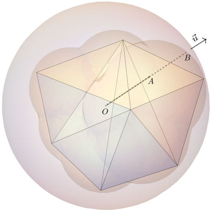
The support function of high-dimensional random polytopes



Pierre Calka

September 13, 2024, Bath

Stochastic Geometry in Action

UNIVERSITÉ DE ROUEN NORMANDIE

LMRS

Outline

Random polytopes in high dimension: state of the art

Results on the support function process

Sketch of proof

Joint work with **Benjamin Dadoun**

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Random polytopes in high dimension: state of the art

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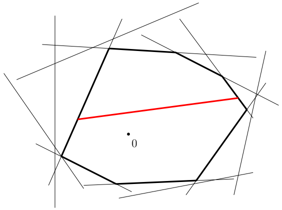
Conjectures of asymptotic convex geometry

Hyperplane conjecture

There exists a universal constant $c > 0$ such that for every d-dimensional convex body K with volume 1, there is a section of K with $(d-1)$-dimensional volume greater than c.

Reference. J. Hörrmann, D. Hug, M. Reitzner & C. Thäle (2015)

The zero-cell of a parametric class of hyperplane tessellations depending on a distance exponent bd^{α} for some $b > 0$ and $\alpha > 1/2$ satisfies the hyperplane conjecture with high probability.



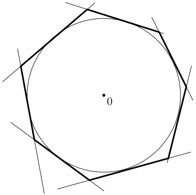
Conjectures of asymptotic convex geometry

Hirsch conjecture

The edge-vertex graph of the boundary of a d-dimensional convex polytope with n facets has a graph diameter at most $(n - d)$.

Reference. G. Bonnet, D. Dadush, U. Grupel, S. Huiberts and G. Livshyts (2022)

The zero-cell of a hyperplane tessellation generated by a Poisson number of points on the unit sphere satisfies the polynomial Hirsch conjecture with high probability.



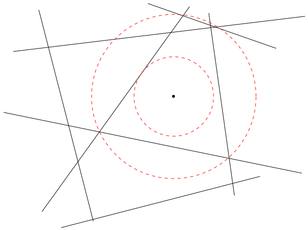
Information theory

One-bit compressed sensing

Recover a signal from the knowledge of belonging to half-spaces for sufficiently many half-spaces

Reference. F. Baccelli and E. O'Reilly (2019)

Asymptotic study of several characteristics of cells from a Poisson hyperplane tessellation with intensity ρd^{α} for some $\rho > 0$ and $\alpha \geq 0$ (inradius, circumscribed radius, mean volume)



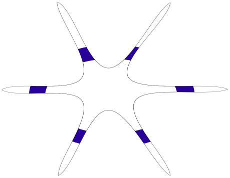
Information theory

Thin-shell concentration

The norm of a radial random vector with log-concave density is concentrated around its mean.

References. O. Guédon & E. Milman (2011), E. O'Reilly (2020)

The zero-cell of a stationary hyperplane tessellation satisfies the thin-shell concentration.

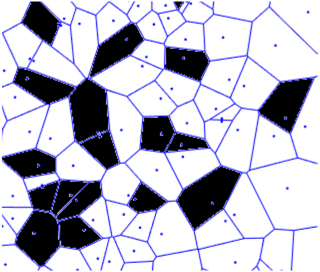


Percolation in high dimension

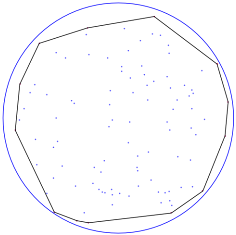
Poisson-Voronoi percolation

The critical probability has a precise equivalent for large d.

