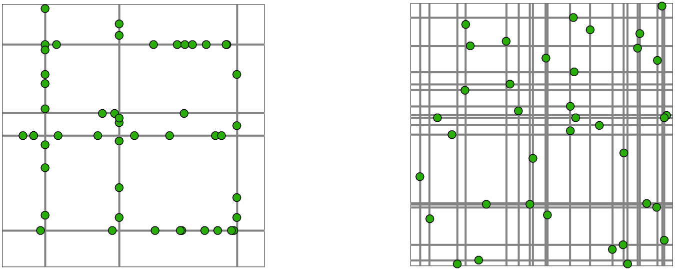
A **Cox process** is a doubly stochastic Poisson process, that is, the intensity measure is itself random.

The **Manhattan Poisson line Cox process** (MPLCP) is the PPP with intensity measure given by the MPLP.

Choose horizontal and vertical lines with coordinates given by independent one dimensional PPPs of intensity λ_l. Then on each line, form a new PPP of intensity λ_c, uniform and independent of the PPP forming the MPLP and of the PPP on the other lines.

The lines in this process represent roads, and the points, facilities.

MPLCP: Illustration and limits



When $\lambda_c/\lambda_l \to \infty$ (left), the

Typical point has local neighbourhood a PPP of intensity λ_c on a line.

Typical intersection has local neighbourhood a PPP of intensity λ_c on two orthogonal lines.

When $\lambda_c/\lambda_l \to 0$ (right), the point process approaches a PPP in the plane with intensity $2\lambda_l\lambda_c$.

Related work

- Guttorp and Thorarinsdottir (ISR, 2012) give early history of Poisson and Cox processes.
- PLP and MPLP: Goudsmit (RMP, 1945).
- k-nearest neighbour distributions, binomial point processes: Evans, Jones and Schmidt (2002).
- Cox processes for Poisson-Voronoi tessellations and PLP: Voss, Gloaguen and Schmidt (2010).
- MPLCP: Baccelli and Zhang (2015).
- Cox processes for PLP, MPLP, stick and lilypond models: Jeyaraj and Haenggi (2021).
- Binomial line Cox process: Shah, Ghatak, Sanyal and Haenggi (2024).

All the above use either Euclidean distance, or a mixed model with different line-of-sight or non-line-of sight signal attenuation.

Spatial model and notation

Def: **Shortest path distance** $\ell(\mathbf{a}, \mathbf{b})$ is the smallest sum of distances along the lines in the process between points **a** and **b**.

The MPLP is $\Phi_l = \Phi_{lh} \cup \Phi_{lv}$. In addition, we denote the x-axis as L_x and the y-axis as L_y.

Due to Slivnyak's theorem, the origin is a

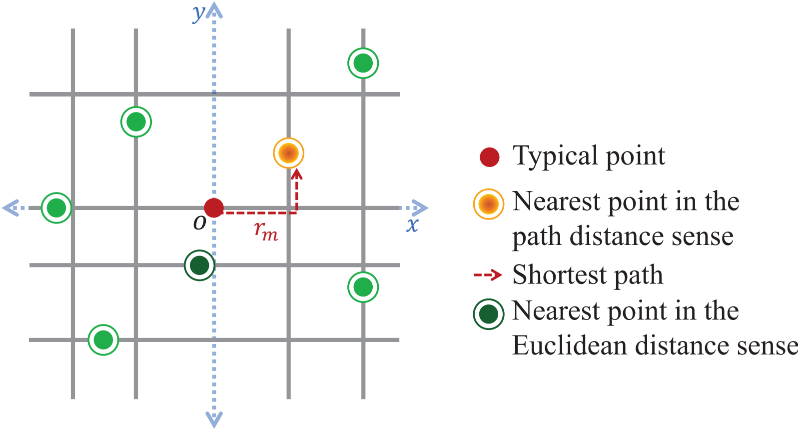
Typical point for $\Phi_{l,typ} = \Phi_l \cup \{L_x\}$, and

Typical intersection for $\Phi_{l,int} = \Phi_l \cup \{L_x, L_y\}$.

The MPLCP is denoted $\Phi_{c,typ}$, $\Phi_{c,int}$ respectively.

We seek the distribution of the shortest path distance from the origin to the nearest point in the MPLCP in each of these cases.

