The anisotropy of 2D Gaussian random fields through their Lipschitz-Killing curvature densities

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Outlines

1. Geometrical functionals
   - Curvature measures
   - Geometry of smooth excursion sets
2. Mean geometry for stationary random fields excursion sets
   - Gaussian stationary random fields
   - Mean geometry and Lipschitz-Killing densities
   - Geometrical spectral moments and ratio of anisotropy
3. Statistical inference
   - Effective level
   - Effective Ratio of anisotropy

Curvature measures

Let $E \subset \mathbb{R}^2$ be a "nice set". Its curvature measures $\Phi_j(E, \cdot)$, for $j = 0, 1, 2$, are defined for any Borel set $U \subset \mathbb{R}^2$ by

- $\Phi_2(E, U) = |E \cap U|$, occupied area
- $\Phi_1(E, U) = \frac{1}{2}\mathcal{H}^1(\partial E \cap U) = \frac{1}{2}\mathrm{Per}(E, U)$, regularity property
- $\Phi_0(E, U) = \frac{1}{2\pi}\mathrm{TC}(\partial E, U)$, connectivity property

where $\mathcal{H}^1(\partial E \cap U)$ is the lenght and $\mathrm{TC}(\partial E, U)$ the total curvature of the positively oriented curve ∂E in U.

For E a compact or convex set and $E \subset U$ also related to Minkowski or intrinsic volumes, widely used in mathematical morphology, convex and integral geometry : Hadwiger (1957), Federer (1959), Santaló (1976), Schneider & Weil (2008),...

Total curvature and Euler characteristic

Theorem (Gauss-Bonnet)

Let $E \subset U$ be a regular region ie $E = \overline{\overset{o}{E}}$ such that $\partial E = \cup_{i=1}^{n} \Gamma_i$ is a finite union of disjoint positively oriented Jordan piecewise regular curves. then

$$\mathrm{TC}(\partial E, U) := \sum_{i=1}^{n} \mathrm{TC}(\Gamma_i, U) = 2\pi\chi(E) \;\; (= \; 2\pi\Phi_0(E, U)) \, ,$$

where $\chi(E) \in \mathbb{Z}$ is the **Euler characteristic** *of E.*

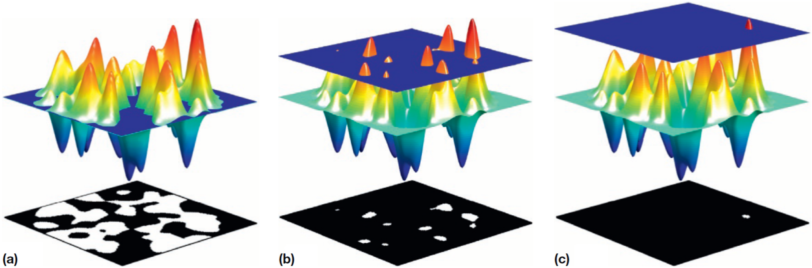
$$\chi(E) = \#\text{connected components} - \#\text{ holes}.$$

Excursion sets

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2. For $t \in \mathbb{R}$, we consider the excursion set of level t

$$E_f(t) := \{x \in \mathbb{R}^2; f(x) \geq t\}.$$

We assume it is observed through U a bounded **open rectangle**.



Credit : BrainMapping : an encyclopedic reference- Topological Inference

Implicit planar curves

Since f is continuous, we have for $t \in \mathbb{R}$

$$\partial E_f(t) = \{x \in \mathbb{R}^2; f(x) = t\},$$

corresponding to a level set of f. Hence if $\nabla f(x) \neq 0$, the unit vector $\nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}$ is the normal vector of $\partial E_f(t)$ at $x \in \partial E_f(t)$ with

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} \left[I_2 - \nu_f(x)\nu_f(x)^T \right] D^2 f(x),$$

where $D^2 f(x)$ is the Hessian matrix. It follows that the **signed curvature** at x is given by

$$\kappa_f(x) = -\langle \nu_f(x)^\perp, D\nu_f(x)\nu_f(x)^\perp \rangle = -\frac{1}{\|\nabla f(x)\|} \langle \nu_f(x)^\perp, D^2 f(x)\nu_f(x)^\perp \rangle.$$