References. P. N. Balister & B. Bollobás (2010), R. Conijn, M. Irlbeck, Z. Kabluchko & T. Müller (2024+)



Random convex hull in the unit ball



- $\mathbb{B}^d :=$ d-dimensional unit ball of volume $\kappa_d := \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$
- $\mathcal{P}_\lambda :=$ homogeneous Poisson point process in $\mathbb{R}^d$ of intensity $\lambda := \lambda(d)$
- $K_\lambda^d :=$ convex hull of $\mathcal{P}_\lambda \cap \mathbb{B}^d$
- Mean number of Poisson points inside $\mathbb{B}^d = \lambda\kappa_d$

Random convex hull in the unit ball: volume

Reference. G. Bonnet, Z. Kabluchko & N. Turchi (2021)

$\mathrm{Vol}_d :=$ d-dimensional Lebesgue measure, $\kappa_d = \mathrm{Vol}_d(\mathbb{B}^d)$

$\log(\lambda\kappa_d)$	$\ll \frac{d}{2}\log d$	$\sim \frac{d}{2}\log\frac{d}{2x}$	$\gg \frac{d}{2}\log d$
$\lim\limits_{d,\lambda\to\infty} \mathbb{E}\frac{\mathrm{Vol}_d(K_\lambda^d)}{\mathrm{Vol}_d(\mathbb{B}^d)}$	0	e^{-x}	1

Extension to the beta polytope

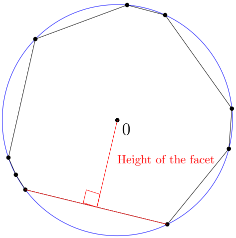
The uniform distribution in $\mathbb{B}^d$ is replaced by the distribution with density proportional to $(1-\|x\|^2)^\beta$ if $\beta > -1$ and the uniform distribution on the unit sphere if $\beta = -1$.

Random convex hull in the unit ball: typical facet height

Reference. G. Bonnet & E. O'Reilly (2022)

$P_n^d :=$ convex hull of n uniform points $X_1, \cdots, X_n$ on the unit sphere
$H_{\text{typ}} :=$ height of the facet generated by $X_1, \cdots, X_d$ conditional on the event $\{X_1, \cdots, X_d$ generates a facet$\}$

$\log(n)$	$\ll d$	$\sim xd$	$\gg d$
H_{typ}	$\overset{\mathbb{P}}{\sim} \sqrt{\frac{2}{d}\log(\frac{n}{d})}$	$\overset{\mathbb{P}}{\to} \sqrt{1-e^{-2x}}$	$\overset{\mathbb{P}}{\sim} 1$



Outline

Random polytopes in high dimension: state of the art

Results on the support function process

Sketch of proof

Support function and radius-vector function

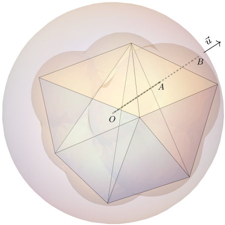
$K_\lambda^d :=$ convex hull of $\mathcal{P}_\lambda \cap \mathbb{B}^d$ with $\mathcal{P}_\lambda$ of intensity λ

Support function of K_λ^d

$h_\lambda^d(u) := \sup\{\langle x, u\rangle : x \in K_\lambda^d\}, \;\; u \in \mathbb{S}^{d-1}$

Radius-vector function of K_λ^d

$\rho_\lambda^d(u) := \sup\{r > 0 : ru \in K_\lambda^d\}, \;\; u \in \mathbb{S}^{d-1}$



Properties

The support function of K_λ^d is the radius-vector function of the flower $\bigcup_{x \in \mathcal{P}_\lambda \cap \mathbb{B}^d} B\left(\frac{x}{2}, \frac{\|x\|}{2}\right)$.

Both functions h_λ^d and r_λ^d characterize the convex polytope K_λ^d.

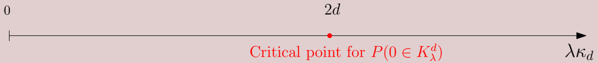
Containing the origin

Reminder on Wendel's calculation (1962)

$\{X_n : n \geq 1\} :=$ sequence of i.i.d. r.v. with a symmetric distribution

$$\mathbb{P}(0 \in \mathrm{Conv}(X_1, \cdots, X_N)) = \mathbb{P}(S_{N-1} \geq d)$$

where $S_{N-1} \stackrel{D}{=} \mathrm{Binomial}(n, \frac{1}{2})$.