Spatial model: Illustration



The typical intersection

The shortest path distance is

$$T_m = \inf_{(x_i, y_i) \in \Phi_{c,int}} \ell(0, (x_i, y_i))$$

with

$$\ell(0, (x_i, y_i)) = |x_i| + |y_i|$$

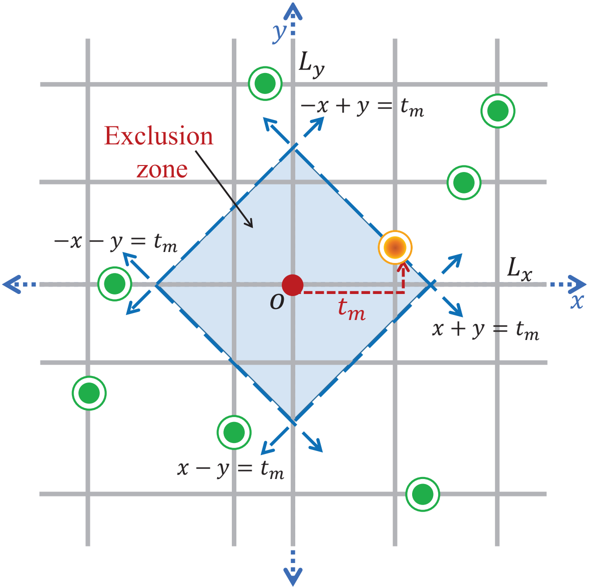
Thus $\mathbb{P}(T_m > t_m) = \mathbb{P}(\Phi_{c,int}(B) = 0)$ with

$$B = \{(x, y) : |x| + |y| < t_m\}$$

The number of horizontal and vertical lines intersecting B are distributed as

$$N_h(B), N_v(B) \sim \mathrm{Po}(2\lambda_l t_m)$$

The typical intersection: Illustration



The typical intersection: Result

Theorem 1 The CDF of T_m is

$$F_{T_m}(t_m) = 1 - \exp\left[-4\lambda_c t_m - 4\lambda_l t_m + \frac{2\lambda_l}{\lambda_c}(1 - e^{-2\lambda_c t_m})\right]$$

Proof ideas:

- Horizontal and vertical lines are iid.
- Condition on $N_h(B)$.
- The number of points on horizontal lines in $\{L_x\} \cup \Phi_{lh}$ are independent
- The void probability for the 1D PPP on a line of length $2t_m - 2y$ is $\exp(-\lambda_c(2t_m - 2y))$.

The typical intersection: Proof

$$
\begin{aligned}
F_{T_m}(t_m) &= 1 - \mathbb{P}(T_m > t_m) \\
&= 1 - \mathbb{P}(N_p(\Phi_{l,int} \cap B) = 0) \\
&= 1 - p^2 \\
p &= \mathbb{P}(N_p((\Phi_{lh} \cup L_x) \cap B) = 0) \\
&= q \sum_{n=0}^{\infty} \mathbb{P}(N_h(B \backslash L_x) = n) \mathbb{P}(N_p(\Phi_{lh} \cap B) = 0 | N_h(B \backslash L_x) = n) \\
&= q \sum_{n=0}^{\infty} \mathbb{P}(N_h(B \backslash L_x) = n) \prod_{j=1}^{n} \mathbb{P}(N_p(\Phi_{h_j} \cap B) = 0) \\
&= q \sum_{n=0}^{\infty} \frac{e^{-2\lambda_l t_m}(2\lambda_l t_m)^n}{n!} \left(\int_0^{t_m} e^{-\lambda_c(2t_m - 2y)} \frac{dy}{t_m} \right)^n \\
q &= \mathbb{P}(N_p(L_x \cap B) = 0) \\
&= e^{-2\lambda_c t_m}
\end{aligned}
$$

The typical intersection: Limits

We have from Theorem 1:

$$F_{T_m}(t_m) = 1 - \exp\left[-4\lambda_c t_m - 4\lambda_l t_m + \frac{2\lambda_l}{\lambda_c}(1 - e^{-2\lambda_c t_m})\right]$$