Coarea formula

By Morse-Sard theorem, the image by f of the set of critical values of f has measure 0 in $\mathbb{R}$. For a.e. level $t \in \mathbb{R}$ and U open bounded,

$$\Phi_1(E_f(t), U) = \frac{1}{2} \int_{\partial E_f(t) \cap U} 1 \mathcal{H}^1(dx)$$

$$\Phi_0(E_f(t), U) = \frac{1}{2\pi} \int_{\partial E_f(t) \cap U} \kappa_f(x) \mathcal{H}^1(dx).$$

The **coarea formula** states that, for any borel function $g : \mathbb{R}^2 \to \mathbb{R}$ s.t $\int_U |g(x)| \|\nabla f(x)\| \, dx < +\infty$,

$$\int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) \, dt = \int_U g(x) \|\nabla f(x)\| \, dx.$$

Let us choose $h : \mathbb{R} \to \mathbb{R}$ a bounded continuous function (test function) such that multiplying $g(x)$ by $h(f(x))$ we get

$$\int_{\mathbb{R}} h(t) \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) \, dt = \int_U h(f(x)) g(x) \|\nabla f(x)\| \, dx.$$

Weak formula for Φ_1 and Φ_0

Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $\int_U |g(x)| \|\nabla f(x)\| \, dx < +\infty$, recall the **Coarea formula** :

$$\int_{\mathbb{R}} h(t) \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) \, dt = \int_U h(f(x)) g(x) \|\nabla f(x)\| \, dx.$$

Coarea formula with $g(x) = 1$:

$$\int_{\mathbb{R}} h(t) \Phi_1(E_f(t), U) dt = \frac{1}{2} \int_U h(f(x)) \|\nabla f(x)\| \, dx.$$

Coarea formula with $g(x) = \kappa_f(x) 1_{\|\nabla f(x)\| > 0}$ for

$$\kappa_f(x) = -\frac{1}{\|\nabla f(x)\|} \langle \nu_f(x)^\perp, D^2 f(x) \nu_f(x)^\perp \rangle, \text{ and } \nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|},$$

$$\int_{\mathbb{R}} h(t) \Phi_0(E_f(t), U) dt = -\frac{1}{2\pi} \int_U h(f(x)) \langle \nu_f(x)^\perp, D^2 f(x) \nu_f(x)^\perp \rangle 1_{\|\nabla f(x)\| > 0} \, dx.$$

Gaussian stationary random fields

Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be an even C^5 function that is of positive type meaning that $\forall k \geq 1, x_1, \dots, x_k \in \mathbb{R}^2, \lambda_1, \dots, \lambda_k \in \mathbb{R}$,

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \rho(x_i - x_j) \geq 0.$$

Then one can find $(\Omega, \mathcal{A}, \mathbb{P})$ a complete probability space and

$$X : \Omega \times \mathbb{R}^2 \to \mathbb{R}$$

such that X is a **centered Gaussian stationary C^2 random field** :

- $\forall \omega \in \Omega,\ x \in \mathbb{R}^2 \mapsto X(\omega, x) \in \mathbb{R}$ is C^2 ;
- $\forall n \in \mathbb{N},\ x_1, \dots, x_n \in \mathbb{R}^2,\ \omega \in \Omega \mapsto (X(\omega, x_1), \dots, X(\omega, x_n)) \in \mathbb{R}^n$ is a centered Gaussian vector of covariance

$$K(x_i, x_j) = \mathrm{Cov}(X(x_i), X(x_j)) = \rho(x_i - x_j).$$

Isotropy

Definition

$X = (X(x))_{x\in\mathbb{R}^2}$ isotropic if, $\forall Q$ rotation, $(X(Qx))_{x\in\mathbb{R}^2}$ has the same law than X.

