M55: Exercise sheet 8

1. Make suitable simplifications to identify the following surfaces:
(i) $S(abcbca^{-1})$, (ii) $S(abca^{-1}b^{-1}c^{-1})$, (iii) $S(abcabc^{-1})$, (iv) $S(abcdec^{-1}da^{-1}b^{-1}e^{-1})$,
(v) $S(abcda^{-1}b^{-1}c^{-1}d^{-1})$, (vi) $S(abcda^{-1}b^{-1}c^{-1}d)$, (vii) $S(a_1a_2\ldots a_na_1^{-1}a_2^{-1}\ldots a_{n-1}^{-1}a_n)$,
(viii) $S(a_1a_2\ldots a_na_1^{-1}a_2^{-1}\ldots a_{n-1}^{-1}a_n^{-1})$.

2. Find the surfaces given by the following triangulations:

(a)	123	234	345	451	512	136	246	356	416	526.
(b)	124	134	246	236	367	347	469	459	698	
	678	457	259	289	578	358	125	238	135.	

3. Show that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

4. Let e, f, v be the number of faces, edges and vertices in a triangulation of a surface. Show that $3f = 2e$.
(Thus f is even.)

5. (a) What is the Euler characteristic of a torus?
(b) What are the possible values of the Euler characteristic of a compact, connected surface?

6. Show that a compact, connected surface admits a triangulation with exactly 4 faces if and only if it is a sphere.

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M55: Exercise sheet 8—Solutions

1. (i)

$$\begin{aligned} S(abcbca^{-1}) &= S(a^{-1}abcbc) && \text{by step 0: start at } a^{-1} \\ &= S(bcbc) && \text{by step 1} \\ &= \mathbb{R}P(2)\#S(c^{-1}c) && \text{by step 2} \\ &= \mathbb{R}P(2) && \text{by step 1.} \end{aligned}$$

(ii)

$$\begin{aligned} S(abca^{-1}b^{-1}c^{-1}) &= T^2\#S(c^{-1}c) && \text{by step 3} \\ &= T^2 && \text{by step 1.} \end{aligned}$$

(iii)

$$\begin{aligned} S(abcabc^{-1}) &= \mathbb{R}P(2)\#S(c^{-1}b^{-1}bc^{-1}) && \text{by step 2} \\ &= \mathbb{R}P(2)\#S(c^{-1}c^{-1}) && \text{cancel } b\text{'s by step 1} \\ &= \mathbb{R}P(2)\#\mathbb{R}P(2) \end{aligned}$$

(iv)

$$\begin{aligned} S(abcdec^{-1}da^{-1}b^{-1}e^{-1}) &= S(dec^{-1}da^{-1}b^{-1}e^{-1}abc) && \text{start at } d \text{ by step 0} \\ &= \mathbb{R}P(2)\#S(ce^{-1}a^{-1}b^{-1}e^{-1}abc) && \text{by step 2} \\ &= 2\mathbb{R}P(2)\#S(b^{-1}a^{-1}ebae) && \text{by step 2} \\ &= 2\mathbb{R}P(2)\#S(ebaeb^{-1}a^{-1}) && \text{start at } e \text{ by step 0} \\ &= 3\mathbb{R}P(2)\#S(a^{-1}b^{-1}b^{-1}a^{-1}) && \text{by step 2} \\ &= 4\mathbb{R}P(2)\#S(bb) && \text{by step 2} \\ &= 5\mathbb{R}P(2). \end{aligned}$$

(v)

$$\begin{aligned} S(abcda^{-1}b^{-1}c^{-1}d^{-1}) &= T^2\#S(c^{-1}d^{-1}cd) && \text{by step 3} \\ &= T^2\#T^2. \end{aligned}$$

(vi)

$$\begin{aligned} S(abcda^{-1}b^{-1}c^{-1}d) &= S(da^{-1}b^{-1}c^{-1}dabc) && \text{start at } d \\ &= \mathbb{R}P(2)\#S(cbaabc) && \text{by step 2} \\ &= 2\mathbb{R}P(2)\#S(b^{-1}a^{-1}a^{-1}b^{-1}) && \text{by step 2} \\ &= 3\mathbb{R}P(2)\#S(aa) && \text{by step 2} \\ &= 4\mathbb{R}P(2). \end{aligned}$$

(vii)

$$\begin{aligned} S(a_1a_2\dots a_na_1^{-1}a_2^{-1}\dots a_{n-1}^{-1}a_n) &= S(a_na_1^{-1}a_2^{-1}\dots a_{n-1}^{-1}a_na_1a_2\dots a_{n-1}) \\ &= \mathbb{R}P(2)\#S(a_{n-1}\dots a_1a_1\dots a_{n-1}) \\ &= \mathbb{R}P(2)\#(n-1)\mathbb{R}P(2) = n\mathbb{R}P(2), \end{aligned}$$

where the last line comes from $n-1$ applications of step 2.

