M55: Exercise sheet 7

More on components

1. Let X be an open subset of $\mathbb{R}^n$ (equipped with the metric topology, of course!).
   (a) Show that each path-component of X is open.
   (b) Deduce that each path-component of X is also closed.
   (c) Show that X is connected if and only if it is path-connected.

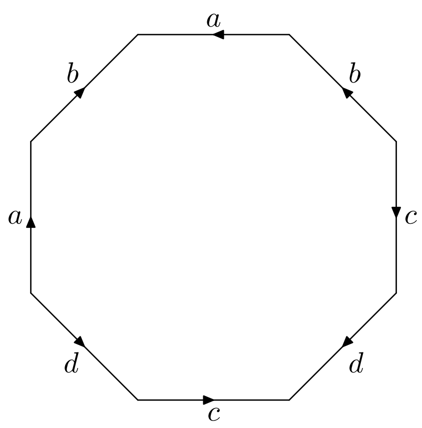
On quotients

2. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if $x = y = 0$ or $xy > 0$.
   (a) How many points are there in $\mathbb{R}/\sim$?
   (b) Write down all the open sets of the quotient topology on $\mathbb{R}/\sim$.
   (c) Deduce that the quotient topology on $\mathbb{R}/\sim$ is not Hausdorff.
   This example shows that a quotient of a Hausdorff space need not be Hausdorff!

3. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. What is the topology on $\mathbb{R}/\sim$?

4. A relation R on a set X may be viewed as a subset R of $X \times X$: $R = \{(x, y) : xRy\}$. Show that if the quotient X/R of a topological space X is Hausdorff then the corresponding subset of $X \times X$ is closed in the product topology.
   Show that the converse is true if the map $p : X \to X/R$ is open.

5. Draw the topological space X obtained as the quotient of the octagon with edges identified as in the diagram below. (X can be embedded nicely in $\mathbb{R}^3$.)



6. Give your favorite alphabet the induced topology from $\mathbb{R}^2$. Now divide it into classes of homeomorphic letters.
   (Think about how you would prove that the classes you find are distinct.)

May 2, 2003

Home page: `http://www.maths.bath.ac.uk/~feb/math0055.html`

M55: Exercise sheet 7—Solutions

1. (a) Fix a path-component $D \subset X$ and $x \in D$. Since X is open, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset X$. However, for any $y \in B_\varepsilon(x)$, there is a path in $B_\varepsilon(x)$ from x to y given by $t \mapsto (1-t)x + ty$. Thus xPy so that $y \in D$ also. Otherwise said, $B_\varepsilon(x) \subset D$ and so D is open.
(b) The complement of D is the union of all the other path components of X and so is open being a union of open sets.
(c) We already know that path-connected sets are connected so the issue is the converse. If a connected X had more than one path-component D then both D and its complement are open and so disconnect X unless that complement is empty. Thus $X = D$ and X is path-connected.

2. (a) There are three points: the equivalence classes of -1, 0 and 1.
(b) Remember that a set is open in the quotient topology if and only if its inverse image by p is open. Writing $[a]$ for the equivalence of a, we enumerate the non-trivial inverse images:
$$p^{-1}(\{[-1]\}) = (-\infty, 0) \qquad p^{-1}(\{[0]\}) = \{0\} \qquad p^{-1}(\{[1]\}) = (0, \infty)$$
$$p^{-1}(\{[-1],[0]\}) = (-\infty, 0] \qquad p^{-1}(\{[0],[1]\}) = [0, \infty) \qquad p^{-1}(\{[-1],[1]\}) = \mathbb{R} \setminus \{0\}.$$
We therefore conclude that the open sets in $\mathbb{R}/\sim$ are $\{\emptyset, \{-1\}, \{1\}, \{-1, 1\}, \mathbb{R}/\sim\}$.
(c) This topology is clearly not Hausdorff since the only open set containing zero is $\mathbb{R}/\sim$.

3. The topology here is the indiscrete topology. Indeed, let $U \subset \mathbb{R}/\sim$ be a non-empty open set and contemplate $p^{-1}(U)$. Let $x \in p^{-1}(U)$. There is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset p^{-1}(U)$ since p is open. Now let $y \in \mathbb{R}$ and choose $q \in \mathbb{Q}$ so that $|x - y - q| < \varepsilon$ (possible since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then $y - q \in (x-\varepsilon, x+\varepsilon) \subset p^{-1}(U)$ whence $y \in p^{-1}(U)$ since $y \sim y + q$. Thus $p^{-1}(U) = \mathbb{R}$ giving $U = \mathbb{R}/\sim$. Thus there is only one non-empty set in $\mathbb{R}/\sim$ and we conclude that the quotient topology is the discrete topology.

4. Suppose that X/R is Hausdorff and that $(x, y) \notin R$. Thus $x \not R y$ so that $p(x) \neq p(y)$. Thus there are disjoint open sets U, V in X/R with $p(x) \in U$ and $p(y) \in V$. Then $p^{-1}(U) \times p^{-1}(V)$ is a basic open set in $X \times X$ containing (x, y) and disjoint from R ($(u, v) \in R$ if and only if $p(u) = p(v)$). It follows that R is closed since its complement is open.
For the converse, let $p(x) \neq p(y)$ be distinct points in X/R so that $(x, y) \notin R$. If R is closed there is a basic open set $A \times B$ containing (x, y) and disjoint from R. It is then easy to check that $p(A)$ and $p(B)$ are disjoint sets in X/R with $p(x) \in p(A)$ and $p(y) \in p(B)$ and since p is open, both $p(A)$ and $p(B)$ are open. Thus X/R is Hausdorff.

5. X is a connected sum of two tori.

6. We take as our alphabet:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

and then the separate homeomorphism classes are

{A R} {B} {C G I J L M N S U V W Z} {D O} {E F T Y} {H K} {P} {Q} {X}