If $\limsup_{d\to\infty} \frac{\lambda\kappa_d}{d} < 2$,

$$\lim_{d\to\infty} \mathbb{P}(0 \in K_\lambda^d) = 0$$

If $\liminf_{d\to\infty} \frac{\lambda\kappa_d}{d} > 2$,

$$\lim_{d\to\infty} \mathbb{P}(0 \in K_\lambda^d) = 1$$

The support function in one direction

Equivalent of $h_\lambda^d(u)$ when $d \to \infty$

$\log(\lambda\kappa_d)$	$\ll d$	$\sim dx$	$\gg d$
$h_\lambda^d(u)$	$\overset{\mathbb{P}}{\sim} \sqrt{\frac{2}{d}\log(\lambda\kappa_d)}$	$\overset{\mathbb{P}}{\to} \sqrt{1-e^{-2x}}$	$\overset{\mathbb{P}}{\sim} 1 - \frac{1}{2}(\lambda\kappa_d)^{-\frac{2}{d+1}}$

Convergence in distribution of the rescaled support function

$$d\left(\log \frac{1}{\sqrt{1-\left(h_\lambda^d(u)\right)^2}} - \frac{1}{d+1}\log \lambda\kappa_d\right) + \log\sqrt{\mathfrak{m}(d)} \xrightarrow{D} \text{Gumbel}$$

where

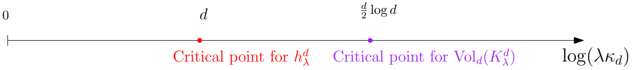
$\log(\lambda\kappa_d)$	$\ll d$	$\sim dx$	$\gg d$
$\mathfrak{m}(d)$	$4\pi\log(\lambda\kappa_d)$	$2\pi d(1-e^{-2x})$	$2\pi d\left(1-(\lambda\kappa_d)^{-\frac{2}{d+1}}\right)$

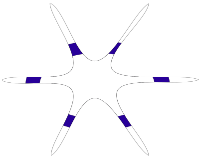
Comparison with the results on the mean volume

Reference. G. Bonnet, Z. Kabluchko & N. Turchi (2021)

$\log(\lambda\kappa_d)$	$\ll \frac{d}{2}\log d$	$\sim \frac{d}{2}\log\frac{d}{2x}$	$\gg \frac{d}{2}\log d$
$\lim\limits_{d,\lambda\to\infty} \mathbb{E}\frac{\mathrm{Vol}_d(K_\lambda^d)}{\mathrm{Vol}_d(\mathbb{B}^d)}$	0	e^{-x}	1

$\log(\lambda\kappa_d)$	$\ll d$	$\sim dx$	$\gg d$
$h_\lambda^d(u)$	$\overset{\mathbb{P}}{\sim} \sqrt{\frac{2}{d}\log(\lambda\kappa_d))}$	$\overset{\mathbb{P}}{\to} \sqrt{1-e^{-2x}}$	$\overset{\mathbb{P}}{\sim} 1-\frac{1}{2}(\lambda\kappa_d)^{-\frac{2}{d+1}}$

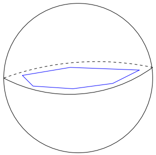




The support function in several directions

For $m \le d$ fixed,

$$h_\lambda^{d,m} := \inf_{u \in \mathbb{S}^{d-1} \cap \mathbb{R}^m} h_\lambda^d(u)$$



$\log(\lambda\kappa_d)$	$\ll d$	$\sim dx$	$\gg d$
$h_\lambda^{d,m}$	$\overset{\mathbb{P}}{\sim} \sqrt{\frac{2}{d}\log(\lambda\kappa_d))}$	$\overset{\mathbb{P}}{\to} \sqrt{1-e^{-2x}}$	$\overset{\mathbb{P}}{\sim} 1-\frac{1}{2}(\lambda\kappa_d)^{-\frac{2}{d+1}}$

$$\mathfrak{a}(d;m) - \mathfrak{b}(d;m)\log\frac{1}{\sqrt{1-\left(h_\lambda^{d,m}\right)^2}} \xrightarrow{D} \text{Gumbel}$$

where

$\mathfrak{a}(d;m) = (m-1)\mathfrak{s}(d)\log\frac{A_m\lambda\kappa_d}{\mathfrak{s}(d)} - (m-1)\mathfrak{s}(d)^2 - (m-1)\log\mathfrak{s}(d) - \log B_m$,

$\mathfrak{b}(d;m) = (m-1)d\,\mathfrak{s}(d)$ and $\mathfrak{s}(d) = \log\sqrt{d\left((\lambda\kappa_d)^{\frac{2}{d}}-1\right)}$.