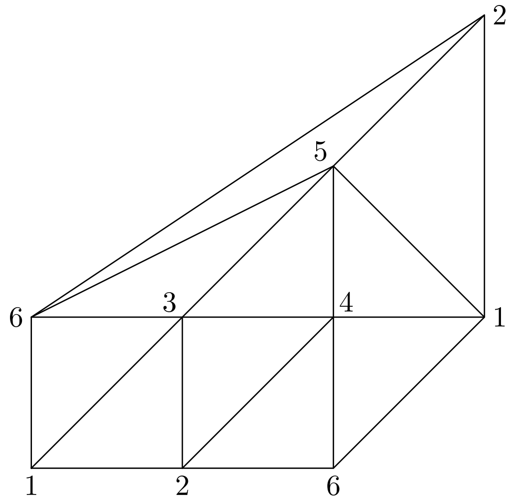
(viii)

$$\begin{aligned} S(a_1 a_2 \dots a_n a_1^{-1} a_2^{-1} \dots a_{n-1}^{-1} a_n^{-1}) &= T^2 \# S(a_3^{-1} \dots a_n^{-1} a_3 \dots a_n) \\ &= T^2 \# S(a_3 \dots a_n a_3^{-1} \dots a_n^{-1}) \end{aligned}$$

now iterate to get

$$= \begin{cases} \frac{n}{2} T^2 & \text{for } n \text{ even;} \\ \frac{n-1}{2} T^2 \# S(a_n a_n^{-1}) = \frac{n-1}{2} T^2 = & \text{for } n \text{ odd.} \end{cases}$$

2. (a) Glue the triangles together to get something like:



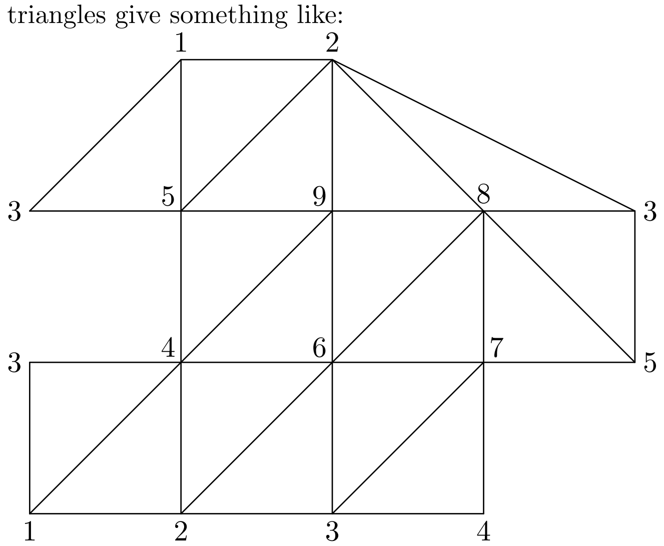
(you may well end up with a different polygon).
Read around the boundary to get the ordered list of vertices:

1 6 2 1 6 2

so that the surface is homeomorphic to $S(abcabc)$. But

$$\begin{aligned} S(abcabc) &= \mathbb{R}P(2) \# S(c^{-1} b^{-1} bc) && \text{by step 2} \\ &= \mathbb{R}P(2) && \text{by two applications of step 1.} \end{aligned}$$

(b) This time the triangles give something like:



The list of vertices reads:

1 3 4 5 3 1 2 3 5 4 3 2

so that our surface is homeomorphic to $S(abcda^{-1}efd^{-1}c^{-1}b^{-1}f^{-1}e^{-1})$. Now

$$\begin{aligned} S(abcda^{-1}efd^{-1}c^{-1}b^{-1}f^{-1}e^{-1}) &= T^2 \# S(f^{-1}e^{-1}efd^{-1}c^{-1}cd) && \text{by step 3} \\ &= T^2 && \text{by multiple applications of step 1.} \end{aligned}$$

3. First triangulate each surface: say that S_i has f_i faces, e_i edges and v_i vertices. The connected sum is formed by removing the interior of two triangles and identifying corresponding edges (and vertices). The triangulations of S_1 and S_2 then fit together to give a triangulation of $S_1 \# S_2$ with $f_1 + f_2 - 2$ faces, $e_1 + e_2 - 3$ edges and $v_1 + v_2 - 3$ vertices. Arithmetic now gives

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

4. Each face contributes three edges but each edge lies on exactly two faces (otherwise we would not have a surface). Thus $3f = 2e$.

5. (a) To compute $\chi(T^2)$ we need a triangulation of T^2. Luckily, one is provided by question 2(b) which has 18 sides (and so 27 vertices by question 4) and 9 vertices. We conclude that $\chi(T^2) = 0$.
(b) All surfaces are build out of the basic surfaces which have Euler characteristics $\chi(S^2) = 2$, $\chi(\mathbb{R}P(2)) = 1$ and $\chi(T^2) = 0$ (as we have just proved). From question 3, we see that, for any surface S,

$$\begin{aligned} \chi(S \# T^2) &= \chi(S) - 2 \\ \chi(S \# \mathbb{R}P(2)) &= \chi(S) - 1 \end{aligned}$$

so that the connected sum of g tori has Euler characteristic $2 - 2g$ while the connected sum of g $\mathbb{R}P(2)$'s is $2 - g$.
This accounts for all compact connected surfaces so we learn that the possible values of the Euler characteristic are any integers less than or equal to two.

6. Certainly the sphere has a triangulation with four faces (the tetrahedral one!).
For the converse, let S be such a surface with Euler characteristic χ and v the number of vertices in this triangulation with four faces. By question 4, there are 6 edges in this triangulation so that

$$\chi = 4 - 6 + v = v - 2.$$

Moreover, $v \geq 3$ (each triangle has three distinct vertices) giving $\chi \geq 1$. In view of question 5, this only leaves two possibilities, $\chi = 2$ (in which case $S \cong S^2$) or $\chi = 1$. In this last case, we have $v = 3$ which means that all four triangles have the same vertices. But since at least two must have an edge in common, this breaks the rules for triangulations. Thus $v = 4$ and $S \cong S^2$.