1. (a) What is a *topological space*?

   (b) What is the *discrete topology* on a set?

   Let X be a finite set and $\mathscr{T}$ a topology on X for which all singleton sets $\{x\}$, $x \in X$, are closed. Show that $\mathscr{T}$ is the discrete topology.

   Does the same conclusion hold if X is countable? Give a proof or find a counter-example.

   (c) Let A be a subset of a topological space X.

   What is the *closure* $\overline{A}$ of A?

   Show that A is closed if and only if $A = \overline{A}$.

   Show that a map $f : X \to Y$ of topological spaces is continuous if and only if, for any set $A \subset X$, we have
$$f(\overline{A}) \subset \overline{f(A)}$$

2. (a) Let X and Y be topological spaces.

   What is the *product topology* on $X \times Y$?

   Show that the product topology is the unique topology on $X \times Y$ with the property that, for any map $h = (f, g) : Z \to X \times Y$ of a topological space Z, h is continuous if and only if $f : Z \to X$ and $g : Z \to Y$ are continuous.

   (b) What does it mean to say that a topological space is *Hausdorff*?

   Let $f : X \to Y$ be a map of topological spaces. The *graph* $\Gamma(f)$ of f is given by
$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\} \subset X \times Y.$$

   Let Y be Hausdorff and f continuous. Show that $\Gamma(f)$ is a closed subset of $X \times Y$ with respect to the product topology.

   Conversely, suppose that $\Gamma(f)$ is closed. Does it follow that f is continuous? Give a proof or provide a counter-example.

3. (a) What does it mean to say that a topological space is

   (i) *compact*?

   (ii) *normal*?

   Show that a closed subset of a compact topological space is compact.

   Show that a compact Hausdorff topological space is normal.

   (b) What does it mean to say that a topological space is *connected*?

   Let X, Y be connected topological spaces and $A \subsetneq X$, $B \subsetneq Y$ proper subsets.

   Prove that $X \times Y \setminus A \times B$ is connected.

   [You may assume without proof that X is homeomorphic to any $X \times \{y\}$, $y \in Y$, and similarly for Y.]

4. (a) Explain why $T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

   (b) Which surface is determined by the following triangulation?

124	235	346	457	561	672
713	134	245	356	467	571
126	237				