Rk : A stationary Gaussian random field is isotropic iff $\rho(Qx) = \rho(x)$ for all Q rotation and $x \in \mathbb{R}^2$

Exple : $\rho(x) = \exp(-\frac{\gamma_1}{2}x_1^2)\exp(-\frac{\gamma_2}{2}x_2^2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$ and $\gamma_1, \gamma_2 \in (0, +\infty)$
Note that $X(x) \sim \mathcal{N}(0,1)$ (standard field) and $\nabla X(x) \sim \mathcal{N}(0, \Gamma_{\nabla X})$ with $\Gamma_{\nabla X} = \mathrm{diag}(\gamma_1, \gamma_2)$

$$X \text{ isotropic iff } \gamma_1 = \gamma_2 \quad \text{and} \quad \Delta = \gamma_2 I_2.$$

Rk : Any stationary C^1 Gaussian random field may be written as

$$Y = m + \sigma X \circ Q,$$

with X standard with $\rho_X = \frac{1}{\sigma^2}\rho_Y$ and $\Gamma_{\nabla X} = \frac{1}{\sigma^2} Q \Gamma_{\nabla Y} Q^T$.

Excursion sets

Let $X = (X(x))_{x \in \mathbb{R}^2}$ be a C^2 stationary Gaussian random field. We consider the excursion set of level $t \in \mathbb{R}$

$$E_X(t) := \{x \in \mathbb{R}^2; X(x) \geq t\}.$$

$\gamma_1 = \gamma_2 = 0.005$ $t = 0$ $t = 1$ $t = 1.96$

$\gamma_1 = 0.002$

Mean geometry for excursion sets

First note that by stationarity

$$\mathbb{E}[\Phi_2(E_X(t),U)] = \mathbb{E}\left(\int_U 1_{X(x)\geq t}dx\right) = |U|\mathbb{P}(X(0)\geq t).$$

Moreover, taking expectation it follows that for all h bounded continuous, writing $\nabla X(0) = \|\nabla X(0)\|\nu_X(0)$ a.s., since $\mathbb{P}(\|\nabla X(0)\| = 0) = 0$,

$$\begin{aligned}\int_{\mathbb{R}} h(t)\mathbb{E}[\Phi_1(E_X(t),U)]dt &= |U| \times \frac{1}{2}\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right)\\ \int_{\mathbb{R}} h(t)\mathbb{E}[\Phi_0(E_X(t),U)]dt &= |U| \times \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\langle\nu_X(0)^\perp, D^2X(0)\nu_X(0)^\perp\rangle\right)\end{aligned}$$

We therefore consider **LK densities** :

$$C_j^*(X,t) = \frac{1}{|U|}\mathbb{E}[\Phi_j(E_X(t),U)].$$

Stationarity

We write $X_j = \partial_j X$, $X_{ij} = \partial_{ij}^2 X$ for $1 \leq i, j \leq 2$. Since

$$\begin{aligned}
\rho(x) = \mathrm{Cov}(X(x), X(0)) &= \mathbb{E}(X(x)X(0)) = \mathrm{Cov}(X(x+y), X(y)), \\
\partial_i \rho(x) = \mathrm{Cov}(X_i(x), X(0)) &= \mathrm{Cov}(X_i(0), X(-x))), \text{ and} \\
\partial_{ij}^2 \rho(x) = \mathrm{Cov}(X_{ij}(x), X(0)) &= -\mathrm{Cov}(X_i(0), X_j(-x))).
\end{aligned}$$

Recall ρ is even and therefore $\partial_i \rho(0) = 0$ for $1 \leq i \leq 2$ implies

$X(0)$ is **independent** from $\nabla X(0) = (X_1(0), X_2(0))$

and similarly, $\nabla X(0)$ is **independent** from $D^2 X(0)$.
Since,

$$\gamma_i = \mathrm{Var}(\partial_i X(0)) = -\partial_{ii}^2 \rho(0) = -\mathrm{Cov}(X_{ii}(0), X(0)).$$

Hence $X_{ii}(0) + \gamma_i X(0)$ is **independent** from $X(0)$. Since we assume that $\mathrm{Cov}(X_1(0), X_2(0)) = 0$ it also implies

$X(0)$ **independent** from $X_{12}(0)$.