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Results on the support function process

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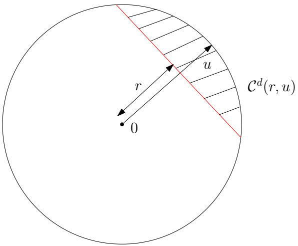
Distribution tail of the support function in one direction



$$\mathbb{P}\left(h_\lambda^d(u) \le r\right) = \mathbb{P}(\mathcal{P}_\lambda \cap \mathcal{C}^d(r,u) = \emptyset) = e^{-\lambda \mathrm{Vol}_d(\mathcal{C}^d(r,u))}$$
$$= \exp\left(-\frac{\lambda\kappa_{d-1}}{2}\mathrm{B}\left(1-r^2; \tfrac{d+1}{2}, \tfrac{1}{2}\right)\right),$$

where
$$\mathrm{B}(x;p,q) := \int_0^x v^{p-1}(1-v)^{q-1}\,\mathrm{d}v, \quad x \in (0,1),\ p,q > 0,$$
lower incomplete beta function

Asymptotics of the incomplete beta function

$$\mathrm{B}(x; p, q) := \int_0^x v^{p-1}(1-v)^{q-1}\,\mathrm{d}v, \quad x \in (0,1),\ p, q > 0$$

If $p \gg \dfrac{|q-1|x}{1-x}$, then

$$\mathrm{B}(x; p, q) = \frac{x^p(1-x)^{q-1}}{p}\left[1 + O\left(\frac{|q-1|x}{p(1-x)}\right)\right].$$

If $p \gg \dfrac{|q-1|}{1-x}$, then

$$\frac{\mathrm{B}(x; p, q)}{\mathrm{B}(1; p, q)} = \frac{x^p[(1-x)p]^{q-1}}{\Gamma(q)}\left[1 + O\left(\frac{|q-1|}{p(1-x)}\right)\right].$$

Reference. Temme (1996)

Proof of the one-dimensional result

$$\mathbb{P}\left(h_\lambda^d(u) \le r\right) = \exp\left(-\frac{\lambda\kappa_{d-1}}{2}\mathrm{B}\big(1-r^2; \frac{d+1}{2}, \frac{1}{2}\big)\right)$$

Step 1. Combine previous estimates with Stirling's formula to get

$$-\log\mathbb{P}\left(h_\lambda^d(u) \le r\right) = \exp\left(\log\lambda\kappa_d + \tfrac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} + o(1)\right)$$

Step 2. Calibrate $r := r(d,\tau)$ so that

$$\log\lambda\kappa_d + \frac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} = -\tau.$$

The calculation depends on $\lim_{d\to\infty}\frac{1}{d}\log\lambda\kappa_d$.

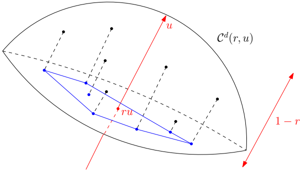
Bonus: consequence for the radius-vector function

$$\rho_\lambda^d(u) := \sup\{r > 0 : ru \in K_\lambda^d\}, \quad u \in \mathbb{S}^{d-1}, \quad u \in \mathbb{S}^{d-1}$$

Step 0. Naturally, $\rho_\lambda^d(u) \leq h_\lambda^d(u)$

Step 1. Geometric observation

If ru is in the convex hull of the projection of the Poisson points from the cap $\mathcal{C}^d(r,u)$ onto the cap basis, then $\rho_\lambda^d(u) \geq r$.



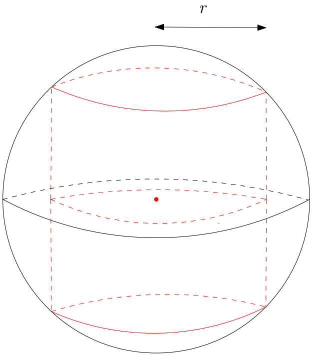
Step 2. Estimate such probability thanks to Wendel's formula.