When $\frac{\lambda_c}{\lambda_l} \to \infty$,

$$F_{T_m}(t_m) \approx 1 - \exp(-4\lambda_c t_m)$$

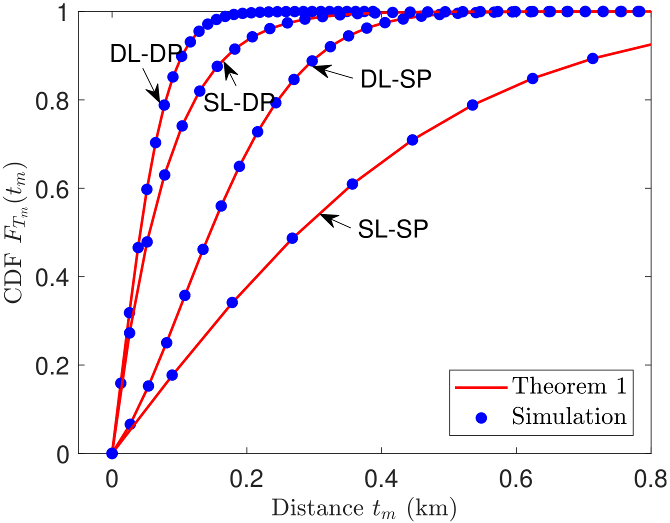
as expected for two perpendicular lines with a PPP of intensity λ_c.

When $\frac{\lambda_c}{\lambda_l} \to 0$, expanding the inner exponential gives

$$F_{T_m}(t_m) \approx 1 - \exp(-4\lambda_l \lambda_c t_m^2)$$

as expected for a PPP in the plane of intensity $2\lambda_l\lambda_c$ and using Manhattan distance.

The typical intersection: Numerics

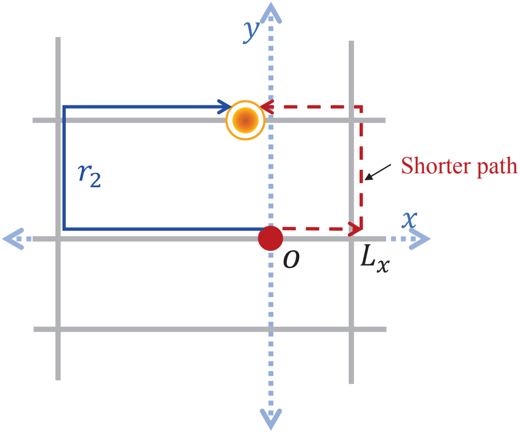


DL: $\lambda_l = 10$, SL: $\lambda_l = 1$, DP: $\lambda_c = 3$, SP: $\lambda_c = 0.5$

The typical point

Now the path distance is denoted R_m.

- The shortest path may back-track in the x-direction
- The shortest distance is not just the L^1 norm



The typical point: Definitions

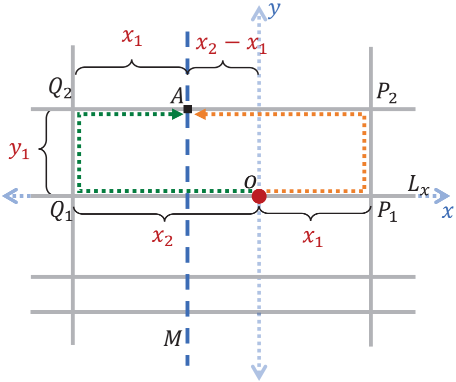
Distance to nearest intersections along the x-axis, S_l, S_r:

$$f_{S_i}(s_i) = \lambda_l \exp(-\lambda_l s_i), \quad 0 \leq s_i < \infty, \quad i \in \{l, r\}$$

Minimum and maximum:

$$f_{X_1}(x_1) = 2\lambda_l \exp(-2\lambda_l x_1) \qquad X_1 = \min\{S_l, S_r\}$$

$$f_{X_2}(x_2) = 2\lambda_l \exp(-\lambda_l x_2)(1 - \exp(-\lambda_l x_2)) \qquad X_2 = \max\{S_l, S_r\}$$



The typical point: More definitions

Distance to nearest point of the MPLCP on L_x in direction X_j: D_j:

$$f_{D_j}(d_j) = \lambda_c \exp(-\lambda_c d_j), \quad j \in \{1, 2\}$$

Distance to the nearest point of the MPLCP on the same side of the auxiliary line M as X_j: $R_j = X_j + W_j$

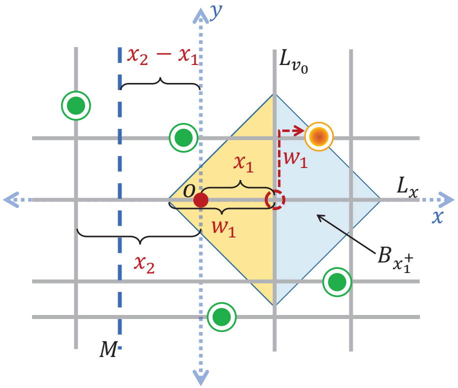
Events:

- $\mathcal{E}_1 : D_1 \leq X_1, D_2 > X_2 \qquad R_m = D_1$
- $\mathcal{E}_2 : D_1 \leq X_1, D_2 \leq X_2 \qquad R_m = \min(D_1, D_2)$
- $\mathcal{E}_3 : D_1 > X_1, D_2 > X_2 \qquad R_m = \min(R_1, R_2)$
- $\mathcal{E}_4 : D_1 > X_1, D_2 \leq X_2 \qquad R_m = \min(R_1, D_2)$

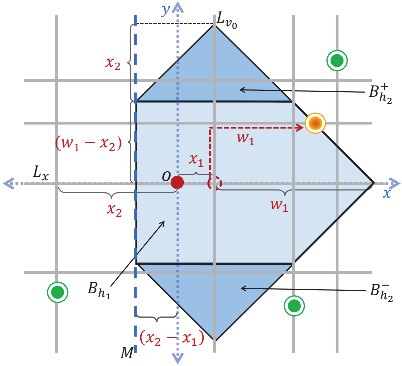
The calculations for D_1 and D_2 are relatively straightforward.

The typical point: Exclusion zones

$w_1 \leq x_2$



$w_1 > x_2$



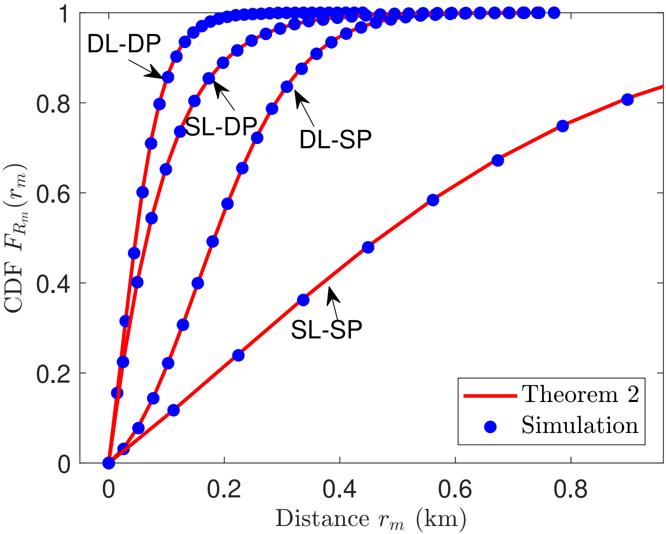
The typical point: Result

Theorem 2 The CDF of R_m is

$$F_{R_m}(r_m) = \int_0^\infty \int_0^\infty \sum_{i=1}^{4} F_{R_m}(r_m | \mathcal{E}_i, X_1, X_2) \times \mathbb{P}(\mathcal{E}_i | X_1, X_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

where each of the above nine functions is given explicitly in terms of exponential functions. The F_{R_m} are defined piecewise for r_m in the intervals $[0, x_1)$, $[x_1, x_2)$, $[x_2, x_1 + x_2)$, $[x_1 + x_2, \infty)$.

The typical point: Numerics



DL: $\lambda_l = 10$, SL: $\lambda_l = 1$, DP: $\lambda_c = 5$, SP: $\lambda_c = 0.5$

Typical intersection for k nearest neighbours

For the typical intersection, we seek the CDF of the distance to the kth nearest neighbour

$$F_{R_k}(r) = 1 - \sum_{j=0}^{k-1} P_j$$

in terms of the probability of k facilities in B

$$P_k = \int_{4r}^{\infty} \frac{e^{-\lambda_c l}(\lambda_c l)^k}{k!} f_{L_t}(l) dl$$

in turn written in terms of the probability density of total length of road in B

$$L_t = 4r + \sum_{i=1}^{N} L, \qquad L \sim \mathrm{Unif}[0, 2r], \qquad N \sim \mathrm{Po}(4\lambda_l r)$$

Typical intersection for k nearest neighbours: Result

Theorem 3 The moment generating function of the number of points of the Manhattan Poisson Line Cox Process in the r-radius Manhattan ball from the typical intersection $B(r)$ is

$$\mathbb{M}_{N_p(\Phi_{l,int}\cap B)}(t) = \exp\left(4r\lambda_c(e^t-1) + 4\lambda_l r\left(\frac{e^{2r\lambda_c(e^t-1)}-1}{2r\lambda_c(e^t-1)} - 1\right)\right)$$