Gaussian Lipschitz-Killing (LK) densities

$$\begin{aligned}\int_{\mathbb{R}} h(t) C_1^*(X,t)dt &= \frac{1}{2}\mathbb{E}\left(\mathbb{E}(h(X(0))\|\nabla X(0)\| \,|\, X(0))\right)\\ \frac{1}{2}\mathbb{E}\left(h(X(0))\mathbb{E}(\|\nabla X(0)\|)\right) &= \int_{\mathbb{R}} h(t)\frac{1}{2}\mathbb{E}(\|\nabla X(0)\|)\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt.\end{aligned}$$

with for $e_\theta = (\cos(\theta), \sin(\theta))$,

$$\mathbb{E}(\|\nabla X(0)\|) = \frac{1}{4}\int_0^{2\pi} \mathbb{E}(|\langle \nabla X(0), e_\theta\rangle|)d\theta,$$

$\langle \nabla X(0), e_\theta\rangle \sim \sqrt{\gamma_1\cos^2(\theta)+\gamma_2\sin^2(\theta)}\mathcal{N}(0,1)$ and $\mathbb{E}(|\mathcal{N}(0,1)|) = \sqrt{\frac{2}{\pi}}$.

Proposition

$C_1^*(X,t) = \frac{1}{4}\sqrt{\gamma_{\mathrm{Per}}}e^{-t^2/2}$, *a.e.* $t\in\mathbb{R}$, *where*

$$\gamma_{\mathrm{Per}} = \left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\gamma_1\cos^2(\theta)+\gamma_2\sin^2(\theta)}d\theta\right)^2.$$

Gaussian Lipschitz-Killing (LK) densities

Let $\nu_X(0) = (\cos(\Theta), \sin(\Theta))$ with Θ independent from $X(0)$, $D^2X(0)$,

$$\begin{aligned}
&\int_{\mathbb{R}} h(t) C_0^*(X,t)dt \\
&= \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\left[X_{11}(0)\sin^2(\Theta) + X_{22}(0)\cos^2(\Theta) - X_{12}(0)\sin(2\Theta)\right]\right) \\
&= \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\left[-\gamma_1 X(0)\mathbb{E}\left(\sin^2(\Theta)\right) - \gamma_2 X(0)\mathbb{E}\left(\cos^2(\Theta)\right)\right]\right) \\
&= \int_{\mathbb{R}} h(t)\frac{1}{2\pi}\mathbb{E}(\gamma_1 \sin^2(\Theta) + \gamma_2\cos^2(\Theta))\frac{t}{\sqrt{2\pi}}e^{-t^2/2}dt.
\end{aligned}$$

Proposition

$C_0^*(X,t) = \frac{1}{(2\pi)^{3/2}}\gamma_{\mathrm{TC}}\, t e^{-t^2/2}$, *a.e.* $t \in \mathbb{R}$, *where*

$$\gamma_{\mathrm{TC}} = \mathbb{E}(\gamma_1\sin^2(\Theta) + \gamma_2\cos^2(\Theta)) = \sqrt{\gamma_1\gamma_2}.$$

Rk : if $\gamma_1 = \gamma_2$ then $\gamma_{\mathrm{TC}} = \gamma_{\mathrm{Per}} = \gamma_2$ and $\nu_X(0) \sim \mathcal{U}(S^1)$.

Summary

Theorem

For X C^2 stationary Gaussian standard random field

$$
\begin{aligned}
C_0^*(X,t) &= \gamma_{\mathrm{TC}} \frac{1}{(2\pi)^{3/2}}\; t\, e^{-\frac{t^2}{2}} \;\; a.e. \\
C_1^*(X,t) &= \sqrt{\gamma_{\mathrm{Per}}}\,\frac{1}{4}\; e^{-\frac{t^2}{2}} \;\; a.e. \\
C_2^*(X,t) &= 1-\Psi(t) \;\; \textit{for } \Psi(t) = \int_{-\infty}^{t} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du
\end{aligned}
$$