When $\log(\lambda\kappa_d) \gg d$, $(\lambda\kappa_d)^{\frac{2}{d+1}}\left(1-\rho_\lambda^d(u)\right) \xrightarrow{\mathbb{P}} \frac{1}{2}$

The support function in several dimensions: geometric interpretation

For $m \leq d$ fixed, $h_\lambda^{d,m} := \inf_{u \in \mathbb{S}^{d-1} \cap \mathbb{R}^m} h_\lambda^d(u)$

Geometric condition for having $h_\lambda^{d,m} \geq r$?



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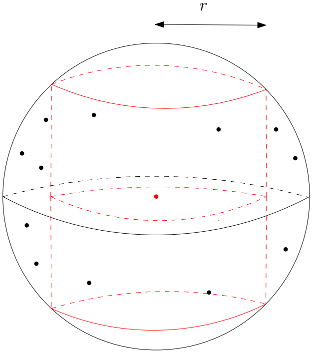

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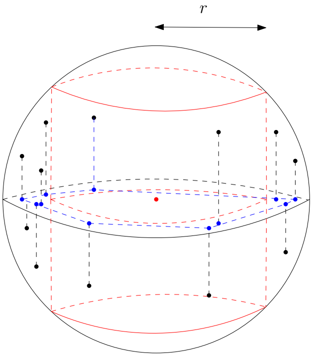

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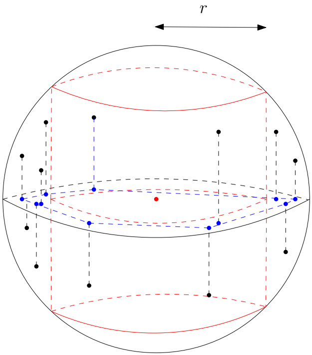
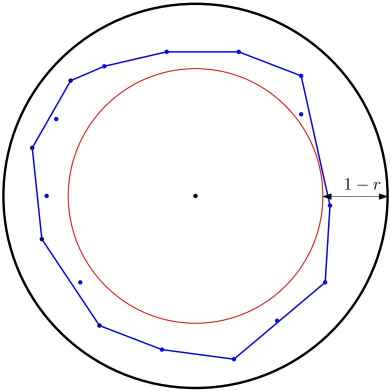
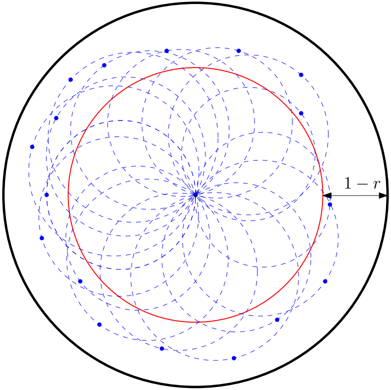


$h_\lambda^{d,m} \geq r \iff$ the min of the support function of the blue convex hull is $\geq r$

Link with the coverage of the sphere



Link with the coverage of the sphere



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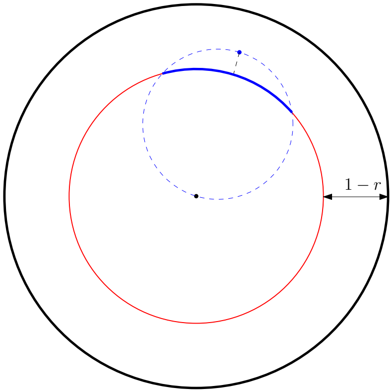

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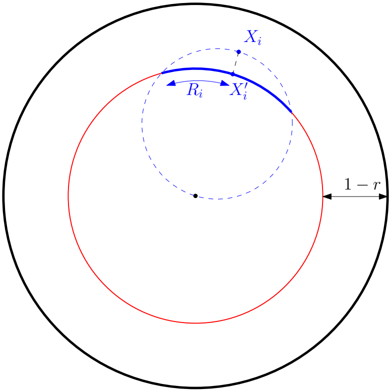

Link with the coverage of the sphere

Geometric observation

$h_\lambda^{d,m} \geq r$ iff the sphere $r\mathbb{S}^{m-1}$ is covered by random spherical caps

- with centers X_i', $i \geq 1$, at the projections of the points X_i, $i \geq 1$, from $\mathbb{B}^m \setminus r\mathbb{B}^m$ onto $r\mathbb{S}^{m-1}$
- with geodesic radii $R_i = \arccos \frac{r}{\|X_i\|}$, $i \geq 1$.

