University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MATH0055: INTRODUCTION TO TOPOLOGY

Friday 31 May 2002, 09.30–11.30

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

Examiners will attach importance to the number of well-answered questions.

1. (a) Let X be a set. What is a *topology* on X?

   (b) Let A be a subset of a topological space $(X, \mathscr{T})$.
   What is the *induced topology* on A?
   What is the *closure* $\overline{A}$ of A?
   Show that A is dense as a subset of $\overline{A}$ equipped with the induced topology.

   (c) Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces.
   Let $A \subset X$ and $B \subset Y$. Show that the closure $\overline{A \times B}$ of $A \times B$ with respect to the product topology on $X \times Y$ is given by

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

   (You may state without proof any standard result that you use.)

2. (a) What does it mean to say that a topological space is

   (i) *Hausdorff*?
   (ii) *normal*?
   (iii) *compact*?

   (b) Show that a compact subset of a Hausdorff topological space is closed.

   (c) Show that a closed subset of a normal topological space is normal (with respect to the induced topology).

   (d) Show that the metric topology on a metric space is normal.

3. (a) Let A be a subset of a topological space $(X, \mathscr{T})$.

   (i) What does it mean to say that A is *connected*?

   (ii) Suppose that A is connected. Show that its closure $\overline{A}$ is also connected.

   (iii) If A is connected, does it follow that A has both connected interior and connected boundary?
   Justify your answer with a proof or counterexample.

   (iv) Conversely, if A has both connected interior and connected boundary, does it follow that A is connected?
   Justify your answer with a proof or counterexample.

   (b) Recall that a topological space is *totally disconnected* if its only non-empty connected subsets are the singleton sets.
   Show that any set with the discrete topology is totally disconnected.
   Is the converse true? That is, does a totally disconnected topological space necessarily have the discrete topology?
   Justify your answer with a proof or counterexample.

4. (a) What is a *surface*?

   (b) Explain why the Klein bottle is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$.

   (c) Which surface is determined by the following triangulation?

123	256	341	451
156	268	357	468
167	275	374	476
172	283	385	485