- If one knows that $t \mapsto C_1^*(X,t)$ or $t \mapsto C_0^*(X,t)$ are continuous then a.e. is enough ! In Berzin, Latour, Leon (2017) general assumptions to ensure that $u \mapsto C_1^*(X,t)$ is continuous ;
- For isotropic stationary C^3 Gaussian field the formulas hold for all level (weakest assumptions cf Adler, Taylor (2007)) with

$$\gamma_{\mathrm{TC}} = \gamma_{\mathrm{Per}} = \gamma_2 = \gamma_1.$$

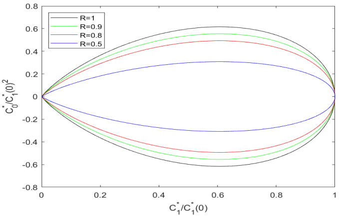
Ratio of anisotropy

Proposition

$\min(\gamma_1, \gamma_2) \leq \gamma_{\mathrm{TC}} \leq \gamma_{\mathrm{Per}} \leq \max(\gamma_1, \gamma_2)$ and $\gamma_{\mathrm{TC}} = \gamma_{\mathrm{Per}}$ iff $\gamma_1 = \gamma_2$.

Defining $\mathrm{R} = \frac{\gamma_{\mathrm{TC}}}{\gamma_{\mathrm{Per}}} \in \left[\frac{\min(\gamma_1,\gamma_2)}{\max(\gamma_1,\gamma_2)}; 1\right]$ and plot the **Almond curve of anisotropy** $\{(x(t), y(t)); t \in \mathbb{R}\}$

$$x(t) = \frac{C_1^*(X,t)}{C_1^*(X,0)} = e^{-t^2/2} \text{ and } y(t) = \frac{C_0^*(X,t)}{(C_1^*(X,0))^2} = \frac{16}{(2\pi)^{3/2}} \mathrm{R}\, te^{-t^2/2}.$$



with $C_1^*(X,0) = 4\sqrt{\gamma_{\mathrm{Per}}}$. See also Klatt, Hörmann, Mecke (2021) for inspiration

Statistical inference

$\gamma_1 = \gamma_2 = 0.005$ $\quad t = 0$ $\quad t = 1$ $\quad t = 1.96$

In simulation we can compute for $t \in \mathbb{R}$ and $j = 0, 1, 2,$

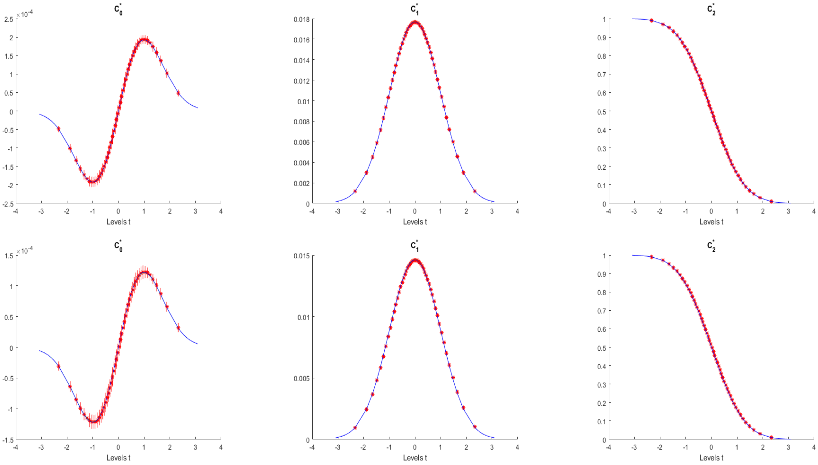
$$\hat{C}_j(X, t) = \frac{\Phi_j(E_X(t), U)}{|U|}, \text{(empirically accessible)}$$

with $\mathbb{E}(\hat{C}_j(X, t)) = C_j^*(X, t)$. Under good assumptions on X (at least C^3 and good decay on ρ and derivatives) we should have

$$\hat{C}_j(X, t) \underset{U \nearrow \mathbb{R}^2}{\longrightarrow} C_j^*(X, t) \text{ a. s. } \text{ with asymptotic normality.}$$

Some Ref on CLT : Spodarev (2012), Estrade, Leoń (2016), Müller (2017), Kratz Vadlamani (2018), Reddy et al (2018), Berzin (2021).