Probabilistic observation

- Each X_i is the projection of a point uniformly distributed into $\mathbb{B}^d \setminus \{x \in \mathbb{B}^d : x_1^2 + \cdots + x_m^2 \geq r^2\}$.
- The set of X_i' along the sphere $r\mathbb{S}^{m-1}$ is a homogeneous Poisson point process with explicit intensity $\underset{d\to\infty}{\longrightarrow} \infty$.
- The common distribution of the independent geodesic radii R_i is explicit and $R_i \underset{d\to\infty}{\longrightarrow} 0$.

Janson's covering result

Reference. Janson (1986)

Cover the sphere $\mathbb{S}^{m-1}$ with spherical patches

- whose centers are at points of a homogeneous Poisson point process along $\mathbb{S}^{m-1}$ of intensity Λ,
- whose geodesic radii are εR_i with $\{R_i, i \geq 1\}$ sequence of i.i.d. variables with fixed law and finite $(m+\eta)$-th moment,
- such that Λ and ε satisfy the asymptotic relation

$$c_1 \varepsilon^{m-1} \Lambda + (m-1)\log(\varepsilon) - (m-1)\log(-\log(\varepsilon)) + c_2 \longrightarrow \tau.$$

Then the probability of covering $\mathbb{S}^{m-1}$ converges to $e^{-e^{-\tau}}$.

Strategy

Step 1. Calculate the intensity Λ

$$\Lambda = \lambda \kappa_d \cdot \frac{\mathrm{B}\left(1 - r^2; 1 + \frac{d-m}{2}, \frac{m}{2}\right)}{\mathrm{B}\left(1 + \frac{d-m}{2}, \frac{m}{2}\right)}$$

Step 2. Calculate ε and the common distribution of the rescaled geodesic radii

$$\varepsilon = \frac{1}{\sqrt{d}} \frac{\sqrt{1 - r^2}}{r}$$

Step 3. Calibrate r so that the asymptotic relation is satisfied.

Strategy

Step 1. Calculate the intensity Λ

$$\Lambda = \lambda \kappa_d \cdot \frac{\mathrm{B}\left(1 - r^2; 1 + \frac{d-m}{2}, \frac{m}{2}\right)}{\mathrm{B}\left(1 + \frac{d-m}{2}, \frac{m}{2}\right)}$$

Step 2. Calculate ε and the common distribution of the rescaled geodesic radii

$$\varepsilon = \frac{1}{\sqrt{d}} \frac{\sqrt{1 - r^2}}{r}$$

Step 3. Calibrate r so that the asymptotic relation is satisfied.

Problem! The distribution of the rescaled geodesic radii still depends on ε.

Slight refinement of Janson's result

Keep the same assumptions as in Janson's theorem

BUT with geodesic radii equal to $\varepsilon R_i^{(\varepsilon)}$
where the r.v. $R_i^{(\varepsilon)}$ are i.i.d. and $\overset{D}{=} R^{(\varepsilon)}$.

Assume that there exists a r.v. R such that for every $w > 0$,

$$\sup_{\rho>0} \frac{\mathbb{P}(R^{(\varepsilon)} > \rho)}{\mathbb{P}\left(\left[1 + \frac{w}{\log\frac{1}{\varepsilon}}\right] R > \rho\right)} \leq 1$$

and

$$W_1\left(\log R^{(\varepsilon)}, \log R\right) \underset{\varepsilon\to 0}{=} o\left(\frac{1}{\log^2\frac{1}{\varepsilon}}\right).$$

Then the same conclusion for the coverage probability holds.

Thank you for your attention!