from which (by differentiation) we find

$$P_0 = \exp(-4r\lambda_c - 4\lambda_l r(1-a))$$

$$P_1 = 4P_0(r\lambda_c + r\lambda_l(a - e^{-2r\lambda_c}))$$

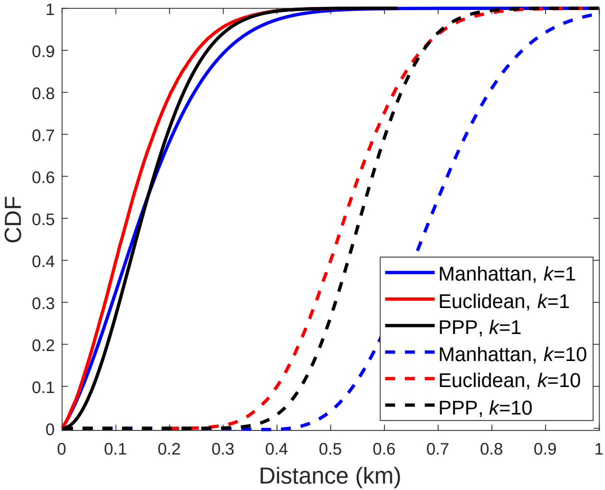
$$P_2 = 4P_0\left[2(r\lambda_c + r\lambda_l(a - e^{-2r\lambda_c}))^2 + \lambda_l r(a - e^{-2r\lambda_c} - r\lambda_c e^{-2r\lambda_c})\right]$$

$$a = \frac{1-e^{-2\lambda_c r}}{2\lambda_c r}$$

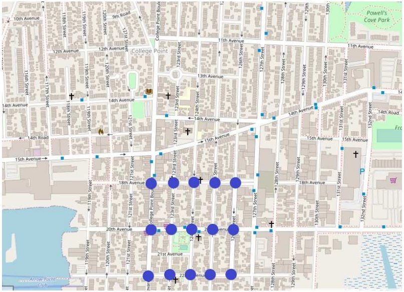
Typical intersection for k nearest neighbours: Proof

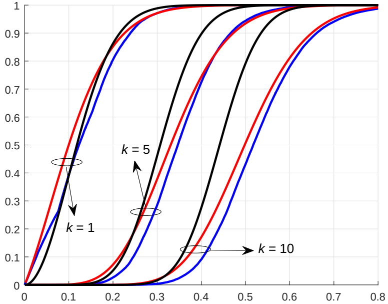
$$\begin{aligned}
\mathbb{M}_{L_t}(t) &= \mathbb{E}(e^{tL_t}) \\
&= \mathbb{E}_N(\mathbb{E}(e^{tL_t}|N)) \\
&= \mathbb{E}_N(e^{4rt}\mathbb{M}_L(t)^N) \\
&= e^{4rt}\mathbb{E}_N\left(\left(\frac{e^{2rt}-1}{2rt}\right)^N\right) \\
&= \exp\left(4rt + 4\lambda_l r\left(\frac{e^{2rt}-1}{2rt}-1\right)\right) \\
\mathbb{M}_{N_p(\Phi_{l,int}\cap B)}(t) &= \mathbb{E}(e^{N_p(\Phi_{l,int}\cap B)t}) \\
&= \mathbb{E}_{L_t}\left(\mathbb{E}(e^{N_p(\Phi_{l,int}\cap B)t}|L_t)\right) \\
&= \mathbb{E}_{L_t}(e^{\lambda_c(e^t-1)L_t}) \\
&= \mathbb{M}_{L_t}(\lambda_c(e^t-1))
\end{aligned}$$

Typical intersection for k nearest neighbours: Numerics



Typical intersection for k nearest neighbours: Real map





Location: New York City (source: openstreetmap.org).
Blue: Simulations. Red: Theory. Black: PPP model.
$\lambda_c = 1\ \mathrm{km}^{-1}$.

Summary

We have found the path distributions for a typical intersection, including for the kth distance and a typical point, with more accurate results than Euclidean distance or a PPP model.

Possible extensions:

- Other road layouts: PLP, perhaps with restrictions, random tessellations.
- Weighted networks (speed limits)
- Directed networks (one way streets)

Further reading

- G. Last, and M. Penrose, *Lectures on the Poisson process* Cambridge University Press (2017).
- S. N. Chiu, D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications* John Wiley & Sons (2013).
- V. V. Chetlur, H. S. Dhillon and C. P. Dettmann, *Shortest path distance in Manhattan Poisson line Cox process* J. Stat. Phys. **181**, 2109-2130 (2020).
- K. Koufos, H. S. Dhillon, M Dianati and C. P. Dettmann, *On the k nearest-neighbor path distance from the typical intersection in the Manhattan Poisson line Cox process* IEEE Trans. Mob. Comput. **22**, 1659-1671 (2023).