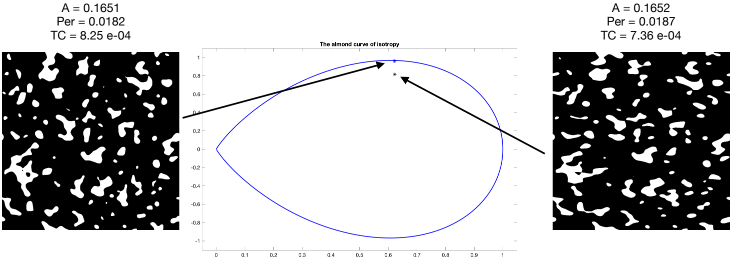
Statistical inference



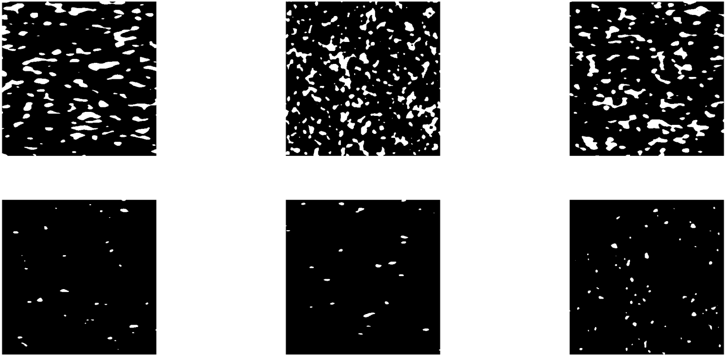
C_0^* C_1^* C_2^*

First line : $\gamma_1 = \gamma_2 = 0.005$ and Second line $\gamma_1 = 0.001$

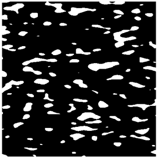
Is it anisotropic ?



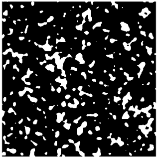
Is it anisotropic ?



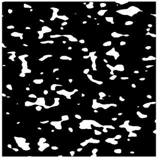
Is it anisotropic ?



$R = 0.8$



$R = 1$



$R = 0.9$

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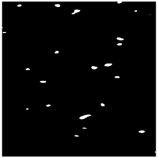
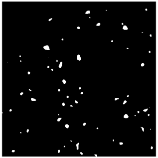


$R = 0.8$



$R = 1$

Effective level

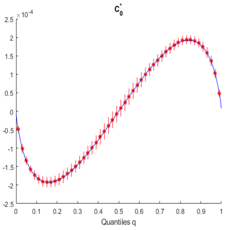
For $t \in \mathbb{R}$ unknown, following Di Bernardino and Duval (2020), define the effective level as

$$\hat{t} = \Psi^{-1}(1 - \hat{C}_2(X, t)),$$

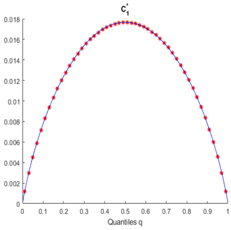
Note that for the quantile $t = \Psi^{-1}(q)$ for $q \in (0,1)$ one has $C_2^*(X, t) = 1 - q$ and set

$$\hat{q} = 1 - \hat{C}_2(X, t) \text{ such that } \hat{t} = \Psi^{-1}(\hat{q}).$$

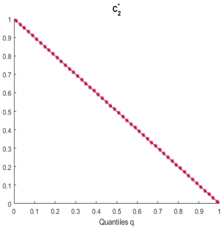
We can consider $C_j^*(X, \Psi^{-1}(q))$, $j = 0, 1, 2$.



$C_0^* \circ \Psi^{-1}$



$C_1^* \circ \Psi^{-1}$



$C_2^* \circ \Psi^{-1}$

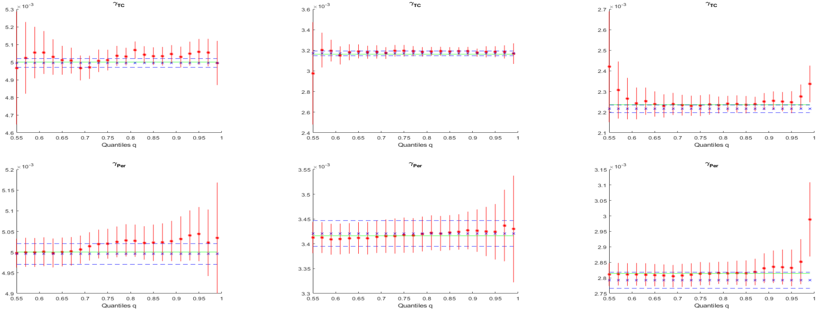
Effective $\gamma_{\rm Per}$ and $\gamma_{\rm TC}$

Using that

$$C_0^*(X,t) = \gamma_{\rm TC} \frac{1}{(2\pi)^{3/2}}\; t\, e^{-\frac{t^2}{2}} \text{ and } C_1^*(X,t) = \sqrt{\gamma_{\rm Per}}\frac{1}{4}\; e^{-\frac{t^2}{2}},$$

define for $\hat{t} > 0$ or $\hat{q} > 1/2$

$$\hat{\gamma}_{\rm TC} = \hat{C}_0(X,t) \times (2\pi)^{3/2}\; \hat{t}^{-1}\, e^{\frac{\hat{t}^2}{2}} \text{ and } \hat{\gamma}_{\rm Per} = \hat{C}_1(X,t)^2 \times 16\; e^{\hat{t}^2}.$$

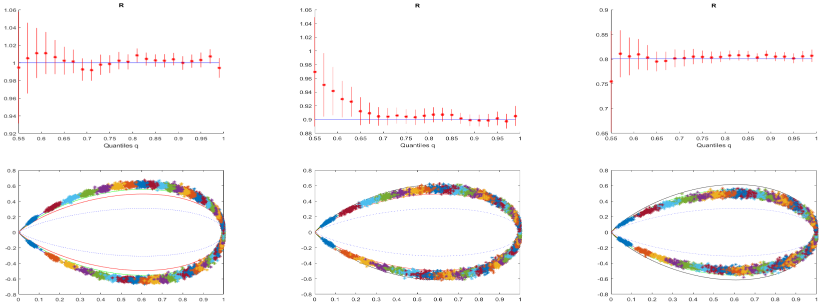


R = 1 R = 0.9 R = 0.8

Effective Ratio of anisotropy

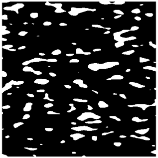
We finally define

$$\hat{\mathrm{R}} = \frac{\hat{\gamma}_{\mathrm{TC}}}{\hat{\gamma}_{\mathrm{Per}}} = \frac{\hat{C}_0(X,t)}{\hat{C}_1(X,t)^2} \times \frac{(2\pi)^{3/2}}{16} \;\hat{t}^{-1}\, e^{-\frac{\hat{t}^2}{2}}$$

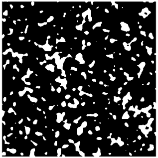


$\mathrm{R} = 1$ $\mathrm{R} = 0.9$ $\mathrm{R} = 0.8$

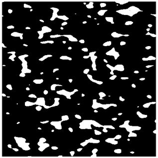
Is it anisotropic ?



$\hat{\mathrm{R}} = 0.7672$

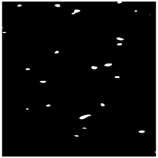


$\hat{\mathrm{R}} = 1.012$

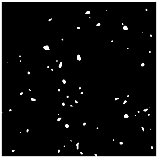


$\hat{\mathrm{R}} = 0.8826$

$\hat{\mathrm{R}} = 0.9029$



$\hat{\mathrm{R}} = 0.7372$



$\hat{\mathrm{R}} = 0.9762$

Conclusion and perspectives

Conclusion :

- New geometrical equivalent of spectral moments
- Anisotropy estimation available from one excursion set
- Extension in dimension d with mean curvature, numerical evaluation for $d = 3$

Perspective :

- Second order and higher moment properties
- Control of bias induced by discrete simulation/estimation
- Extension for fractional Gaussian fields

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